FRESHMAN MATHEMATICS

SLOBIN and WILBUR'S

FRESHMAN MATHEMATICS

THIRD EDITION

REVISED BY

C. V. NEWSOM

Assistant Commissioner for Higher Education, The State of New York · Formerly Professor of Mathematics, Oberlin College

RINEHART & COMPANY, INC.

PUBLISHERS · NEW YORK

Preface

This new edition is a second revision of *Freshman Mathematics* by H. L. Slobin and W. E. Wilbur. The original plan of presenting algebra, trigonometry, and analytical geometry as a tandem course—to permit adequate preparation in each subject, to permit the use of arithmetic and algebra in trigonometry, and of arithmetic, algebra, and trigonometry in analytical geometry—is maintained in this revision. Further, the general aim is still to present these subjects so that the student may have a real understanding of the fundamental principles and processes involved and of the values of these subjects vocationally and culturally. It is hoped that the book will give the student an adequate foundation in mathematics, irrespective of his educational objectives, and that it will prove even more useful as a teaching instrument than previous editions.

The second edition of *Freshman Mathematics* has been almost entirely rewritten in this revision. Special attention has been given to the readability of the material, and many expositions have been revised to make them more lucid. Although the tradition of content and treatment of the earlier editions has been maintained, new trends and emphases have been recognized.

In Book I, Algebra, the chapter on infinite series appearing in the previous edition has been deleted, and a brief chapter on inequalities has been added. Book I contains nearly 1,200 exercises for the student.

Book II, Trigonometry, has been revised in line with the growing trend toward analytic trigonometry. Book II contains about 500 exercises for the student.

Book III, Analytic Geometry, continues to treat more than the conic sections. The general equation of the second degree and curve fitting are covered extensively. Book III contains about 800 exercises for the student.

The reviser and publishers express their appreciation of the many helpful suggestions that have come to them from the users of the previous edition. Especial recognition is due Professor J. S. Taylor of the University of Pittsburgh and Mrs. Ruth Smyth of Wooster College, who have made noteworthy contributions to the present revision.

Tables I to IV are taken from the *Rinehart Mathematical Tables* by Harold D. Larsen.

C. V. Newsom

Albany, New York
January, 1949

Table of Contents

Chapter 10 · QUADRATIC FUNCTIONS AND EQUATIONS

Chapter 11 · SYSTEMS INVOLVING QUADRATIC EQUATIONS

Chapter 12 · INTEGRAL RATIONAL FUNCTIONS

Chapter 13 · LOGARITHMS

BOOK II · TRIGONOMETRY

Chapter 1 · TRIGONOMETRIC FUNCTIONS

Chapter 2 · TRIGONOMETRIC IDENTITIES AND CONDITIONAL EQUATIONS

Chapter 3 · SOLUTION OF TRIANGLES

Chapter 3 · EQUATIONS OF LOCI

Chapter 4 · THE STRAIGHT LINE

Chapter 5 · THE CIRCLE

Chapter 6 · THE ELLIPSE

Chapter 7 · THE HYPERBOLA

Chapter 8 · THE PARABOLA

Chapter 9 · THE GENERAL EQUATION OF THE SECOND DEGREE

Chapter 10 · CURVE FITTING

Chapter 11 · PARAMETRIC EQUATIONS

Chapter 12 · SOLID ANALYTIC GEOMETRY

TABLES

Book I · ALGEBRA

1 Measurement and Number

1. MEASUREMENT

The concept of measurement is most important in the life of every person and especially in the career of the scientist. Quantities which are measurable are frequently called *magnitudes;* among them are lengths, areas, volumes, speeds, pressures, and temperatures. Obviously not all lengths are the same, nor all areas, nor all pressures. In order to be precise in our description of the extent or size of such magnitudes, it is important that we devise ways of measuring them.

2. UNITS OF MEASUREMENT

To measure a magnitude, it is essential first to select an appropriate amount of the quantity as a unit of measurement. Thus, a length may be expressed in terms of the foot as a unit; an age may be expressed in terms of the year as a unit; and a person's wealth may be expressed in terms of the dollar as a unit.

3. DIRECT MEASUREMENT

When a magnitude is measured by direct comparison with the amount adopted as the unit, the method of measurement is described as a *direct measurement.*

4. INDIRECT MEASUREMENT

To measure the area of a rectangle, it is convenient to make direct measurements to determine the length and width and then find the area by a simple computation. So the measure of the area is determined indirectly and is spoken of as an *indirect measurement.* Obviously, such measurements as the radius of the earth, or the weight of the earth, or our distance from the sun are obtained as indirect measurements. Indeed, most measurements are of this kind. Indirect measurements always depend upon direct measurements.

5. MEASUREMENT AND NUMBER

The measure of a quantity is expressed by means of a number. Thus, the age of a person is designated by a number of years and the weight of a person by a number of pounds.

If a chosen unit is contained an integral number of times in some quantity under consideration, the desired measurement is obtained by merely *counting* the number of times that the unit is contained in the magnitude being measured. In such a case, the measure is given by a whole number, which we shall call a *positive integer*. The positive integers may be represented consecutively by points equally spaced on a straight line, beginning with some arbitrary starting point denoted by zero, as shown in Figure 1. The distance between any two points corresponding

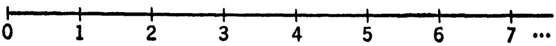


Fig. 1

to consecutive integers is the unit of measurement upon the line.

It often happens that a specified unit is not contained an integral number of times in a quantity being measured. For example, let us assume that we are measuring the length of a line AB in terms of a unit CD (Figure 2), and it happens that CD is contained seven times in AB up to the

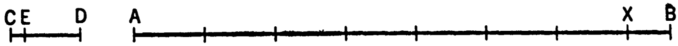


Fig. 2

point X and that the segment XB is less than CD. Obviously, the length of AB is more than 7 units and less than 8 units.

If XB is less than one half of the unit CD, then 7 is said to be the length of AB to the nearest integer. If, on the other hand, XB is greater than one half of the unit CD, then 8 is said to be the length of AB to the nearest integer. If XB is just one half of the unit CD, then 8 is usually considered to be the length of AB to the nearest integer; the number 8 is selected instead of 7 because, in case of a choice between an odd or even integer, the number that is even is usually chosen.

If we wish to get a closer approximation to the length of AB, we may divide the unit CD into any number of equal parts and proceed to measure the segment XB. For illustrative purposes, let CE be one of the 5 equal parts of CD. If the part CE is contained in XB an exact number of times, say three times, then the total length of AB is $7\frac{3}{5}$ units, or $\frac{38}{5}$ units. If it happens that CE is not contained exactly in XB, but is contained twice and the remainder is less than one half of CE, we say that the length of AB is $7\frac{2}{5}$ or $\frac{37}{5}$, to the nearest fifth of the unit. It is apparent that this process may be continued to any desired precision.

6. RATIONAL NUMBERS

It is evident that the length of a line segment, measured in terms of any linear unit, may always be expressed, either exactly or approximately, as

the quotient of two integers. Of course, each integer may be regarded as a quotient in which the divisor, or denominator, is 1. *A quotient of two integers is called a rational number.*

7. IRRATIONAL NUMBERS

We must not assume that ultimately it is possible to express exactly the measurement of any quantity in terms of a given unit by means of a rational number. For instance, the length of the hypotenuse AB of the right triangle given in Figure 3 cannot be measured in terms of the unit AL by writing down a rational number; that is, the ratio AB/AL cannot be expressed as a rational number.

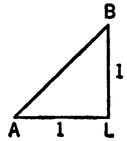


FIG. 3

This may be proved as follows: *Assume* that the length of AB can be expressed by a rational number p/q, where p and q are integers that have no common integral divisor other than 1. Hence, at least one of the numbers p and q must be odd. Since $\overline{AB}^2 = \overline{AL}^2 + \overline{LB}^2 = 1 + 1 = 2$, it follows that

$$\frac{p^2}{q^2} = 2,$$

or

$$p^2 = 2q^2.$$

The last equation shows that p^2 is an even number since it is equal to the product of 2 and an integer. It is demonstrable that if p^2 is even, p must be even; so p may be written in the form $2m$, where m is an integer. After replacing p by $2m$, it follows that

$$4m^2 = 2q^2,$$

and

$$q^2 = 2m^2.$$

The last equation shows that q^2 is an even number; hence, q is an even number. Thus, p and q are both even numbers, which is contrary to the hypothesis which implied that at least one of the numbers p or q is odd.

Since the assumption that the length AB can be expressed by a rational number p/q leads to a contradiction, it follows that the assumption is false.*

In order to have a number that corresponds to the length AB when measured in terms of AL, a special symbol must be created. The length is designated by $\sqrt{2}$. When we assign the symbol $\sqrt{2}$ to AB, we merely mean that its length x satisfies the equation $x^2 = 2$.

The number $\sqrt{2}$ is an irrational number. By definition, an irrational number is any number of arithmetic which is not a rational number. It can be shown that the nth root of any number which is not a perfect nth

* This type of argument is described as "reduction to an absurdity." It is used frequently in mathematical analysis.

power of a rational number is an irrational number. Also, the number π and many other numbers met in mathematical analysis are irrational.

In elementary geometry it is shown that the segment AB, corresponding to $\sqrt{2}$, may be constructed exactly with the aid of ruler and compasses and, hence, may be laid off on the line AF (Figure 4). In general, however,

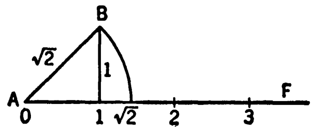


FIG. 4

the segments corresponding to irrational numbers cannot be constructed by using a ruler and the compasses. Although we are unable to construct exactly the segments corresponding to most irrational numbers, we shall assume that to every line segment measured from A on AF there corresponds a positive number (rational or irrational), and, conversely, to every positive number there corresponds a line segment measured from A on AF.

EXERCISES 1

1. The number π is sometimes given as $3\frac{1}{7}$. If it is known that π is irrational, can this value be correct?

2. Is 16.3 rational or irrational? Justify your reply.

3. Show that $3\sqrt{2}$ is irrational.

SUGGESTION: Assume that $3\sqrt{2}$ is rational, and then show that this assumption leads to a contradiction of the statement that $\sqrt{2}$ is irrational.

4. Show that $\sqrt{2} + 5$ is irrational.

Note the suggestion in Exercise 3.

5. Designate the numbers in the following list which are irrational:

$$3;\quad 1/\sqrt{2};\quad 31/47;\quad \sqrt[3]{27};\quad \sqrt{18};\quad \pi + 7;\quad 5.16;\quad \sqrt{5}$$

6. If the radius of a circle is a rational number, is the area rational?

8. THE DECIMAL NUMBER NOTATION

The student is already familiar with the common number notation that employs a base of 10; since the base is 10, the notation is called the *decimal* notation. Thus, the numeral 325 denotes $3(100) + 2(10) + 5(1)$. In fact, the 3 is said to be the *hundreds'* digit; 2 is the *tens'* digit; and 5 is the *units'* digit. Similarly, 243.652 denotes $2(100) + 4(10) + 3(1) + 6(\frac{1}{10}) + 5(\frac{1}{100}) + 2(\frac{1}{1000})$.

Scientists and businessmen, in solving problems resulting in either rational or irrational numbers, frequently find it convenient to express the answers in the decimal notation. The student will recall that the decimal equivalent of a common fraction may be obtained, at least approximately,

by dividing the numerator by the denominator. Thus, the numbers $\frac{1}{3}$, $\frac{1}{7}$, $\frac{3}{11}$, $\frac{4}{5}$, and $\frac{15}{8}$, when converted into the decimal notation to the nearest thousandths, are, respectively, 0.333, 0.143, 0.273, 0.800, and 1.875. It will be noted that $\frac{1}{7}$ expressed as a decimal *to the nearest thousandth* is 0.143 rather than 0.142. Conversely, the numbers 0.125, 0.346, and 0.028 in the decimal notation may be expressed, respectively, as $\frac{125}{1000}$, $\frac{346}{1000}$, and $\frac{28}{1000}$; upon simplification these fractions become $\frac{1}{8}$, $\frac{173}{500}$, and $\frac{7}{250}$.

Irrational numbers may be expressed in the decimal notation to any desired degree of approximation. Frequently this requires the use of various tables. More will be said about these tables when they are needed.

9. APPROXIMATE NUMBERS

It is usually assumed by the scientist that any measurement can be made only approximately. Thus, he does not regard a weight of 27.2 lb as having been determined exactly; rather, he thinks of the measurement 27.2 lb as being closer to the true weight than 27.1 lb or 27.3 lb. If the weight could be measured to a greater degree of precision, the result might be 27.1836 lb or 27.1836275 lb.

Rational numbers that represent measurements belong to the class of approximate numbers. Frequently, it is important to be able to *round off* such numbers to some desired degree of precision. When the number 27.1836275 is rounded off to tenths, it is 27.2; when rounded off to four decimal places, it is 27.1836. In general, when rounding off a number to some desired precision, the last digit retained is unchanged if the portion of the number to be dropped is less than one half a unit in the last position retained; the last digit retained is increased by 1 if the portion of the number dropped is more than one half a unit in the last position retained. In the special case when the part of a number to be dropped is exactly one half a unit in the last position retained, the last retained digit is unchanged or increased by 1, whichever will make it even. It is apparent, therefore, that 27.1836275 rounded off to three decimal places is 27.184 since the part to be dropped, namely, 0.0006275, is more than one half a unit in the third decimal place. When rounded off to six decimal places, the result is 27.183628; the last digit is increased by 1 to make it even since the part discarded is *exactly* one half a unit in the sixth place.

In dealing with numbers representing measurements, or with approximate numbers generally, it is frequently necessary to speak of *significant digits*. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are always significant. All zero digits are significant, except for any consecutive zeros immediately adjacent to the decimal point in the case of a decimal fraction less than 1 and probably in the case of an integer. Thus, all digits in the numbers 23.06 ft, 1.008 in., and 0.860 lb are significant. On the other hand, the first two zeros in 0.0060 ft are not significant. Also, the last two digits in the number 80,600 miles would usually not be regarded as significant. Any zeros involved in the writing of an integer are significant if they are the

result of accurate determination; they are not significant if they are used merely to locate the decimal point.

10. NEGATIVE NUMBERS

We often have occasion to measure a quantity in a definite direction. For example, temperatures are measured above and below the starting point, zero. The temperatures above zero are distinguished from those below zero by designating the former by positive numbers and the latter by negative numbers. In general, if positive numbers designate the measurement of a quantity in a certain direction, then negative numbers are used to designate the measurement of the quantity in the opposite direction.

11. REAL NUMBERS

The positive and negative, rational and irrational, numbers and zero constitute the real numbers, and they may be represented graphically by points on a line as typified by the integers in Figure 5. Those to the right of the origin O are usually designated as positive and are marked $+$ (or are unmarked), and those to the left of the origin are usually designated as negative and are marked $-$.

FIG. 5

12. MAGNITUDE OF REAL NUMBERS

Of two numbers indicated on a horizontal scale with positive direction to the right, that one which lies to the right of the other is said to be the greater, and the one which lies to the left is said to be the less. Thus, by reference to Figure 5, 4 is greater than 1; also -3 is greater than -4.

EXERCISES 2

1. Arrange the following numbers in ascending order of magnitude.

$$2;\quad \pi^*;\quad -\sqrt{2};\quad 0;\quad -1;\quad 5\tfrac{1}{2};\quad -3\pi;\quad \sqrt{3}$$

2. Round off the approximate number 62.630255 (*a*) to four decimal places; (*b*) to five decimal places; (*c*) to thousandths; (*d*) tenths; (*e*) units.

3. List the significant digits in each of the following: (*a*) 93,000,000 miles; (*b*) 620.6 ft; (*c*) 0.01 in.; (*d*) 20.004 kg; (*e*) 500 lb.

4. Arrange the following numbers in descending order of magnitude:

$$4\pi;\quad 3\tfrac{1}{4};\quad -3.1;\quad \tfrac{37}{3};\quad -\tfrac{23}{8};\quad 12.3741;\quad -\tfrac{311}{101}$$

5. (*a*) When rounded off to two decimal places, which of the following approximate numbers can be written in the form 15.64? 15.641? 15.638? 15.6349? 15.645? 15.635? 15.6465?

(*b*) Describe the permissible range to the measurement denoted by 15.64 ft.

6. Round off each of the following approximate numbers to two decimal places, and list the significant digits originally appearing in each number:

3.00625 kg; 0.0051 ft; 32.075 mi; 1.004 yd; 0.0150 m; 0.0219 g

* The number π is approximately 3.14159265.

2 The Fundamental Operations Applied to Literal Number Symbols

13. LITERAL NUMBER SYMBOLS

The student already knows from his knowledge of elementary algebra that it often is convenient to use letters in addition to the numerical symbols of arithmetic to designate numbers. We shall assume that he already has some knowledge of the operations of addition, subtraction, multiplication, division, and the finding of powers and roots as they apply to literal number symbols. However, some very fundamental topics involving the use of algebraic symbols are reviewed in the sections which follow in this chapter.

14. DIVISION BY ZERO

When we consider the division of a by b, where $b \neq 0$, we seek the number x such that $bx = a$; that is, division is the inverse of multiplication.

However, if $b = 0$ and $a \neq 0$, it is impossible to find a number x such that $bx = a$, since the product of any number multiplied by 0 is 0. Consequently, division by zero when the dividend is not zero is impossible.

If, when $b = 0$, a is also 0, then we have $bx = a$ *for all values of* x. In this case, x is said to be indeterminate. In general, therefore, division by zero is excluded.

EXERCISES 3

The Substitution of Numbers in Literal Expressions

Find the values of the following expressions:

1. $\left(\dfrac{a-b}{a+b} + 2b\right)^2 - 2ab,$ if $a = 6$, $b = 2$

Solution: After substituting the given values for a and b in the original expression, we have

$$\left(\frac{6-2}{6+2} + 2 \cdot 2\right)^2 - 2 \cdot 6 \cdot 2 = \left(\frac{4}{8} + 4\right)^2 - 24 = \frac{81}{4} - 24 = -\frac{15}{4}$$

2. $\dfrac{a^2 + b^2 - 2b}{a+b},$ if $a = 3$, $b = 1$

3. $\left(\dfrac{a}{b} + \dfrac{b}{a}\right)^2 - \left(\dfrac{a^3}{b^3} + \dfrac{b^3}{a^3}\right),$ if $a = 3$, $b = 2$

4. $(a-2)(a+2)(a-1)(a+1),$ if $a = 3$

5. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$, if $a = 2$, $b = 3$, $c = 4$

6. $\dfrac{1}{a} + \dfrac{1}{bc} - \dfrac{3}{2a}$, if $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$

7. $\sqrt{a^2 + 2ab + b^2}$, if $a = 5$, $b = 7$

8. $\sqrt{a^2 + b^2}$, if $a = 12$, $b = 5$

9. $\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}$, if $a = 3$, $b = 2$

10. $\sqrt[3]{a^3 + 2ab}$, if $a = 2$, $b = 14$

11. $\dfrac{\frac{2a}{3b}}{c} + \dfrac{\frac{3b}{2a}}{2c} + \dfrac{b}{\frac{2}{c}}$, if $a = 1$, $b = 2$, $c = 3$

12. $\dfrac{a - b}{\sqrt{a^2 + b^2}} + \dfrac{a + b}{\sqrt{a^2 - 2ab + b^2}}$, if $a = 4$, $b = 3$

13. $\dfrac{\frac{a}{b}}{\frac{c}{d}} + \dfrac{\frac{3a}{2b}}{\frac{4c}{3d}}$, if $a = 5$, $b = 2$, $c = -2$, $d = 3$

14. Find the value of V if $V = \dfrac{4\pi r^3}{3}$, if $\pi = 3.1416$, $r = 3$.

15. Find the value of S if $S = 4\pi r^2$, if $\pi = 3.1416$, $r = 5$.

16. Find the value of C if $C = 2\pi r$, if $\pi = 3.1416$, $r = 6$.

In the problems which follow, obtain any required roots by referring to Table 4 in the Appendix.

17. $\sqrt{a^2 + b^2}$, if $a = 12$, $b = 4$

18. $\sqrt[3]{a^3 + b^3}$, if $a = 3$, $b = 2$

19. $\sqrt{s(s - a)(s - b)(s - c)}$, if $a = 3$, $b = 5$, $c = 6$; and $s = \dfrac{a + b + c}{2}$

20. $\sqrt{\dfrac{2s}{g}}$, if $s = 71$, $g = 32$

15. ABSTRACT NUMBERS, DENOMINATE NUMBERS, DIMENSIONS

We have already observed that in making a measurement a number n is employed to express the ratio of a quantity q and some chosen unit u of that quantity. This fact may be expressed by

$$n = \frac{q}{u}.$$

The number n is referred to as an *abstract number*, while nu, the number of units of the quantity under consideration, is referred to as a *denominate number*. Thus, 2 is an abstract number, but 2 ft is a denominate number. As already indicated, the measurement implied by a denominate number is usually to be regarded as approximate.

If a length d contains n units of length L, we may write

$$d = nL.$$

In this relation, n is an abstract number, and nL is a denominate number (a length) by virtue of the factor L.

If an area A contains n square units, we may write

$$A = nL^2,$$

where L^2 represents a square unit. In this relation n is an abstract number, and nL^2 is a denominate number (an area) by virtue of the factor L^2.

If a volume V contains n cubic units, we may write

$$V = nL^3,$$

where L^3 represents a cubic unit. In this equality, n is an abstract number and nL^3 is a denominate number (a volume) by virtue of the factor L^3.

If a time t contains n units of time T, we may write

$$t = nT.$$

Thus, n is an abstract number, and nT is a denominate number (a time) by virtue of the factor T.

If a body of mass m contains n units of mass M, we may write

$$m = nM.$$

In this relation, n is an abstract number, and nM is a denominate number (a number of units of mass) because of the factor M.

The symbols L, L^2, L^3, T, and M, as employed above, are referred to as the dimensions (dimensional symbols) of length, area, volume, time, and mass, respectively. Similar symbols have been introduced for dealing with virtually all the magnitudes with which man is concerned.

By means of the dimensional symbols associated with magnitudes that are measured directly, it is possible to construct dimensional symbols to be associated with magnitudes measured indirectly. Thus, if the sides of a rectangle are 2 ft and 3 ft, the area is (2 ft)(3 ft) = 6 ft^2, that is, 6 sq ft.

Similarly, if we designate s/t by v, where s represents a distance and t a time, we note that since s is of dimension L and t is of dimension T, then the dimensional symbol to be associated with v is L/T (unit of length per unit of time).

If we designate m/V by ρ, where m represents a mass and V a volume, we observe that since m is of dimension M and V is of dimension L^3, then the dimensional symbol for ρ is M/L^3 (unit of mass per unit of volume).

If a body of weight 6 lb is lifted through the distance 4 ft, the work involved is (6 lb)(4 ft) = 24 ft-lb.

Dimensional symbols are employed to considerable advantage by the physical scientist in converting denominate numbers from one system of units to another.

Thus, the denominate number 5 yd may be converted to feet or to inches as follows:

$$5(\text{yd}) = 5(3\text{ ft}) = 15\text{ ft} = 15(12\text{ in.}) = 180\text{ in.}$$

Similarly, the denominate number 5 sq yd, or 5 $(\text{yd})^2$, may be converted to square feet or to square inches as follows:

$$\begin{aligned} 5(\text{yd})^2 = 5(3\text{ ft})^2 = 5(9)(\text{ft})^2 &= 45(\text{ft})^2 = 45(12\text{ in.})^2 \\ &= 45(144)(\text{in.})^2 = 6480(\text{in.})^2. \end{aligned}$$

The denominate number 5 days may be converted to hours, minutes, or seconds as follows:

$$\begin{aligned} 5\text{ days} = 5(24\text{ hr}) &= 120\text{ hr} = 120(60\text{ min}) = 7200\text{ min} \\ &= 7200(60\text{ sec}) = 432{,}000\text{ sec}. \end{aligned}$$

It is important to note that formulas or equations involving denominate numbers cannot have any valid significance unless each term is expressed in the same units and has the same dimensions. Thus, a formula such as $V = 5h + 3h^2$, where V is a volume and h a length, has no significance, since the terms V, $5h$, and $3h^2$ are of dimensions L^3, L, L^2, respectively. The volume V of a frustum of a pyramid is

$$V = \tfrac{1}{3}h(B + b + \sqrt{Bb}),$$

where h is the altitude and B and b are the areas of the two bases. The terms V, $\frac{1}{3}hB$, $\frac{1}{3}hb$, $\frac{1}{3}h\sqrt{Bb}$ are each of dimension L^3, but the formula has significance only when each term is expressed in the same cubic units.

16. TRANSFORMATION OF SIMPLE FORMULAS

The simple formula (1) $A = LH$ may be written in the forms (2) $L = A/H$ and (3) $H = A/L$. If we are given the numerical values of L and H to obtain A; the form (1) is most useful. If we are given the numerical values of A and H to obtain L, the form (2) is most serviceable; and if we are given the numerical values of A and L to obtain H, the form (3) is most useful. It is evident that the ability to transform a given formula into various other forms is important for the scientist's needs. Furthermore, practice in making the transformations provides excellent drill in the use of the fundamental operations of algebra.

A particular transformation of a formula may be impossible or may involve more advanced mathematics than is at the command of the student. So at this time we shall confine ourselves to those simple transformations that depend upon the following six axioms:

(1) *If equal numbers are added to equal numbers, the sums are equal.*

(2) *If equal numbers are subtracted from equal numbers, the remainders are equal.*

(3) *If equal numbers are multiplied by equal numbers, the products are equal.*

(4) *If equal numbers are divided by equal numbers (exclusive of zero), the quotients are equal.*

(5) *The same powers of equal numbers are equal.*

(6) *The same roots of equal numbers are equal.*

Axiom 6 means that in the case of the extraction of the square root, for example, the positive numbers that are the roots of equal numbers are equal, and the negative numbers that are the roots of equal numbers are equal.

In later chapters we treat the general theory of exponents, radicals, and the solution of equations. We shall assume at this point, however, that the student is already familiar with enough algebra to apply the six axioms to the exercises that follow in this chapter.

Illustration 1: Let us obtain an explicit formula for each of the letters if

$$L = \frac{Mt - g}{t}.$$

Given:
$$L = \frac{Mt - g}{t}. \tag{1}$$

Hence, multiplying each member of (1) by t (Axiom 3),

$$Lt = Mt - g. \tag{2}$$

Also, subtracting Mt from each member of (2) (Axiom 2),

$$Lt - Mt = -g. \tag{3}$$

Therefore,
$$t(L - M) = -g,$$

and dividing each member of (3) by $L - M$ (Axiom 4),

$$t = \frac{-g}{L - M}. \tag{4}$$

From (3),
$$-Mt = -g - Lt \qquad \text{(Why?)}.$$

So,
$$M = \frac{g + Lt}{t} \qquad \text{(Why?)}.$$

Also,
$$g = Mt - Lt \qquad \text{(Why?)}.$$

Illustration 2: Let us obtain a formula for each of the letters involved in the equation,

$$I = \sqrt{r^2 + P^2 + L^2},$$

where all the quantities designated by the various letters are positive.

Squaring each member of the given equation (Axiom 5), we have

$$I^2 = r^2 + P^2 + L^2,$$

from which the student can readily obtain the following:

$$r = \sqrt{I^2 - P^2 - L^2} \qquad \text{(Axioms 2 and 6);} \tag{1}$$

$$P = \sqrt{I^2 - r^2 - L^2} \qquad \text{(Why?);} \tag{2}$$

$$L = \sqrt{I^2 - r^2 - P^2} \qquad \text{(Why?).} \tag{3}$$

Illustration 3: Let us obtain a formula for each of the letters appearing in the relation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

After multiplying each member of the given equation by Rr_1r_2 (Axiom 3), we have

$$r_1r_2 = Rr_2 + Rr_1,$$

from which the student can readily obtain

$$R = \frac{r_1r_2}{r_1 + r_2}; \tag{1}$$

$$r_1 = \frac{Rr_2}{r_2 - R}; \tag{2}$$

$$r_2 = \frac{Rr_1}{r_1 - R}. \tag{3}$$

Illustration 4: If $x^3 + 3x^2y = 5$, it is comparatively simple to show that $y = \dfrac{5 - x^3}{3x^2}$, but it requires a considerable mastery of algebraic processes to obtain a formula for x. This illustration is cited as an example which requires knowledge of mathematics beyond elementary algebra.

EXERCISES 4

Obtain an explicit formula for each of the letters involved in the following relations. The symbols $>$ and $<$ appearing in several problems are the symbols of inequality; the symbol always points toward the smaller quantity; thus, $a > 0$ is read "a is greater than zero," and $a < 0$ means "a is less than zero."

1. $x + y = u + v$
2. $xy = wv$
3. $x^2y = n^2z;\ x > 0,\ n > 0$
4. $V = \dfrac{4\pi r^3}{3}$
5. $A = 4\pi r^2;\ r > 0$
6. $K = \dfrac{Wv^2}{64.4};\ v > 0$
7. $V = \dfrac{2d}{d_1 - d_2}$
8. $S = \dfrac{n}{2}(a + l)$
9. $P = \dfrac{E^2}{R};\ E > 0$
10. $F = \dfrac{M_1M_2}{d^2};\ d > 0$

11. $t = 6.28\sqrt{\dfrac{L}{32.2}}$; $L > 0$

12. $v = c\sqrt{2gh}$; $g > 0, h > 0$

13. $\sqrt[3]{x} + y = d^2 + 4$; $d > 0$

14. $C = \dfrac{mv^2}{r}$; $v > 0$

15. $\dfrac{M}{EI} = \dfrac{1}{R}$

16. $T = \dfrac{EJ\theta}{l}$

17. $P = \dfrac{\pi^2 EI}{l^2}$; $l > 0$

18. $S = \dfrac{3wx^2}{bd^2}$; $x > 0, d > 0$

19. $p = \left(m\dfrac{S_c}{S_t} + 1\right)d$

20. $E = \dfrac{\phi PNm}{(p)10^8}$

21. $R_t = R_1 + R_2\left(\dfrac{N_1}{N_2}\right)^2$; $N_1 > 0$, $N_2 > 0$

22. $v = v_0\sqrt{1 + at}$; $t > 0, a > 0$

23. $\dfrac{P}{A} = \dfrac{S}{l + q\left(\dfrac{l}{d}\right)^2}$; $l > 0$; $d > 0$

24. $r_1 = \dfrac{r_2}{n - (n - 1)r_2}$

25. If $S = P(1 + i)^n$, where n is a positive integer, obtain formulas for P and i.

17. THEOREMS AND FORMULAS FROM GEOMETRY

For purpose of reference we give the following formulas and theorems from geometry:

(1) If the sides and the hypotenuse of a right triangle are a, b, c, respectively, then,

$$a^2 + b^2 = c^2.$$

This is known as the **Pythagorean theorem.**

(2) The area of a parallelogram (including a rectangle or a square) is the product of the base by the altitude; that is,

$$A = bh.$$

(3) (*a*) The area of a triangle is half the product of the base by the altitude; thus,

$$A = \tfrac{1}{2}bh.$$

(*b*) The area of a triangle is $\sqrt{s(s - a)(s - b)(s - c)}$, where a, b, and c are the three sides and s is half the perimeter.

(4) The area of a trapezoid is equal to the product of the altitude by half the sum of the parallel sides. So, if the parallel sides are designated by b and B, and the altitude by h,

$$A = \frac{h(B + b)}{2}.$$

(5) The area of a circle of radius r is πr^2, and its circumference is $2\pi r$.

(6) The area of a sector of a circle is equal to half the product of its

radius by the arc length of the sector; that is, where s is the arc length,

$$A = \tfrac{1}{2}rs.$$

(7) The volume of a rectangular parallelopiped is equal to the product of its three dimensions.

(8) The volume of a prism (or a cylinder) is equal to the product of its base by its altitude; that is,

$$V = Bh.$$

(9) The volume of a pyramid (or a cone) is equal to one third the product of its base by its altitude; that is, $V = \tfrac{1}{3}Bh$.

(10) The lateral area of a regular pyramid (or a cone) is equal to one half the product of its slant height by the perimeter of its base.

(11) The volume V of a frustum of a pyramid (or a cone) is

$$V = \tfrac{1}{3}h(B + b + \sqrt{Bb}),$$

where h is the altitude, and B and b are the areas of the two bases.

(12) If r is the radius and h is the altitude of a right circular cylinder, then,

$$\begin{aligned} \text{Lateral area} &= 2\pi rh, \\ \text{Total area} &= 2\pi r(h + r), \\ \text{Volume} &= \pi r^2 h. \end{aligned}$$

(13) If r is the radius, h is the altitude, and s is the slant height of a right circular cone, then,

$$\begin{aligned} \text{Lateral area} &= \pi rs, \\ \text{Total area} &= \pi r(s + r), \\ \text{Volume} &= \tfrac{1}{3}\pi r^2 h. \end{aligned}$$

(14) If r is the radius of a sphere, then,

$$\begin{aligned} \text{Surface area} &= 4\pi r^2, \\ \text{Volume} &= \tfrac{4}{3}\pi r^3. \end{aligned}$$

EXERCISES 5

(Any square roots needed in the solution of the following problems may be obtained by reference to Table 4 in the Appendix.)

1. From the statement of the Pythagorean theorem, obtain a formula for one side of a right triangle in terms of the hypotenuse and the other side.

2. Find the length of one of the equal sides of an isosceles triangle whose base is 10 cm and whose area is 76 sq cm.

3. Determine the area of an equilateral triangle whose sides are each 8 in.

4. The area of an isosceles triangle is 96 sq in. Its height is 8 in. Find the length of the equal sides.

5. A square and an equilateral triangle have the same perimeter. How do their areas compare?

6. (*a*) Obtain a formula for the radius of a sphere in terms of its surface area.
(*b*) Obtain a formula for the radius of a sphere in terms of its volume.

7. The length of the hypotenuse of a right triangle is 74 ft. One leg is 31 ft. Find the other leg.

8. A box (rectangular parallelopiped) has the inside dimensions 3 ft, 4 ft, and 6 ft. What is the longest steel rod that can be placed inside the box?

9. In an isosceles triangle the equal sides are each 16 ft. The base is 10 ft. Find the height.

10. The diameter of a circular opening is 50 in. Find its circumference in feet and its area in square feet.

11. Find the area of a sector of a circle if the central angle of the sector is 72° and the radius of the circle is 6 in.

12. Determine the area of the largest square that can be inscribed in a circle of area 225π sq ft.

13. Find the volume of a right prism whose base is a regular hexagon (six sides) if the altitude of the prism is 3 ft and one side of the base is 8 in.

14. A conical pile of sand is 300 ft in circumference at the base and is 40 ft high. Find the number of cubic yards of sand that it contains.

15. Find the number of cubic yards of concrete required for a concrete pier in the form of a frustum of a pyramid if the bases are squares 30 in. and 20 in. on a side, respectively, and the pier has an altitude of 12 ft.

16. Show that the formula for the area of a sector of a circle gives the area of the complete circle when the arc length of the sector is taken as the circumference of the circle.

17. A water tank consists of a cylinder with a hemispherical base. Find the volume of the tank in cubic feet if the altitude of the cylinder is 25 ft and the diameter of the base of the cylinder is 12 ft.

18. The tank in Exercise 17 is covered with a conical roof whose altitude is 6 ft. Find the total surface of the tank.

3 Review Topics of Elementary Algebra

18. FACTORS

When two or more quantities are multiplied together to form a product, each of the quantities is called a *factor of the product.* Any factor is called a *coefficient* of the product of the remaining factors. Thus, the factors of $2ab$ are 2, a, and b; moreover, 2 is the coefficient of ab. Also, $(a - b)$ and $(a + b)$ may be described as the factors of $a^2 - b^2$, since the product of $(a - b)$ and $(a + b)$ is $a^2 - b^2$.

In elementary arithmetic, when we speak of the factors of a positive integer, we refer to the positive integers which are its exact divisors. It is evident that a restricted definition of the term *factor* is being employed. In accord with this restricted definition, a positive integer which has no factors other than itself and the number 1 is called a *prime number.*

19. SPECIAL PRODUCTS

Certain combinations of algebraic factors are met so frequently in practice that it is a convenience to know the forms of their products. Likewise, if a given algebraic expression is in a form identified as a *special product,* it is frequently desirable to know its factors. In anticipation of either case, the student will find it advantageous to memorize the following equalities. Every algebraist should be able to write immediately the right member of each of the following relations if the left member is given. Conversely, he should be able to write immediately the left member if the right member is given.

$$(a + b)(a - b) = a^2 - b^2 \tag{1}$$

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{2}$$

$$(a - b)^2 = a^2 - 2ab + b^2 \tag{3}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{4}$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \tag{5}$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3 \tag{6}$$

$$(a + b)(a^2 - ab + b) = a^3 + b^3 \tag{7}$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \tag{8}$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab \tag{9}$$

The previous products may all be verified by actual multiplication. Moreover, it is evident that the left member of each equality will reduce to the same numerical value as the corresponding right member for any set of numbers that may be assigned to the literal symbols; such equalities are known as *algebraic identities*.

In the previous identities, each of the various letters may symbolize any expression. Thus, in

$$(3x + 2y)(3x - 2y),$$

we may consider $3x = a$ and $2y = b$.

Hence, $(3x + 2y)(3x - 2y) = (3x)^2 - (2y)^2$ [by identity (1)];

that is, $(3x + 2y)(3x - 2y) = 9x^2 - 4y^2$.

Also in

$$(3x + 5y)^3,$$

we may consider $3x = a$ and $5y = b$.

Hence, by identity (4),

$$(3x + 5y)^3 = (3x)^3 + 3(3x)^2(5y) + 3(3x)(5y)^2 + (5y)^3;$$

that is, $(3x + 5y)^3 = 27x^3 + 135x^2y + 225xy^2 + 125y^3$.

In considering the product

$$(2x + 3y + 5z)(2x + 3y - 5z),$$

we may take $2x + 3y = a$ and $5z = b$.

Therefore, by identity (1)

$$(2x + 3y + 5z)(2x + 3y - 5z) = (2x + 3y)^2 - (5z)^2.$$

But by identity (2)

$$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2.$$

So, in conclusion,

$$(2x + 3y + 5z)(2x + 3y - 5z) = 4x^2 + 12xy + 9y^2 - 25z^2.$$

EXERCISES 6

Special Products of the Form $(a + b)(a - b)$

Obtain each of the following products directly by reference to identity (1), listed in Section 19:

1. $(x + 2y)(x - 2y)$

2. $(16 - 8)(16 + 8)$

3. $(3x + 7y)(3x - 7y)$

4. $(a - b - c)(a - b + c)$

SUGGESTION: First write the product of Exercise 4 in the form

$$[(a - b) - c][(a - b) + c].$$

5. $\left(x - \frac{4}{y}\right)\left(x + \frac{4}{y}\right)$

6. $\left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{1}{x} + \frac{1}{y}\right)$

7. $(a + b + c)(a - b - c)$

SUGGESTION: Note that $-b - c$ may be written as $-(b + c)$.

8. $(x + 2y + z)(x + z - 2y)$

9. $(x - y - z)(x - y + z)$

10. $\left(\frac{x}{3} + \frac{y}{4} + z\right)\left(\frac{x}{3} + \frac{y}{4} - z\right)$

11. Verify the identities obtained in the exercises above by multiplication.

EXERCISES 7

Factors of Expressions of the Form $a^2 - b^2$

The algebraic expressions listed in this section may all be regarded as resulting from the multiplication of two factors in the form $(a + b)$ and $(a - b)$. In each case, what are the factors?

1. $x^2 - 4$
2. $a^2 - 4x^2y^2$
3. $16b^2 - 4a^2$
4. $49x^4 - 36y^4$
5. $(x + y)^2 - 16(x - y)^2$
6. $(x + 3)^2 - (y - 3)^2$
7. $a^2 - b^2 - 2bc - c^2$
8. $x^2 - 2xy + y^2 - z^2$
9. $16x^2 + 24xy + 9y^2 - 144z^2$
10. $81 - 4x^4$

20. EXPANSIONS OF THE FORM $(a \pm b)^2$

As already observed, the square of an algebraic expression of two terms, called a *binomial,* follows the law $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Thus,

$$(2x + 3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$$
$$= 4x^2 + 12xy + 9y^2.$$

Also,

$$(w - 2v)^2 = w^2 - 2(w)(2v) + (2v)^2$$
$$= w^2 - 4wv + 4v^2.$$

EXERCISES 8

Expand each of the following:

1. $(2x - 3y)^2$
2. $(2w + v)^2$
3. $(2s - 5t)^2$
4. $\left(\frac{x}{2} + \frac{y}{3}\right)^2$
5. $\left(a - \frac{b}{2}\right)^2$
6. $(6 - x)^2$
7. $\left(4t + \frac{s}{2}\right)^2$
8. $(98)^2 = (100 - 2)^2$

21. FACTORS OF EXPRESSIONS IN THE FORM $a^2 \pm 2ab + b^2$

From our experience with the previous section, it is apparent that a *trinomial,* an algebraic expression of three terms, is a perfect square when it appears in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$; the factors of the

former are $(a + b)^2$ and of the latter are $(a - b)^2$. Thus, $x^2 - 6xy + 9y^2$ is a perfect square, since $-6xy$ is in the form $-2ab$, where a is the positive square root of x^2 and b is the positive square root of $9y^2$; consequently, its factors are $(x - 3y)^2$. The following exercises may be factored in the same manner.

EXERCISES 9

1. $x^4 + 2x^2y^2 + y^4$
2. $1 - 4xy + 4x^2y^2$
3. $16 - 24x + 9x^2$
4. $4 - 4x + x^2$
5. $9 + 12x + 4x^2$
6. $\frac{1}{9} + \frac{2x}{3} + x^2$
7. $\frac{x^2}{9} - \frac{xy}{6} + \frac{y^2}{16}$
8. $4x^2 - 6xy + \frac{9y^2}{4}$
9. $x^2 + 4xy + 4y^2$
10. $a^2x^2 + 2abx + b^2$

22. EXPRESSIONS REDUCIBLE TO THE FORM $a^2 - b^2$

Many expressions to be factored may first be reduced to a difference of two squares. The following illustrations represent some typical situations.

Illustration 1: Factor $a^4 + a^2b^2 + b^4$.

By adding and then subtracting a^2b^2, there results

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 - ab)(a^2 + b^2 + ab). \end{aligned}$$

Illustration 2: Factor $a^4 - a^2b^2 + 16b^4$.

By adding and then subtracting $9a^2b^2$, we have

$$\begin{aligned} a^4 - a^2b^2 + 16b^4 &= (a^4 + 8a^2b^2 + 16b^4) - 9a^2b^2 \\ &= (a^2 + 4b^2)^2 - (3ab)^2 \\ &= (a^2 + 4b^2 - 3ab)(a^2 + 4b^2 + 3ab). \end{aligned}$$

Illustration 3: Factor $a^4 + 4b^4$.

$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab). \end{aligned}$$

EXERCISES 10

Factor the following expressions:

1. $x^4 + 4x^2 + 16$
2. $1 + a^2 + a^4$
3. $x^4 + 4x^2y^2 + 16y^4$
4. $x^4 + 4x^2y^2 - 12y^4$
5. $81 + 9x^2 + x^4$
6. $x^4 - 7x^2 + 81$
7. $x^4 - 17x^2y^2 + 16y^4$
8. $25x^4 + 24x^2y^2 + 36y^4$
9. $9t^4 + 21t^2s^2 + 25s^4$
10. $x^4 + 64y^4$
11. Verify by multiplication the factors obtained in the previous exercises.

EXERCISES 11

Expansions of the Form $(a \pm b)^3$

Make the following expansions by reference to the appropriate standard forms in Section 19:

1. $(x + 2y)^3$
2. $(3x - y)^3$
3. $\left(y + \frac{x}{2}\right)^3$
4. $(2y - 3x)^3$
5. $\left(\frac{y}{2} + \frac{x}{3}\right)^3$
6. $\left(y - \frac{3x}{2}\right)^3$
7. $(4 - x)^3$
8. $(3x - 2y)^3$
9. $\left(a + \frac{3b}{2}\right)^3$
10. $\left(4x - \frac{y}{2}\right)^3$
11. Verify by multiplication the identities obtained in the exercises above.

EXERCISES 12

Factors of Expressions in the Form $a^3 \pm b^3$

Factor the following expressions by reference to the forms in Section 19 for the sum or the difference of two cubes:

1. $(2x)^3 + y^3$
2. $27 + 8x^3$
3. $1 - 27x^3y^3$
4. $8x^3 + a^3b^3$
5. $a^6 + 8$
6. $z^3 - 8$
7. $x^6y^3 - 1$
8. $125 - c^6$
9. $x^3 - a^6y^6$
10. $1 - 27x^6$

EXERCISES 13

Expansions of the Form $(a + b + c)^2$

Expand the following squares of trinomials after reviewing standard form 8, Section 19:

1. $(x + 2y + z)^2$
2. $(x - y - z)^2$
3. $(x + 2y + 3z)^2$
4. $(2x - 3y + z)^2$
5. $(4 - x + y)^2$
6. $\left(x + \frac{y}{2} + \frac{z}{2}\right)^2$
7. $(3x - y - 2z)^2$
8. $\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right)^2$

23. FACTORS OF EXPRESSIONS OF THE FORM $x^2 + (a + b)x + ab$

The trinomial $x^2 + (a + b)x + ab$ has the factors $(x + a)$ and $(x + b)$. Thus, to factor $x^2 - 7x + 12$, we may consider $ab = 12$ and $a + b = -7$. It is then necessary to determine two numbers a and b whose product is 12 and whose sum is -7. If this can be done by inspection, the factors are readily found. In the particular problem under consideration, the two numbers are evidently -3 and -4. Hence, $x^2 - 7x + 12$ possesses the factors $(x - 3)$ and $(x - 4)$.

EXERCISES 14

Factor each of the following expressions:

1. $x^2 - 3x - 10$
2. $x^2 - \frac{1}{2}x - \frac{1}{2}$
3. $x^2 + (3 + \sqrt{3})x + 3\sqrt{3}$
4. $x^2 + (\sqrt{2} - 1)x - \sqrt{2}$
5. $x^2 - 7x - 30$
6. $x^2 - \frac{9}{2}x - \frac{5}{2}$
7. $x^2 + ax - 2a^2$
8. $x^2 + \dfrac{3a}{2}x + \dfrac{a^2}{2}$
9. $x^2 - (a + b)x + ab$
10. $x^2 - 6x - 7$
11. $x^2 - 6x + 9$

24. FACTORS OF EXPRESSIONS OF THE FORM $ax^2 + bx + c$

Expressions of this general type, when $a \neq 1$, may frequently be factored by trial.

For example, to factor $2x^2 + x - 15$, let us attempt to find two x coefficients whose product is 2, and two other numbers whose product is -15, which may then be arranged within two factors so that the sum of their cross product is 1. Thus, the above trinomial factors into $(2x - 5)(x + 3)$.

EXERCISES 15

Factor the following trinomials:

1. $10x^2 - 13x - 3$
2. $3x^2 - 10x + 3$
3. $15x^2 + 73x - 10$
4. $15x^2 - 7x - 2$
5. $15x^2 + x - 2$
6. $x^2 - 7x + 10$
7. $10 + 3x - x^2$
8. $10 + z - 2z^2$
9. $3x^2 - 17xy + 20y^2$
10. $12x^2 - 13xy - 4y^2$

25. FACTORS OF EXPRESSIONS OF THE FORM $ax + ay + bx + by$

Such expressions are readily factored by grouping the terms in such a manner as to show a common binomial factor; thus,

$$ax + ay + bx + by = a(x + y) + b(x + y).$$

Since each of the two terms now appearing contains the factor $(x + y)$, the given expression factors into $(x + y)(a + b)$.

EXERCISES 16

Factor the following expressions:

1. $a^2x + a^2y - b^2x - b^2y$
2. $xyz - abz + cxy - abc$
3. $x^3 - 3x^2y + 3y - x$
4. $cd^2 + cb^2 - c^2b^2 - c^2d^2$
5. $1 - x + x^2 - x^3$
6. $2\sqrt{3} - 20x + \dfrac{\sqrt{3}}{2}y - 5xy$

MISCELLANEOUS EXERCISES 17

Factor the following expressions:

1. $x^2 - 7x + 12$

2. $x^4 + 2x^2 - 8$

3. $x^2 - 3xy + 2y^2$

4. $10x^2 - 40x + 30$

5. $x^2 + 4x - 21$

6. $x^2y + 23xy - 50y$

7. $x^2 - 4x - 12$

8. $a^6 - 17a^3 + 70$

9. $4y^2 + 40y + 36$

10. $6x^2 + 17x + 12$

11. $5y^2 + 14y + 8$

12. $9x^2 + 6x - 8$

13. $9x^2 - 6x - 8$

14. $\dfrac{3x^2}{2} + \dfrac{3xy}{2} - 3y^2$

15. $9x^2 + 12xy + 4y^2$

16. $9y^2 + 37xy + 4x^2$

17. $6y^2 + 22y + 12$

18. $9x^2 + 18xy - 27y^2$

19. $16x^4 + 8x^2 - 3$

20. $x^4 - 3x^3 + 4x^2 - 12x$

21. $2cd - c^2 - d^2$

22. $18a^2x^2 - 24a^2x - 10a^2$

23. $a^5 - a^4 - a^3 + a^2$

24. $5d^2 - 5cd - 10c^2$

25. $x^4 - 17x^2y^2 + 16y^4$

26. $x^4 - 28x^2y^2 + 16y^4$

27. $2a^2x^2 - 6xy - 3by^2 + a^2bxy$

28. $24a^3 - 81b^3$

29. $4x^2 + 6yz - y^2 - 9z^2$

30. $x^4 - 8x^2y^2 + 16y^4$

31. $128 + 54x^3$

32. $27a^3 - 54a^2b + 36ab^2 - 8b^3$

33. $39x^2y^2 - 9x^4 - 25y^4$

34. $a^6 - 26a^3 - 27$

35. $x^2 - 4xy + 4y^2 - 3xz + 6yz$

26. SPECIAL CASE OF BINOMIAL THEOREM

We have seen that

$$(a + b)^2 = a^2 + 2ab + b^2,$$

and

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Similarly, we can show by actual multiplication that

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

and

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The previous identity for $(a + b)^5$ may be written

$$(a + b)^5 = a^5 + \frac{5}{1}a^4b + \frac{5 \cdot 4}{1 \cdot 2}a^3b^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}a^2b^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}b^5.$$

A product of consecutive positive integers, starting with 1, is known as a *factorial product;* for instance, $1 \cdot 2 \cdot 3 \cdot 4$ is known as *factorial 4* and is written $\underline{|4}$ or $4!$. (We shall not use the exclamation sign, however.) Similarly, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ is called *factorial 5* and is written $\underline{|5}$. Of course,

1 may also be written as $\underline{|1}$. Consequently, we may write

$$(a+b)^5 = a^5 + \frac{5}{\underline{|1}} a^4 b + \frac{5 \cdot 4}{\underline{|2}} a^3 b^2 + \frac{5 \cdot 4 \cdot 3}{\underline{|3}} a^2 b^3$$
$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{\underline{|4}} ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{\underline{|5}} b^5.$$

A general formula similar to this expansion for $(a+b)^5$ may be found for $(a+b)^n$, if n is a positive integer. In fact,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{\underline{|3}} a^{n-3} b^3$$
$$+ \cdots + \frac{n(n-1)(n-2)\cdots 2}{\underline{|n-1}} ab^{n-1}$$
$$+ \frac{n(n-1)(n-2)\cdots 1}{\underline{|n}} b^n.$$

This formula, known as the binomial theorem for positive integral powers, will be accepted at this point without proof. The student should test the formula for $n = 2, 3, 4$.

Illustration: Expand $(2x - 3y)^6$ by the binomial theorem.

$$(2x - 3y)^6 = (2x)^6 + \frac{6}{\underline{|1}} (2x)^5(-3y)^1 + \frac{6 \cdot 5}{\underline{|2}} (2x)^4(-3y)^2$$
$$+ \frac{6 \cdot 5 \cdot 4}{\underline{|3}} (2x)^3(-3y)^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{\underline{|4}} (2x)^2(-3y)^4$$
$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{\underline{|5}} (2x)^1(-3y)^5 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{\underline{|6}} (-3y)^6$$
$$= 64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3$$
$$+ 4860x^2y^4 - 2916xy^5 + 729y^6.$$

EXERCISES 18

Obtain each of the following expansions by the binomial theorem:

1. $(x + 2y)^4$
2. $(a + 1)^5$
3. $(3a + 5b)^4$
4. $(a + b)^7$
5. $\left(\frac{3}{2} + \frac{x}{3}\right)^6$
6. $(x + y)^9$
7. $(y + 2x)^8$
8. $(10 - 1)^3$
9. $(1 + 0.04)^5$
10. $(1 + 0.02)^6$

27. DEGREE OF A POLYNOMIAL

Although the terms *binomial* and *trinomial* have been employed in previous discussions, a definition has not yet been given of the more general term, *polynomial*. A polynomial in certain variables, for example, x, y, z is a sum of terms of the type $kx^ay^bz^c$, in which k is a constant and

a, b, and c are positive integers. A monomial is a polynomial of one term; a binomial is a polynomial of two terms; and a trinomial is a polynomial of three terms.

The degree with respect to certain variables of a single term of a polynomial is the sum of the exponents of the designated variables occurring in the term. The degree of a polynomial with respect to certain variables is the maximum degree associated with any single term of the polynomial. Thus,

$$3x^2y + 7xy^5 + 4y^3$$

is a polynomial of three terms, that is, a trinomial. The first term is of third degree with respect to the variables x and y, the second is of sixth degree, and the third is of third degree; so the polynomial is of sixth degree with respect to x and y. Similarly, the same polynomial is of second degree with respect to x, and is of fifth degree with respect to y.

28. HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

The highest common factor (HCF) of two or more polynomials with integral coefficients is the polynomial of highest degree with integral coefficients that can be divided into all of them without a remainder. In practice, an HCF of two or more polynomials with integral coefficients can be determined by resolving each expression into its polynomial factors of lowest degree that have integral coefficients and then writing the product of all common factors. If a common factor is repeated two or more times in all the given polynomials, it should appear in the HCF to the lowest power to which it appears in any expression.

A lowest common multiple (LCM) of two or more polynomials is the polynomial of lowest degree which contains each of the given expressions as a factor. In practice, an LCM of two or more polynomials may be found by factoring each expression completely, as explained in the case of the HCF, and then taking the product of all of their different factors, using each factor the greatest number of times that it occurs in any of the polynomials.

EXERCISES 19

Find an HCF and an LCM for each of the following collections of polynomials:

1. $x^2 + xy$, $x^3 + y^3$, and $x^2 - 3xy - 4y^2$

SUGGESTION: The factors of the three given polynomials are

$$x^2 + xy = x(x + y),$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$

and $$x^2 - 3xy - 4y^2 = (x + y)(x - 4y).$$

Therefore, the HCF is $(x + y)$, and the LCM is $x(x + y)(x^2 - xy + y^2)(x - 4y)$.

2. $6m^2n$, $-4mnx^3$, and $12mn^2x$

3. $ax - ay + bx - by$, $a^2 + ab + b^2$, and $a^2 + ab$
4. $a^3 + 8$, $3a^2 + 5a - 2$, and $a^2 - 4$
5. $1 - x$, $x - 1$, $x^2 - 1$, $x^4 - 1$, and $x^8 - 1$
6. $a^2 + 2ab + b^2 - c^2$ and $a^2 - b^2 - 2bc - c^2$
7. $6x^2 - 54$, $7(x - 3)^2$, and $3x^2 - 6x - 9$

29. ALGEBRAIC FRACTIONS

An algebraic fraction is the indicated quotient of two algebraic expressions. Thus, a/b implies that a is divided by b, where a and b may denote any algebraic expressions.

Throughout this text, in all expressions involving denominators, no denominator is permitted to be zero. Thus, in dealing with each of the following fractions there are restrictions on x as indicated: (1) $\frac{3}{x}$, $x \neq 0$; (2) $\frac{1}{x - 2}$, $x \neq 2$; (3) $\frac{1}{x^2 - 9}$, $x \neq \pm 3$; (4) $\frac{1}{a - b}$, $a \neq b$.

In simplifying and combining fractions we make use of the following principles which should already be familiar to the student.

I. *The value of a fraction is not changed by multiplying or dividing both the numerator and the denominator by the same number, excluding zero. Such an operation upon a fraction is equivalent to multiplying it by* 1.

II. *Changing the sign of either the numerator or the denominator of a fraction is equivalent to changing the sign of the fraction. Thus,*

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}.$$

Exercise: Justify that $\frac{-a}{b} = \frac{a}{-b}$ by employing Principle I, given above.

III. *The algebraic sum of two fractions with a common denominator is a fraction whose numerator is the algebraic sum of the numerators of the given fractions and whose denominator is the common denominator.*

IV. *Two fractions that do not have a common denominator may be changed to equivalent fractions having a common denominator through the use of Principle I, and their sum may then be found as in III. The lowest common denominator of two or more fractions is the LCM of their denominators.*

V. *The product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.*

VI. *To divide one fraction by another, invert the divisor and multiply. This is equivalent to the multiplication of numerator and denominator by the reciprocal of the denominator.*

VII. *The expression* a/b *is in its lowest terms if* a *and* b *do not contain any common factors. To reduce* a/b *to its lowest terms, we divide both numerator and denominator by their highest common factor.*

Illustration 1: Perform the following indicated operations and reduce the result to the simplest form:

$$\frac{2}{x^2 - 3x + 2} + \frac{2}{x^2 - x - 2} - \frac{1}{x^2 - 1}.$$

Solution: After factoring the denominators, we have

$$\frac{2}{(x-1)(x-2)} + \frac{2}{(x-2)(x+1)} - \frac{1}{(x-1)(x+1)}.$$

Since the last fraction may be regarded as

$$+\frac{-1}{(x-1)(x+1)},$$

the desired sum becomes

$$\frac{2(x+1)}{(x-1)(x-2)(x+1)} + \frac{2(x-1)}{(x-1)(x-2)(x+1)} + \frac{-(x-2)}{(x-1)(x-2)(x+1)}$$
$$= \frac{2(x+1) + 2(x-1) - (x-2)}{(x-1)(x-2)(x+1)}$$
$$= \frac{3x+2}{(x-1)(x-2)(x+1)}.$$

Illustration 2: Perform the following indicated operations and reduce the result to the simplest form:

$$\frac{6x^2 - ax - 2a^2}{ax - a^2} \cdot \frac{x - a}{9x^2 - 4a^2} \div \frac{2x + a}{3ax + 2a^2}.$$

In accordance with the principles previously stated, we have

$$\frac{6x^2 - ax - 2a^2}{ax - a^2} \cdot \frac{x - a}{9x^2 - 4a^2} \cdot \frac{3ax + 2a^2}{2x + a}$$
$$= \frac{(3x - 2a)(2x + a)}{a(x - a)} \cdot \frac{(x - a)}{(3x - 2a)(3x + 2a)} \cdot \frac{a(3x + 2a)}{(2x + a)}.$$

Since the factors in the numerator are the same as the factors in the denominator, the product is 1.

Illustration 3: Perform the following indicated operations and reduce the result to the simplest form:

$$\frac{16x^2 - 9a^2}{x^2 - 4} \div \left(\frac{4x - 3a}{x - 2} \cdot \frac{x + 2}{4x + 3a}\right).$$

After inverting the fraction appearing within the parentheses, we have

$$\frac{16x^2 - 9a^2}{x^2 - 4} \cdot \frac{(x-2)(4x+3a)}{(4x-3a)(x+2)}$$
$$= \frac{(4x-3a)(4x+3a)}{(x-2)(x+2)} \cdot \frac{(x-2)(4x+3a)}{(4x-3a)(x+2)}$$
$$= \frac{(4x+3a)^2}{(x+2)^2}.$$

Illustration 4: Perform the following indicated operations and reduce the result to the simplest form:

$$\left(\frac{16x^2 - 9a^2}{x^2 - 4} \div \frac{4x - 3a}{x - 2}\right) \cdot \frac{x+2}{4x+3a}.$$

In this case we have

$$\left(\frac{16x^2 - 9a^2}{x^2 - 4} \cdot \frac{x - 2}{4x - 3a}\right) \frac{x+2}{4x+3a} = \frac{4x+3a}{x+2} \cdot \frac{x+2}{4x+3a} = 1.$$

In Illustrations 3 and 4 the use of parentheses determines the sequence of operations to be performed. Unless parentheses are used, some arbitrary rule is required to define the sequence of operations.

EXERCISES 20

Reduce each of the following expressions to lowest terms:

1. $\dfrac{2a^2x^3}{12ax^7}$
2. $\dfrac{x+1}{x+1+(x+1)^2}$
3. $\dfrac{x^3+y^3}{x^2-xy+y^2}$
4. $\dfrac{x^4-1}{x^3+1}$
5. $\dfrac{2a^2x^2 - a^3x - 6a^4}{x^3 - 3ax^2 + 2a^2x}$

Perform the following indicated operations and reduce to lowest terms:

6. $\dfrac{3}{a^2+2a+1} - \dfrac{4a}{a^2-1}$
7. $\dfrac{2}{x+1} - \dfrac{3}{x-1} + \dfrac{4}{x+3}$
8. $x - 2 - \dfrac{x+1}{x^2-2}$
9. $\dfrac{3}{x+y} - \dfrac{2}{(x+y)^2} + \dfrac{x-y}{(x+y)^3}$
10. $\dfrac{5}{3x-3} - \dfrac{8}{5x-15}$
11. $\dfrac{5x-4}{x-2} + \dfrac{x^2-2x-17}{x^2-5x+6}$
12. $\dfrac{3a}{3a-2} \cdot \dfrac{9a^2-4}{9a^2}$
13. $\dfrac{(a-b)^2}{a+b} \cdot \dfrac{a^2-b^2}{a^2+b^2}$
14. $\dfrac{1}{x^2}\left(\dfrac{x}{y} - \dfrac{y}{x}\right)\left(\dfrac{x^3}{x-y}\right)^2 \div \left(1 - \dfrac{y}{x}\right)$

15. $\left(\frac{1}{x} - \frac{1}{y}\right)\left(1 - \frac{2y}{x} + \frac{y^2}{x^2}\right) \div (x - y)^2$

16. $\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \frac{1}{r_1^2 - r_2^2}$

17. $\dfrac{\frac{x^2}{y^2} - \frac{y^2}{x^2}}{x + \frac{y^2}{x}}$

18. $\dfrac{\frac{x}{x - 1} - 1}{1 + \frac{x}{1 - x}}$

19. $\left(1 - \frac{ab}{a^2 - ab + b^2}\right)\left(1 - \frac{ab}{a^2 + 2ab + b^2}\right) \div \frac{a^3 - b^3}{a^3 + b^3}$

20. $\left(\dfrac{\frac{a^2 + ax}{2x}}{a^2 - x^2}\right)\left(\frac{(a + x)^2}{4ax}\right) - 1$

21. $\left(1 - \frac{1 - a}{1 + a} + \frac{1 + 2a^2}{1 - a^2}\right)\left(\frac{a + 1}{2a + 1}\right)$

22. $\dfrac{\frac{a + b}{ab}\left(\frac{1}{a} - \frac{1}{b}\right) - \frac{b + c}{bc}\left(\frac{1}{c} - \frac{1}{b}\right)}{\frac{a + c}{ac}\left(\frac{1}{a} - \frac{1}{c}\right)}$

23. $\left(a - 3 + \frac{-5}{y + 1}\right) \div \left(2 - \frac{7y + 2}{y^2 - 1}\right)$

24. $\frac{x}{x - y} + \frac{y}{x + y} + \frac{x^2 + y^2}{y^2 - x^2}$

25. $\frac{1}{a - x} - \frac{3}{a + x} + \frac{2a}{x^2 - a^2}$

26. $\left(\dfrac{2x^2 - 3x + 1}{\frac{1}{x} - 1}\right)\left(\frac{1}{x^2} - \frac{2}{x}\right)$

27. $\dfrac{(y - 5x)\left(\frac{24y}{y - 5x}\right) - (5y - x)\left(\frac{24x}{y - 5x}\right)}{(y - 5x)^2}$

28. $\frac{(x - 2)(2x - x^2)}{(x - 2)^2} \div \frac{2x^2}{x - 2}$

29. $\dfrac{2x(x - 2) - (x - 4)[2x + 2(x - 2)]}{4x^2(x - 2)^2}$

30. $\dfrac{(2x - a)^2[2a^2(x - a) + 2a^2x] - 8a^2x(x - a)(2x - a)}{(2x - a)^4}$

31. $\frac{-2a^3(a - 3x)}{(2x - a)^3} + \frac{2a^2(x - a)}{(2x - a)^2}$

32. $\left(\frac{a - 2}{a - 1}\right)\left(\frac{a - 1}{a - 2} - \frac{4a}{a - 5}\right)\left(\frac{a + 2}{3a - 5} - \frac{a - 4}{a + 1}\right)$

4 Constants, Variables, and Graphical Representation

30. CONSTANTS, VARIABLES, AND FUNCTIONS

A *variable* is a symbol which may represent any one of a collection of numbers. Thus, r, the radius of a circle, is a variable, for it may stand for any positive number. A symbol which denotes only one number is given the special name *constant*. Thus, t, the temperature of the air throughout an experiment, is a constant if the air temperature is maintained at 72°. Also, π is a constant when it is employed in the usual sense as the ratio of the circumference to the diameter of a circle. It is important to note, therefore, that since letters may be either constants or variables, it is frequently necessary to know just what they are in any particular formula or equation.

When we assign different values to the variable r, the radius of a sphere, we see that V, the volume, assumes different corresponding values. Thus, V is a variable; it is a variable related to r by means of the particular formula $V = \frac{4}{3}\pi r^3$. When two variables are so related that the choice of a particular number to be assigned to the first variable determines the value or values of the second variable, the second variable is said to be a function of the first variable. By means of the formula $V = \frac{4}{3}\pi r^3$, we may assign a value to r and thereby determine V; so, V is said to be a function of r. This fact may be denoted symbolically by $V = f(r)$. Of course, in the special formula under consideration, we may first assign values to V and determine r; then r would be a function of V. Symbolically, this could be written $r = f(V)$.

The variable to which we assign numerical values is said to be the *independent variable;* the other variable is said to be the *dependent variable,* or function of the independent variable.

The symbols, $f(x)$, $F(x)$, $\phi(x)$, are commonly used to represent functions of the variable x. Hence, if y is a function of x, we may write

$$y = f(x).$$

Similarly, if w is so intimately related to u that w is a function of u, we may write $w = f(u)$, or perhaps $w = F(u)$.

If

$$y = \frac{x^3 - 3}{x + 1},$$

y has a value corresponding to each number that may be assigned to x, except $x = -1$; so we write

$$y = f(x) = \frac{x^3 - 3}{x + 1}; \quad x \neq -1.$$

When $x = 1$, we say

$$f(1) = \frac{1^3 - 3}{1 + 1} = -1;$$

when $x = 2$, we write

$$f(2) = \frac{2^3 - 3}{2 + 1} = \frac{5}{3};$$

and when $x = a$, the value of the function is

$$f(a) = \frac{a^3 - 3}{a + 1}.$$

Definitions: (1) A function of x of the form

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Kx + L,$$

where n is a positive integer and $A, B, C, \cdots, K, L$ are any real numbers, is defined as a rational integral function of x, or as a polynomial in x.

(2) Every function which can be expressed either as a rational integral function, or as a quotient of two rational integral functions, is called a *rational function*.

EXERCISES 21

1. Given $f(x) = x^2 + 3x + 5$, find $f(0), f(1), f(-1), f(a + b)$.

2. Given $\phi(x) = \dfrac{x}{(x - 1)(x - 2)}$, find $\phi(0)$, $\phi(3)$, $\phi(-3)$, $\phi(y + 2)$. Why can you not find $\phi(1)$ or $\phi(2)$?

3. Given $F(x) = x^3 + 2x + 1$, find $F(1)$, $F(2)$, $F(a)$, $F(w - 1)$.

4. Given $f(x) = \dfrac{x + \frac{1}{x}}{2}$, find $f(1), f(-1), f(3)$. Why can you not find $f(0)$? For this function show that $f(x) = f(1/x)$.

5. Given $f(x) = x^3 + 3x$, show that $f(x + h) - f(x) = 3(x^2 + 1)h + 3xh^2 + h^3$.

6. If $F(x) = x^2$, show that $\dfrac{F(b) - F(a)}{b - a} = b + a$.

7. Any value of x for which $f(x) = 0$ is defined as a zero of the function $f(x)$. Show that $x = 1$, $x = 2$, and $x = 3$ are zeros of the function

$$y = f(x) = (x - 1)(x - 2)(x - 3).$$

8. Assuming that every rational integral function of x of nth degree may be resolved into a constant times n factors of the type $(x - c)$, how many zeros does the function possess?

31. GRAPHICAL REPRESENTATION

We have already observed that a functional relationship involving two variables may be given through the medium of a formula. Another important method of displaying a relationship between a variable x and the variable y, where $y = f(x)$, is through the use of a graph.

The type of graphical representation treated in this book is based on two perpendicular lines $X'X$ and $Y'Y$, intersecting at O (Figure 6). The lines are called the *axes of reference*, or, if the variables under consideration are x and y, they may be designated as the x axis and y axis, repectively. The point of intersection O is called the *origin*. We adopt a convenient

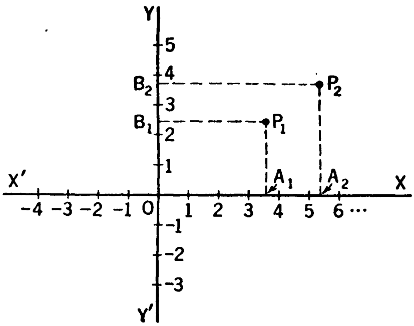


FIG. 6

scale of measurement upon the horizontal axis. Then, corresponding to *positive* numerical values of a variable x, we locate points on $X'X$ measured from O in the direction OX, and, corresponding to *negative* numerical values of x, we locate points on $X'X$ measured from O in the direction OX'.

Similarly, we adopt a convenient scale of measurement upon the vertical axis, and, corresponding to the *positive* numerical values of the function (dependent variable) y, we locate points on $Y'Y$ measured from O in the direction OY, and, corresponding to *negative* numerical values of y, we locate points on $Y'Y$ measured from O in the direction OY'.

Thus, relative to such a system of axes and the scale of measurement adopted upon each axis, the two numbers assigned respectively to the independent variable and the corresponding value of the function may be employed to locate a point in the plane. To be more specific, if the value of an independent variable x is represented by the segment OA_1, and the corresponding value of the function y is denoted by OB_1, the point P_1 is determined as shown. That is, through the point B_1 on the y axis we draw a line parallel to the x axis, and through the point A_1 on the x axis we draw a line parallel to the y axis, then the intersection of the pair of lines determines the desired point in the plane. The lengths OA_1 and OB_1 (or A_1P_1) are called the coordinates of the point P_1, and P_1 may be designated

by the ordered pair of numbers (OA_1, A_1P_1). Similarly, the lengths of OA_2 and A_2P_2 are called the *coordinates* of the point P_2, and P_2 may be denoted by (OA_2, A_2P_2). The value of the independent variable is called the *abscissa* of the point, and the corresponding value of the function is called the *ordinate* of the point. In designating a point by its coordinates, the abscissa is *always* written first, and the ordinate appears second.

This system of representing points by coordinates is called the *rectangular*, or *Cartesian, system*. For purposes of illustration, four points possessing the indicated coordinates have been located in Figure 7.

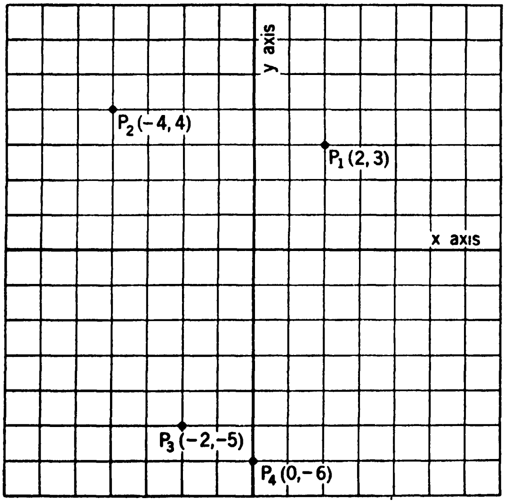


FIG. 7

EXERCISES 22

1. Draw two perpendicular axes and locate the following points: (0, 0), (1, 0), (0, 5), (−1, −2), (−10, 2), (5, −3), (−7, −1), (−3, 0), (2.5, −4.5).

2. Determine the numerical distance between the two points designated by (3, −2) and (7, 1).

3. Show that the following points are all located upon the same circle: (−3, 4), (4, −3), (−5, 0), $(1, \sqrt{24})$, (0, −5). What is the center, and what is the radius of the circle?

4. Determine the length of a diagonal of the rectangle having the vertices (5, 1), (−5, 1), (−5, −2), (5, −2).

The following examples should indicate to the student further important uses of the type of graphical representation now under consideration.

EXAMPLE 1: The statistics in the following table pertain to the number of students (y) who study a certain course in the various years (x).

x (Year)	y (Students)	x (Year)	y (Students)
1913	3	1922	14
1914	7	1923	15
1915	5	1924	16
1916	6	1925	18
1917	0	1926	22
1918	0	1927	26
1919	5	1928	32
1920	7	1929	50
1921	6		

If we represent these data graphically, we obtain Figure 8.

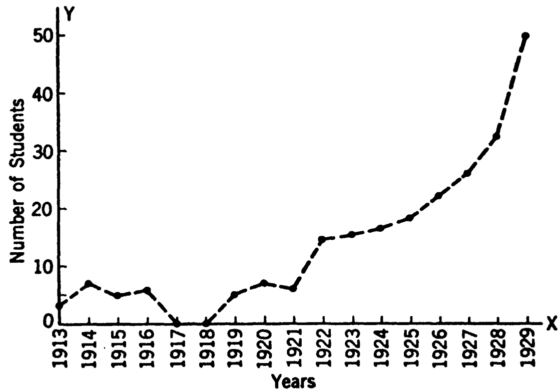


Fig. 8

For purposes of visualization, the points corresponding to the given data have been connected by the dotted straight lines. By referring to the graph, it may be observed that there was an uninterrupted annual increase in the enrollment from 1921 to 1929; that there was an increase during the period from 1918 to 1920; that there was a period of fluctuation from 1913 to 1917; that there was no enrollment during the period of the First World War; and that there was a slight loss from 1920 to 1921.

Example 2: If in a vacuum a body falls from rest, the distance s (ft) covered in the corresponding time t (sec) is given approximately by the following table:

t (Sec)	s (Ft)
1	16
2	64
3	144
4	256
5	400
6	576

A consideration of the numbers designating the distances indicates that they may be written as $16(1)^2$, $16(2)^2$, $16\ 3)^2$, $16(4)^2$, and so on. Consequently, the scientist observes that the relationship between s and t, as indicated by these particular measurements, may be indicated algebraically by $s = 16t^2$. We may therefore assume this law as a hypothesis and then subject it to further verification at various heights and for various other distances and also for bodies of various sizes. In general, it is found that close to the surface of the earth this law holds true, regardless of the size of the body.

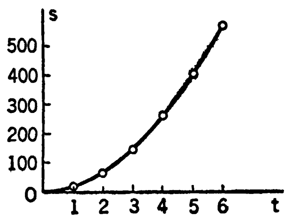


FIG. 9

If we now graph the data given in the previous table and any additional pairs of values (t, s), where t is positive, that satisfy the equation $s = 16t^2$, we have the continuous curve given in Figure 9. If a sufficiently large number of points corresponding to values of t and s that satisfy the equation $s = 16t^2$ are determined, an accurate curve corresponding to the equation $s = 16t^2$ may be drawn; then the curve may be used to determine s corresponding to any given t or to determine t corresponding to any given s.

EXAMPLE 3: Graph the function given by the algebraic formula $y = 2x - 3$.

If we assign arbitrary numerical values to x, such as $1, 2, 3, 4, \cdots$, we may tabulate the number pairs, corresponding in each case to the independent variable and the associated value of the function, as shown below:

Independent Variable x	Function $y = 2x - 3$
1	-1
2	1
3	3
4	5
.	.
.	.
.	.

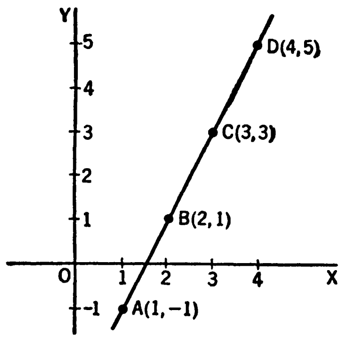


FIG. 10

The points $A(1, -1)$, $B(2, 1)$, $C(3, 3)$, $D(4, 5)$ corresponding to associated values of the variable x and the function $2x - 3$ are represented in Figure 10. The points $A, B, C, D, \cdots$, are joined consecutively by line segments. The totality of all points whose coordinates satisfy the equation is called the *graphical representation of the function.*

32. FIRST-DEGREE FUNCTIONS

A function which is of the form

$$y = mx + b,$$

where m and b are constants and $m \neq 0$, is defined as a general function of the first degree in x.

If $y = 0$, $x = -b/m$; so $x = -b/m$, $m \neq 0$, is described as the zero of the function $y = mx + b$. It is apparent that the coordinates $(-b/m, 0)$ represent the point of intersection of the curve $y = mx + b$ and the x axis.

EXERCISES 23

Locate carefully a few points upon the curve that is the graphical representation of each of the following first-degree functions, and determine the zero of each function. In your arbitrary selection of numbers for x, choose positive and negative integers and fractions.

1. $y = 3x - 6$
2. $y = 5x - 1$
3. $y = 6x$
4. $y = \frac{x}{2}$
5. $y = -3x$
6. $y = -3x - 6$
7. $y = \frac{3x}{4} - 8$
8. $y = \frac{3x}{4} - 6$
9. $y = 4x - 6$
10. $y = 5x - 4$
11. $2y = 3x - 1$

If the student has performed the previous exercises carefully, he must have noticed that for every given function the points seem to lie on some straight line. We shall now show that if we graph the general function

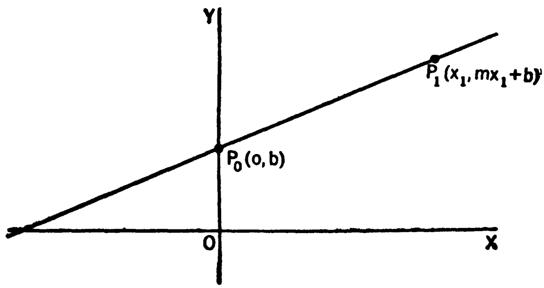


FIG. 11

$y = mx + b$, the points will always lie on a straight line. Hence, we shall have shown that any function of the first degree has a definite straight line as its graph. It is for this reason that the function $y = mx + b$ is frequently referred to as a linear function.

Let us consider the function $y = mx + b$. Let us assign to x the values 0 and x_1. Then, the corresponding values of the function are b and

$mx_1 + b$, respectively. If we indicate the number pairs $(0, b)$ and $(x_1, mx_1 + b)$ as the points P_0 and P_1, respectively, and pass a straight line through the two points, we have the line in Figure 11.

We shall now show that any other pair of corresponding values of the variable x and the function $y = mx + b$ determines a point on the same straight line.

Thus, let us assign to x any value x_2, then the corresponding value of the function is $mx_2 + b$. Let $(x_2, mx_2 + b)$ be the point P_2 in Figure 12; it

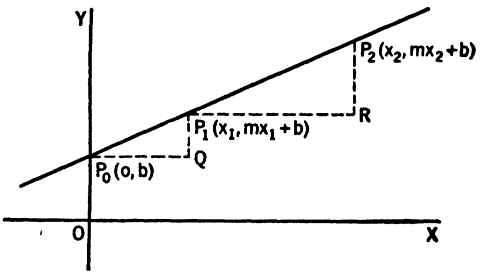


FIG. 12

is not known at the present moment that P_2 is necessarily on the line P_0P_1. Draw the straight lines P_0P_1 and P_1P_2.

From the figure we see that

$$\frac{QP_1}{P_0Q} = \frac{mx_1 + b - b}{x_1} = m,$$

and
$$\frac{RP_2}{P_1R} = \frac{mx_2 + b - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

Consequently,
$$\frac{QP_1}{P_0Q} = \frac{RP_2}{P_1R},$$

and the triangles P_0P_1Q and P_1P_2R are similar. Therefore, angle QP_0P_1 = angle RP_1P_2. Since P_0Q is parallel to P_1R, it follows that line $P_0P_1P_2$ is straight; that is, the point P_2 lies on the straight line through P_0 and P_1.

The number m is called the *slope* of the line $y = mx + b$ and the number b, which is the ordinate of the point where the line cuts the y axis, is called the y intercept.

33. EQUATIONS OF THE FORM $Ax + By + C = 0$

Equations in the form $Ax + By + C = 0$, such as $3x - 4y + 6 = 0$, $6x + 2y - 3 = 0$, $3x - 4y = 0$, may be rewritten in the form $y = mx + b$.

Thus, the equation

$$3x - 4y + 6 = 0$$

may be written

$$-4y = -3x - 6 \quad \text{(subtracting } 3x + 6 \text{ from both members).}$$

This may be transformed further into

$$y = \frac{3x}{4} + \frac{3}{2}. \quad \text{(Dividing each member by } -4\text{)}$$

Similarly, $6x + 2y - 3 = 0$ may be written as $y = -3x + \frac{3}{2}$, and $3x - 4y = 0$ may be written $y = 3x/4$.

If $A = 0$ in the equation $Ax + By + C = 0$, and $B \neq 0$, then $y = -C/B$, irrespective of the value chosen for x; thus, the graph of the equation is a line parallel to the x axis and at a distance $-C/B$ from that axis. The slope of this line is 0, since $m = 0$ when the equation is put in the form $y = mx + b$.

If $B = 0$, $A \neq 0$, then $x = -C/A$, for any value of y. The graph of the equation $x = -C/A$ is a straight line parallel to the y axis and cutting the x axis in $(-C/A, 0)$.

We have thus shown that the graphical representation of any equation of the first degree is a straight line; so, any equation of the form $Ax + By + C = 0$ is called a *linear equation.*

It is evident that the lines representing equations of the form $x = K_1$ and $x = K_2$, $K_1 \neq K_2$, are each parallel to the y axis and, hence, are parallel to each other. The lines representing $y = mx + b$ and $y = mx + c$, where $b \neq c$, may be shown to be parallel to each other as follows:

If we draw the lines $y = mx + b$ and $y = mx + c$, $b \neq c$ (note Figure 13), the distance $OA_1 = b$ and $OA_2 = c$. If distance A_1B_1 is chosen equal

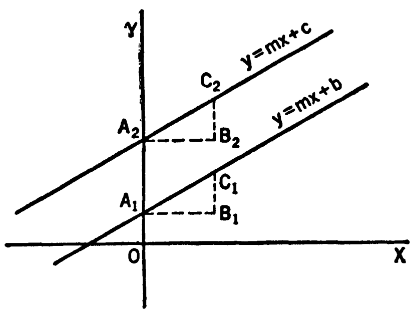


FIG. 13

to A_2B_2 and designated by x_1, then C_1 has the y coordinate $mx_1 + b$, and C_2 has the y coordinate $mx_1 + c$. Consequently, $B_1C_1 = B_2C_2 = mx_1$. There-

fore, the right triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent; hence, the two lines are parallel. If $b = c$ in the equations $y = mx + b$ and $y = mx + c$, the lines are identical.

EXERCISES 24

Write each of the following equations in the form $y = mx + b$, and graph each equation. Which lines of the set are parallel?

1. $3x - 4y + 6 = 0$	**2.** $3x - 4y + 8 = 0$
3. $5x + 2y - 7 = 0$	**4.** $5x + 2y - 15 = 0$
5. $3y = 4x$	**6.** $3y = 4x + 10$
7. $y = x$	**8.** $y = x + 5$
9. $y = -2x + 7$	**10.** $3y + 6x + 14 = 0$

11. Find the area of the parallelogram determined by $y = 2x - 4$, $y = 2x - 12$, $y = 1$, and $y = 8$.

12. Determine the equation of a line passing through the point (0, 3) and having the slope $\frac{3}{2}$.

13. Show that the line representing $y = x + 3$ is inclined at an angle of 45 degrees with the horizontal.

5 First-Degree Equations in One Unknown

34. ROOTS OF AN EQUATION

An equation of the form $f(x) = 0$ is called a *conditional equation* if $f(x)$ does not equal zero for all values of x. If $f(x) = 0$ for all values of x, the equation is called an *identity*. By definition, the values of x which cause $f(x)$ to become zero are the zeros of the function $f(x)$. These values of x are also said to satisfy the equation $f(x) = 0$ and are described as the roots of $f(x) = 0$. Thus, the roots of $x^2 - 7x + 12 = 0$ are $x = 3$ and $x = 4$, since $x = 3$ and $x = 4$ cause the function $x^2 - 7x + 12$ to have the value zero.

The equation $mx + b = 0, m \neq 0$, is an equation of the first degree. The function $mx + b, m \neq 0$, has the value zero when $x = -b/m$; hence, $-b/m$ is a root of $mx + b = 0$.

Theorem. An equation of the first degree $mx + b = 0, m \neq 0$, has only one root, namely, $x = -b/m$. This may be proved as follows:

Assume that x_1 and x_2 are two roots of $mx + b = 0, m \neq 0$; then

$$mx_1 + b = 0 \quad \text{and} \quad mx_2 + b = 0.$$

Hence, $$(mx_1 + b) - (mx_2 + b) = 0 \quad \text{or} \quad m(x_1 - x_2) = 0.$$

Since $m \neq 0$, it follows that $x_1 - x_2 = 0$ and $x_1 = x_2$. Thus, the assumption of two different roots is impossible, and the only root of the equation is the one already described.

35. EQUIVALENT EQUATIONS

The functions $y = x - 3$ and $y = 5x - 15$ are different functions. Moreover, they have different graphs (Figure 14). However, $x = 3$ is the only root of each of the equations $x - 3 = 0$ and $5x - 15 = 0$. Consequently, the two equations are said to be equivalent.

The functions $y = x - 3$ and $y = x^2 - 4x + 3$ are different functions. They have as their corresponding graphs the straight line and the parabolic curve, as shown in Figure 15.

The only root of $x - 3 = 0$ is $x = 3$. There are two roots of $x^2 - 4x + 3 = 0$, namely, $x = 3$ and $x = 1$. These two equations are said to be nonequivalent, even though they have one root, $x = 3$, in

common. In general, two equations that have all their roots in common are said to be *equivalent;* otherwise, they are said to be nonequivalent.

If we consider the equation,

$$x = 2x + 3, \tag{1}$$

and square both members, we have

$$x^2 = 4x^2 + 12x + 9. \tag{2}$$

It can readily be verified that the root of Equation (1) is $x = -3$, while the roots of Equation (2) are $x = -3$ and $x = -1$.

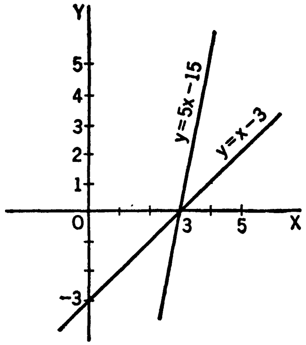


FIG. 14

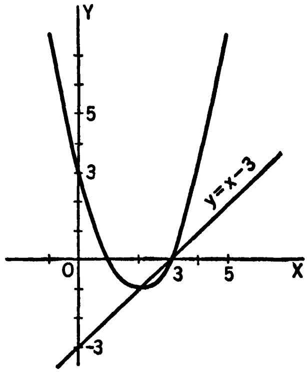


FIG. 15

Hence, Equations (1) and (2) are not equivalent even though Equation (2) was obtained by squaring the members of (1).

Again, if we square both members of the equation

$$\sqrt{x + 1} = x - 1, \tag{3}$$

we have

$$x + 1 = x^2 - 2x + 1. \tag{4}$$

It can readily be verified that $x = 0$ and $x = 3$ are roots of Equation (4), but (3) has only the root $x = 3$, since $x = 0$ does not satisfy (3).

Similarly, if we consider the equation

$$x^2 - 7x + 12 = 0 \tag{5}$$

and divide each member by $x - 3$, we have

$$x - 4 = 0. \tag{6}$$

It can readily be confirmed that the roots of Equation (5) are $x = 3$ and $x = 4$, while the only root of (6) is $x = 4$. Thus, Equations (5) and (6) are not equivalent.

The purpose of the above considerations is to direct attention to the fact that when an equation is derived from another equation by algebraic

means the derived equation is not necessarily equivalent to the original equation. A general answer to the question as to the permissible operations that may be performed upon the members of an equation $f(x) = 0$ to transform it into an equivalent equation $F(x) = 0$ will not be given in this course. It is important to note, however, that we do not divide both members of an equation in x by a polynomial in x, lest we lose possible roots of the original equation. As a general safeguard, all roots of a derived equation should be checked in the original equation, and the values of x that do not satisfy the original equation must be discarded.

Illustration: The equation

$$\frac{8x + 23}{20} - \frac{5x + 2}{3x + 4} = \frac{2x + 3}{5} - 1$$

reduces to $7x - 84 = 0$ after multiplying each member by $20(3x + 4)$; hence, $x = 12$ is a root. This root checks in the original equation.

EXERCISES 25

Solve the following equations and check the roots.

1. $0.05x + 0.02(x - 20) = 28.40$

NOTE: The decimal point may be eliminated by multiplying each member by 100.

2. $0.03(x - 10) - 0.04(50 - x) = 17$
3. $\dfrac{10}{x} + \dfrac{15}{2x} = \dfrac{2}{3}$

SUGGESTION: First, multiply each member by the LCM of the denominators.

4. $21 + \dfrac{3x - 11}{16} = \dfrac{5x - 5}{8} + \dfrac{97 - 7x}{2}$
5. $x + \dfrac{3x - 5}{2} = 12 - \dfrac{2x - 4}{3}$
6. $9x - \dfrac{x - 1}{2} + \dfrac{2x - 2}{3} = 12x - \dfrac{5x - 7}{4}$
7. $\dfrac{a}{x} = b + c$
8. $a + \dfrac{1}{x} = b + c + \dfrac{d}{x}$
9. $x - \dfrac{3x - 3}{5} + 4 = \dfrac{20 - x}{2} - \dfrac{6x - 8}{7} + \dfrac{4x - 4}{5}$
10. $\dfrac{a - x}{b} - \dfrac{4a - x}{c} = a - b$
11. $\dfrac{3x}{b} - \dfrac{x}{c} = m - c$
12. $\dfrac{6x + 7}{9} + \dfrac{7x + 13}{6x + 3} = \dfrac{2x + 4}{3}$
13. $\dfrac{2x + 8}{9} - \dfrac{13x - 2}{17x - 3} + \dfrac{x}{3} = \dfrac{7x}{12} - \dfrac{x + 6}{36}$
14. $\dfrac{x - 1}{x + 1} - \dfrac{x}{x - 2} + \dfrac{4}{x} = 0$

15. $\dfrac{1}{a^2 - 2ax + x^2} - \dfrac{x}{x^2 - a^2} + \dfrac{1}{a + x} = 0$

16. $(a + x)(c - x) - (a - x)(c + x) = 2$

17. $\dfrac{x - 1}{x - \frac{4}{3}} = \dfrac{x + \frac{1}{3}}{x - \frac{2}{3}}$

18. $\dfrac{2x + 5}{x^2 + 9x + 14} = \dfrac{x - 1}{x^2 - x - 6} - \dfrac{5 - x}{x^2 + 4x - 21}$

19. $\dfrac{4}{x + 3} - \dfrac{8x + 3}{9 - x^2} = \dfrac{-3}{3 - x}$

20. If $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$, solve for each letter in terms of the other letters.

36. PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE

The scientist is primarily concerned with mathematics as a tool by means of which he may solve problems arising in his profession. Many practical problems may be expressed mathematically as equations, whereupon the roots may be obtained and properly interpreted. In this chapter we shall consider problems of a practical type which may be expressed mathematically by equations of the first degree.

Illustration 1: How many gallons of a mixture containing 95 per cent alcohol must be added to 50 gal of a solution which is 15 per cent alcohol in order that the resulting mixture shall contain 25 per cent alcohol?

In this type of problem it is first desirable to observe the equality that will become the basis for the equation which is to be solved. It is apparent in this particular problem that one important equality which is involved and which may be symbolized is

Amount of alcohol in the tank at the start + amount of alcohol added = total amount of alcohol finally in the tank.

The first term of this equality is obviously 0.15(50). The second term is not known immediately, since the number of gallons of mixture added to the tank is not known.

Let x = number of gallons of mixture added to the tank.

Then, $0.95x$ = number of gallons of pure alcohol added to that in the tank.

Since there must be $(50 + x)$ gal of final mixture, it follows that

$0.25(50 + x)$ = number of gallons of pure alcohol in final mixture.

Hence, it is now possible to symbolize completely the equality under

consideration, thereby obtaining

$$0.95x + 0.15(50) = 0.25(50 + x),$$
$$0.95x + 7.5 = 12.5 + 0.25x,$$
$$0.7x = 5.0;$$

so

$$x = 7\tfrac{1}{7}\text{ gal.}$$

This result checks with the conditions of the problem as originally given.

The following procedure will be helpful to the student in deriving an equation essential to the solution of a stated problem.

(1) Read the problem carefully and reflect upon it.

(2) Write down in words the fundamental equality that will form the basis for the construction of the equation which is to be solved.

(3) Represent an essential unknown, usually the required quantity, by some letter, such as x.

(4) Express other unknown but necessary quantities only in terms of x and the given quantities.

(5) Completely symbolize the fundamental equality.

Illustration 2: A water tank can be filled by an intake pipe in 3 hr and can be emptied by a drain pipe in 4 hr. How long would it take to fill the tank with both pipes open?

In this problem the basic equality may be chosen as the statement

The surplus of water to be piped into the tank over that drained out in the same time = one tank of water.

It is apparent that the intake pipe can fill one third of the tank in 1 hr, and the drain pipe empties one fourth of the tank in 1 hr. Consequently, the intake pipe gains $(\frac{1}{3} - \frac{1}{4})$ of a tank over the drain pipe in 1 hr.

Let x = time to fill the tank under the conditions of the problem. Then,

$$x(\tfrac{1}{3} - \tfrac{1}{4}) = 1,$$
$$\frac{x}{12} = 1,$$

and

$$x = 12.$$

EXERCISES 26

Solve the following problems:

1. A number consists of two digits, the sum of the digits being 11. If the digits are reversed, the new number is 45 less than the given number. What is the given number?

2. The second digit of a number of two digits is one third the first; and if the number is divided by the difference of its digits, the quotient is 15 and the remainder is 3. Find the number.

3. A train leaves a station and travels at 50 mph. Three hours later another train follows it, traveling at 80 mph. How long before the faster train will overtake the slower train?

4. An airplane travels from A to B at the rate of 180 mph. After it has been gone for 30 min, a second airplane leaves A for B, traveling at the rate of 240 mph, and reaches B 1 hr and 5 min ahead of the first plane. Find the distance from A to B and the time taken by the first plane.

5. A man invests part of a principle of \$2300 at $3\frac{1}{3}\%$ and the balance at $5\frac{1}{4}\%$ and obtains the same income as if he had invested the entire principle at $4\frac{1}{2}\%$. How much does he invest at each rate?

6. A has \$1250 and B has \$500. A spends twice as much money as B and then has three times as much left as B. How much does each spend?

7. An estate of \$1872 is to be divided among a mother, three sons, and two daughters. The mother is to receive three times as much as each daughter, and each son receives one half as much as each daughter. What sum will each receive?

8. If two thirds of a given number is added to one half of it, the sum is 98. Find the number.

9. A beam 28 ft long weighing 500 lb is balanced on a fulcrum by a weight of 200 lb suspended from one end. How far must the fulcrum be placed from this end?

NOTE: The weight of the beam may be assumed concentrated at its center. According to a principle of physics, the weight on one side of the fulcrum multiplied by its distance to the fulcrum must equal the corresponding product obtained on the other side, if there is equilibrium.

10. A beam 20 ft long is balanced on a fulcrum by a weight of 400 lb placed at one end. If the fulcrum is $6\frac{2}{3}$ ft from this end when the beam is balanced, determine the weight of the beam. (See the note in Problem 9.)

11. A crew has bread for a voyage of 50 days if each man eats only $1\frac{1}{2}$ lb a day. After 20 days, 7 men are lost in a storm; this makes it possible for the remainder of the crew to have a daily allowance of $1\frac{4}{5}$ lb for the balance of the voyage. Find the original number of the crew.

12. Oil of two grades is to be mixed; one grade is worth 22 cents a quart, and the other is worth 30 cents a quart. The mixture is to be worth 25 cents a quart. How many gallons of each grade are required to make 500 gal of mixture?

13. A vessel contains 10 gal of an 8 per cent solution of salt. How many gallons of water must be boiled off to make it a 12 per cent solution?

14. A mass of tin and lead weighing 200 lb loses 18 lb when weighed in water. It is known that 50 lb of tin loses 4 lb, and 25 lb of lead loses 3 lb in water. Find the weight of tin and lead in the mass.

15. A reservoir can be filled by one pipe in 30 min and by another pipe in 45 min. A waste pipe empties it in 20 min. If both the filling pipes and the waste pipe are open, how long will it take to fill it?

16. A man can harvest a field of grain in 10 days. He and his son can do it in 8 days. How long would it take the son to harvest the field if he were working alone?

17. A man has three eighths of his money invested at 5% and the remainder at 6%. The total interest amounts to \$180 for the year. What sums are invested at each rate of interest?

18. A transcontinental airline finds that a trip across the country from west to east requires 12 hr, whereas a trip in the other direction requires 13 hr, because of the prevailing winds. If the normal speed of one of the planes in still air is 240 mph, what is the average wind velocity?

19. A man shoots at a metal target, and he hears his rifle bullet strike the target $3\frac{1}{2}$ sec after it was fired. If the bullet travels 2600 fps and sound travels at the rate of 1100 fps, how far away is the target?

6 Variation

37. VARIATION

When two variables y and x are so related that their ratio is a constant, y is said to vary *directly* as x. Of course, this statement is equivalent to the law

$$y = mx,$$

whe e m is a constant. When y varies directly as x, the word *directly* is often implied and not stated. The same relationship between y and x may be expressed by saying that y is proportional to x, since for any two pairs of values (x_1, y_1) and (x_2, y_2) obeying the law $y = mx$,

$$y_1 = mx_1 \qquad \text{and} \qquad y_2 = mx_2;$$

hence we have the proportion

$$\frac{y_1}{y_2} = \frac{mx_1}{mx_2} = \frac{x_1}{x_2}.$$

When the variables y and x are so related that their product is a constant, y is said to vary *inversely* as x. From the algebraic statement

$$yx = m,$$

where m is a constant, we may obtain

$$y = m\frac{1}{x}.$$

Hence, we see that if y varies inversely as x, it varies directly as the variable $1/x$. Also, any two pairs of values (x_1, y_1) and (x_2, y_2) obeying the law $yx = m$ satisfy the relation

$$y_1x_1 = y_2x_2,$$

or

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

Consequently, the same relationship between y and x may be described by saying that y is inversely proportional to x.

A variable z is said to vary jointly as the variables x and y if

$$\frac{z}{xy} = m,$$

or

$$z = mxy,$$

where m is a constant.

A variable z is said to vary directly as the variable x and inversely as the variable y if

$$\frac{zy}{x} = m,$$

or

$$z = m\frac{x}{y},$$

where m is a constant.

The concept of variation has practical value in many problems in both social and physical science. In practical situations it is frequently possible to discover the nature of the law of variation, if such a law is actually present, by the use of experimental methods. It is then possible to determine the constant m by obtaining a single set of related values of the variables involved in the formula.

Illustration 1: It is determined experimentally that within certain limits the amount of elongation of a coiled spring produced by a force acting on one end varies as the amount of the force. If a force of 10 lb produces an elongation of 0.25 in., find m. How much elongation would be produced by a force of 43 lb?

Solution: Let e = amount of elongation in inches,

and P = the number of pounds of force.

Then, since the variation is direct, it follows that

$$\frac{e}{P} = m.$$

After substituting the given values of the variables, we have

$$\frac{0.25}{10} = m,$$

or

$$m = 0.025.$$

Therefore,

$$\frac{e}{P} = 0.025,$$

for all values of e and P when e is measured in inches and P in pounds.

The amount of elongation produced by a force of 43 lb is determined

by substituting 43 for P in the formula just obtained. Hence,

$$e = (0.025)(43) \text{ in.}$$
$$= 1.075 \text{ in.}$$

Illustration 2: The time required to fill a tank with water through a number of pipes of the same diameter, if there is no variation in the supply of water, varies inversely as the product of the number of pipes and the square of the diameter of the pipes. If three pipes 2 in. in diameter can fill this tank in 20 min, how long would it require five pipes, 3 in. in diameter to fill it?

Solution: Let t = time in minutes required to fill the tank,
n = the number of pipes,
and d = diameter in inches.

Then from the nature of the variation as described,

$$tnd^2 = m.$$

After substituting the given values of the variables, we have

$$(20)(3)(2^2) = m,$$

or

$$m = 240.$$

Therefore,

$$tnd^2 = 240.$$

Of course, the constant 240 is only appropriate when t is measured in minutes and d in inches.

The time required for five 3-in. pipes to fill the tank is readily determined by substituting $n = 5$ and $d = 3$ in the formula just obtained. Hence,

$$t = \frac{240}{5(3)^2} = \frac{240}{45} = 5\tfrac{1}{3} \text{ min.}$$

EXERCISES 27

1. The variable u varies directly as v. Moreover, $u = 10$ when $v = 4$. Determine u when $v = 7$.

2. The variable z varies directly as x and inversely as y. If $z = 3$, $x = 9$, and $y = 8$ are related values of the variables, determine y when $z = 1$ and $x = 12$.

3. The distance that a body falls from rest varies as the square of the time during which it falls. If a body falls 402 ft in 5 sec, how long will it take it to fall 1000 ft?

4. The horsepower required to propel a ship in still water varies as the cube of the speed. If the horsepower is 1000 when the speed is 10 knots, what horsepower will be required to produce a speed of 25 knots?

5. When electricity flows through a wire at constant temperature, the wire offers a resistance to the flow of the current which varies directly as the length of wire and inversely as the square of the diameter of its cross section. If a wire 100 ft long and 0.1 in. in diameter has a resistance of 1 ohm, what will be the resistance of a wire 300 ft long and $\frac{1}{4}$ in. in diameter?

6. It is approximately correct that for an observer in an airplane the distance to the horizon varies directly as the square root of the distance of the observer above the ground. If, at a height of 100 ft, the horizon is 12.3 miles distant, what would be the distance to the horizon from a height of 5000 ft?

7. If the temperature of a perfect gas is kept constant, its volume varies inversely as the pressure to which it is subjected (Boyle's law). If 2 cu ft of gas under a pressure of 20 lb per sq in. is forced into a vacuum tank that holds 5 cu ft and is allowed to expand to fill the tank, what will be its pressure?

8. The period of vibration of a pendulum is found to vary directly as the square root of its length. If a pendulum 1 m long ticks seconds, what will be the period of vibration of a pendulum 40 cm long?

9. The strength of a beam having a rectangular cross section varies inversely as its length and directly as its breadth and the square of its depth. If a spruce beam 16 ft long, 6 in. wide, and 8 in. deep will carry safely 1000 lb at the middle, how much will a similar piece of spruce 10 ft long, 4 in. wide, and 6 in. deep carry at the middle when used in the same way?

10. The force with which the earth pulls on a body outside of its surface is found to vary inversely as the square of the distance from its center. If the surface of the earth is 3960 miles from its center, and if a rocket weighing 1 ton at the surface is shot to a height of 100 miles above the surface, what would be its weight at that height?

11. The illumination from a source of light varies inversely as the square of the distance from the source. A book held 20 in. from the source is moved closer. How far must it be moved so that it will receive twice as much illumination?

12. The volume of a cube varies as the cube of the edge. Find the edge of a cube whose volume is double the volume of a cube with a 2-in. edge.

13. The lateral surface of a right circular cylinder varies jointly as the height and radius of the base. Find the ratio of the lateral surface of a cylinder with altitude 10 in. and radius of base 10 in. to the lateral surface of a cylinder with altitude 15 in. and base 5 in.

14. If two right circular cylinders of radius r and equal height are melted and cast into a new right circular cylinder with the same height as each of the original cylinders, show that the radius of the new cylinder is $\sqrt{2}r$.

15. If the radius of a sphere is increased by 10 per cent, by what per cent will its volume be increased?

16. If a plate is mounted in a wind tunnel with its surface at right angles to the direction of the air flow, the pressure varies jointly as the area of the plate and the square of the wind velocity for a given air density. If the pressure on a plate 500 sq in. in area is 4.56 lb when the wind velocity is 30 ft per sec, how much pressure will be exerted on an area of 5 sq ft with a wind velocity of 50 mph?

7 Systems of First-Degree Equations

38. SYSTEMS OF FIRST-DEGREE EQUATIONS

Many practical problems require the determination of a set of two or more unknowns that satisfy a system of equations of the first degree. The following are illustrations of typical problems which may be solved in each case by setting up a system of first-degree equations; also, some well-known methods for the solution of such systems are presented.

Illustration 1: A jeweler wishes to mix 10-carat gold with 18-carat gold to make 30 oz of 12-carat gold. How many ounces of each must be taken?

Solution: Let x = number of ounces of 10-carat gold,

and y = number of ounces of 18-carat gold.

There are two fundamental equalities relating x and y; these are

Number of ounces of 10-carat gold + number of ounces of 18-carat gold = 30 oz. (1)

Number of carats in required amount of 10-carat gold + number of carats in required amount of 18-carat gold = total number of carats. (2)

Since $10x$ = number of carats in x ounces of 10-carat gold,

and $18y$ = number of carats in y ounces of 18-carat gold,

the two basic equalities just given may be symbolized as follows:

$$x + y = 30, \qquad (1)$$

$$10x + 18y = (30)(12) = 360. \qquad (2)$$

We must now seek the pair of values of x and y that satisfy both equations.

Equations (1) and (2) may be solved in several ways. The first method given is frequently described as the *addition or subtraction method.*

$$10x + 10y = 300 \quad \text{Multiplying the members of (1) by 10} \tag{3}$$

$$\underline{10x + 18y = 360} \tag{2}$$

$$8y = 60 \quad \text{Subtracting the members of (3) from those of (2)}$$

Hence, $y = 7\frac{1}{2}$ oz.

After substituting $7\frac{1}{2}$ for y in Equation (1), we have

$$x = 22\tfrac{1}{2} \text{ oz.}$$

By multiplying the members of Equation (1) by 10 and then subtracting them from the corresponding members of Equation (2), we were able to eliminate the x, leaving a simple equation in y to be solved.

Similarly, we could have multiplied the members of (1) by 18 and have eliminated the y by subtracting the members of (2) from the corresponding members of $18x + 18y = 540$.

A second method for the solution of the system is to solve one of the equations for either unknown in terms of the other, and to substitute the resulting expression in the second equation. This is usually known as the *method of substitution.*

Thus, from (1), namely, $x + y = 30$, we obtain

$$y = 30 - x. \tag{4}$$

After substituting this value for y in Equation (2), there results:

$$10x + 18(30 - x) = 360. \tag{5}$$

Equation (5) may be simplified to

$$-8x = -180. \tag{6}$$

Hence, $x = 22\frac{1}{2}$ oz,

and, consequently, $y = 7\frac{1}{2}$ oz.

The next chapter presents a third method for the solution of such a system as we have just considered.

Illustration 2: Solve the following system of equations:

$$4x + 3y + 9z = 53, \tag{1}$$

$$11x - 2y + 8z = 75, \tag{2}$$

$$6x + y + 5z = 47. \tag{3}$$

Since the coefficients of y are small in each equation, it appears desir-

able to eliminate y first; hence, we have

$8x + 6y + 18z = 106$	Multiplying (1) by 2	(4)
$33x - 6y + 24z = 225$	Multiplying (2) by 3	(5)
$41x + 42z = 331$	Adding (4) and (5)	(6)
$11x - 2y + 8z = 75$		(2)
$12x + 2y + 10z = 94$	Multiplying (3) by 2	(7)
$23x + 18z = 169$	Adding (2) and (7)	(8)

Equations (6) and (8) may now be solved as a system for x and z, and then y may be found by substituting the values for x and z in any one of the original equations.

This is left as an exercise for the student. The solution is $x = 5$, $y = 2$, $z = 3$.

Illustration 3: Certain equations that are not of the first degree in the given variables may be converted into equations of the first degree. The following problem results in such a system of equations.

A cistern is filled by three pipes. The first and second will fill it in 72 min, the second and third in 120 min, and the first and third in 90 min. How long will it take each of the pipes to fill it?

Solution: Let x = number of minutes it will take first pipe to fill cistern,

y = number of minutes it will take second pipe to fill cistern,

z = number of minutes it will take third pipe to fill cistern.

Then, $\frac{1}{x}$ = fractional part of cistern filled by first pipe in 1 min,

$\frac{1}{y}$ = fractional part of cistern filled by second pipe in 1 min,

$\frac{1}{z}$ = fractional part of cistern filled by third pipe in 1 min.

Hence, from the given conditions,

$$\frac{72}{x} + \frac{72}{y} = 1, \tag{1}$$

$$\frac{120}{y} + \frac{120}{z} = 1, \tag{2}$$

and

$$\frac{90}{x} + \frac{90}{z} = 1. \tag{3}$$

These equations are not of first degree. However, if we substitute u for $1/x$, w for $1/z$, and v for $1/y$, we have the following equivalent linear system in u, v, and w:

$$72u + 72v = 1, \tag{1'}$$

$$120v + 120w = 1, \tag{2'}$$

$$90u + 90w = 1. \tag{3'}$$

We may now solve this system for u, v, and w and, hence, know x, y, and z.

$$\begin{array}{rcll}
720u + 720v &=& 10. & \text{Multiplying (1') by 10.} \quad (4') \\
\underline{720v + 720w} &=& \underline{\ 6.} & \text{Multiplying (2') by 6.} \quad (5') \\
720u - 720w &=& 4. & \text{Subtracting (5') from (4').} \quad (6') \\
\underline{720u + 720w} &=& \underline{\ 8.} & \text{Multiplying (3') by 8.} \quad (7') \\
1440u &=& 12. & \text{Adding (6') and (7').}
\end{array}$$

Hence, $u = \dfrac{1}{120}$; therefore, $x = 120$ min. Similarly, v and w can be found, and then we can determine y and z.

Obviously, the given system, though not linear, can readily be solved by treating the equations as linear in $1/x$, $1/y$, and $1/z$. Thus,

$$\begin{array}{rcll}
\dfrac{720}{x} + \dfrac{720}{y} &=& 10. & \text{Multiplying (1) by 10.} \quad (4) \\[2ex]
\underline{\dfrac{720}{y} + \dfrac{720}{z}} &=& \underline{\ 6.} & \text{Multiplying (2) by 6.} \quad (5) \\[2ex]
\dfrac{720}{x} - \dfrac{720}{z} &=& 4. & \text{Subtracting (5) from (4).} \quad (6) \\[2ex]
\underline{\dfrac{720}{x} + \dfrac{720}{z}} &=& \underline{\ 8.} & \text{Multiplying (3) by 8.} \quad (7) \\[2ex]
\dfrac{1440}{x} &=& 12. & \text{Adding (6) and (7).}
\end{array}$$

Hence, as before, $x = 120$ min.

After substituting $x = 120$ in Equation (1), we have

$$\frac{72}{y} = 1 - \frac{72}{120}.$$

The solution of this equation provides the result

$$y = 180 \text{ min.}$$

After substituting $y = 180$ in Equation (2), we have

$$\frac{120}{z} = 1 - \frac{120}{180},$$

or

$$z = 360 \text{ min.}$$

NOTE: The student should check the solution in each equation of the original system.

EXERCISES 28

Solve the following systems of equations, and check your solutions:

1. $39x - 8y = 99$
   $52x - 15y = 80$

2. $8x - 5y = 0$
   $13x - 8y = 1$

3. $2x + y = 17$
   $x - 2y = 1$

4. $3x - \dfrac{y-5}{7} = \dfrac{4x-3}{2}$
   $\dfrac{3y+4}{5} - \dfrac{1}{3}(2x-5) = y$

5. $\dfrac{x+1}{y} = \dfrac{1}{3}$
   $\dfrac{x}{y+1} = \dfrac{1}{4}$

6. $0.8x + 0.1y = 0.19$
   $0.6x + 0.9y = 0.39$

7. $k^2x + m^2y = 0$
   $kx + my = k + m$

8. $\dfrac{x+3}{2} + 5y = 9$
   $\dfrac{y+9}{10} - \dfrac{x-2}{3} = 0$

9. $\dfrac{x}{2} - 12 = \dfrac{y}{4} + 8$
   $\dfrac{x+y}{5} + \dfrac{x}{8} - 8 = \dfrac{2y-x}{4} + 27$

10. $\dfrac{4x+5y}{40} = x - y$
    $\dfrac{1}{2} - \dfrac{2x-y}{3} = 2y$

11. $\dfrac{5}{x} - \dfrac{7}{y} = 6$
    $\dfrac{13}{x} + \dfrac{5}{y} = 4$

12. $\dfrac{2}{3x} + \dfrac{9}{2y} = 9\frac{2}{3}$
    $\dfrac{11}{5x} - \dfrac{1}{3y} = 1\frac{8}{15}$

13. $2x - 5y - 7z = 19$
    $5x + 2y - 3z = 33$
    $3x - 7y + 4z = -14$

14. $-x - 13y + 5z = 3$
    $6x + 2y + 3z = -9$
    $3x - 5y - 2z = 15$

15. $\dfrac{2}{x} - \dfrac{3}{y} + \dfrac{4}{z} = -2$
    $\dfrac{5}{x} + \dfrac{6}{y} - \dfrac{2}{z} = 6$
    $\dfrac{3}{x} - \dfrac{5}{y} + \dfrac{2}{z} = \dfrac{1}{3}$

Solve the following problems:

16. Find the rational fraction such that if we add 2 to the numerator, the fraction equals $\frac{1}{2}$, but if we add 3 to the denominator, the fraction equals $\frac{1}{3}$.

17. A man rows 30 miles and back in 12 hr. He can row 5 miles with the stream in the same time that he can row 3 miles against the stream. Find the time required to row up the 30 miles and down, respectively.

18. A power boat whose speed is 30 mph in still water makes a trip of unknown length downstream in 45 min; another boat whose speed is 20 mph in still water makes the same trip in 65 min. Find the length of the trip and the rate of the stream.

19. *A* and *B* together received $346 wages for working 25 and 16 days, respectively. If *A* had worked 20 days and *B* had worked 18 days, they would have received $308. What were the daily wages of each?

20. A grocer offered to sell 50 lb of coffee and 100 lb of sugar for $23, or 10 lb of coffee and 5 lb of sugar for $2.95. Find the price per pound of each.

21. A man receives $3000 yearly interest on his money. If he had loaned the same amount of money at $\frac{1}{2}$ per cent higher interest, he would receive $300 more interest. Find the amount of money which was invested and the rate of interest.

22. The sum of the three angles of any triangle is 180 degrees. If one angle of a triangle exceeds half the sum of the other two angles by 21 degrees and half their difference by 56 degrees, what are the angles?

39. GRAPHICAL REPRESENTATION OF A SYSTEM OF TWO LINEAR EQUATIONS

If we graph on the same set of axes the two straight lines which represent, respectively, the two equations of the following system

$$x + y = 30$$

$$10x + 18y = 360,$$

we obtain Figure 16.

Obviously, the associated values of x and y which satisfy both equations are the coordinates of the point common to both lines. A careful construction would show the coordinates of this common point to be $x = 22\frac{1}{2}$ and $y = 7\frac{1}{2}$. This same system of equations was solved previously in Illustration 1 of Section 38, and the same results were obtained.

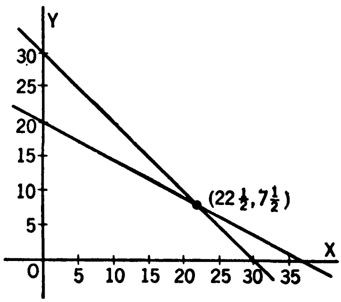


FIG. 16

The method employed in this problem is general in its application; that is, the coordinates of the point of intersection of two straight lines representing a system of two linear equations in two unknowns are the values of x and y which satisfy both equations.

EXERCISES 29

Solve graphically the first six problems of Exercises 28.

40. CONSISTENT SYSTEMS OF LINEAR EQUATIONS

We have just seen that the equations of Section 39 are satisfied by a single pair of values of x and y, and that the lines of the equations are a pair of intersecting lines.

If we attempt to solve the system of equations

$$3x + 4y = 8 \tag{1}$$

$$6x + 8y = 16, \tag{2}$$

we find that both x and y are eliminated, and the result is merely a statement that two equal numbers are equal. It is apparent that Equation (2) furnishes no information about x and y that is not given by Equation (1); for Equation (2) may be obtained from (1) by multiplying each member by 2. The actual equivalence of the two equations is made even more obvious by the fact that their graphs are the same straight line; thus, a pair of values satisfying Equation (1) will also satisfy (2).

Whenever a pair of linear equations are satisfied simultaneously by one or more pairs of values of x and y, they are said to be *consistent.* If they are satisfied simultaneously by only a single pair of values of x and y, they are called *consistent and independent.* If they are both satisfied for all values of x and y that satisfy one of them, they are called consistent, but are dependent. In other words, whenever the lines representing a pair of linear equations intersect in one point or are identical, the equations are consistent. If the lines intersect in only one point the equations are consistent and independent. If the lines are identical, the equations are consistent and dependent.

41. INCONSISTENT SYSTEMS OF LINEAR EQUATIONS

If we attempt to solve the following system of equations:

$$3x + 4y = 7 \tag{1}$$

$$6x + 8y = 15, \tag{2}$$

we meet an unusual situation.

After multiplying the members of (1) by 2, we have Equation (3) which is:

$$6x + 8y = 14. \tag{3}$$

Equations (3) and (2) cannot be true simultaneously, for that would require 14 to equal 15, which is impossible. Hence, we say the pair of Equations (1) and (2), or the equivalent pair of Equations (3) and (2), is an *inconsistent pair of equations.*

If we attempt to solve an inconsistent pair of equations we always find,

as in the previous example, that both x and y are eliminated and the resulting equation requires the equality of two unequal numbers, which is impossible. Hence, there is no common pair of values of x and y that will satisfy both equations.

The slope of each of the lines (1) and (2) is $-\frac{3}{4}$, but the y intercepts are, respectively, $\frac{7}{4}$ and $\frac{15}{8}$. Hence, it is apparent that the lines are distinct and parallel. Whenever two lines are distinct and parallel, the two equations corresponding to the lines are said to be *inconsistent.*

EXERCISES 30

Solve each of the following systems of equations. In each case, state if the system is consistent and independent, consistent and dependent, or inconsistent.

1. $5x - 3y = 7$
   $10x + 6y = 9$

2. $5x - 3y = 7$
   $10x - 6y = 18$

3. $4x - 7y = 9$
   $10x - 14y = 18$

4. $8x + 3y = 5$
   $7x - 2y = 5$

5. $y = 2x + 7$
   $y = 3x - 5$

6. $y = 2x + 7$
   $y = 2x + 8$

7. $y = 3x - 7$
   $3y = 9x - 21$

8. $\frac{x}{5} + \frac{y}{4} = 1$
   $\frac{x}{10} + \frac{y}{8} = 1$

9. $\frac{x}{5} + \frac{y}{4} = 1$
   $\frac{x}{5} - \frac{y}{4} = 1$

10. $7x + 8y = 6$
    $8x - 3y = 5$

11. $x + y = 0$
    $3x - 8 = 0$

12. $5y + 6 = 0$
    $3y - 11 = 0$

13. The perimeter of a triangle is 39 in. One side is 13 in. less than the sum of the other two; and one of these two is three times as large as the difference of the remaining two. Find the length of each side.

14. A and B together can do a piece of work in 20 days; at the end of 12 days, B is called off and A finishes it in 20 days. Find the time in which each could have done the work alone.

15. A and B do a piece of work in 12 days; B and C do the same piece of work in 20 days; A and C do the same piece of work in 15 days. How long will it take each to do the work alone?

16. A gunner fires at a target 500 yd away and hears the bullet strike $2\frac{3}{10}$ sec after he fires. An observer stationed 400 yd from the target and 300 yd from the gunner hears the bullet strike $1\frac{1}{5}$ sec after he hears the report of the rifle. Find the velocity of sound in feet per second and the velocity of the bullet in feet per second.

17. A and B run two quarter-mile races. In the first race A gives B 20 yd start and wins by 5 yd. In the second race A gives B a start of 5 sec and loses by 5 yd. Find the rates of A and B in yards per second.

18. An alloy of metal which weighs 50 lb loses 7 lb when weighed in water. If this alloy is composed of two metals, which we may call A and B; and if it is found that a 50-lb piece of A loses 5 lb when weighed in water, and a 50-lb piece of B loses 10 lb when weighed in water, how much of each metal is there in the alloy?

19. A bar of metal contains 18.22 per cent pure silver, and a second bar contains 10.57 per cent. How many ounces of each bar must be used if, when the parts are melted together, a new bar weighing 100 oz is obtained, of which 15 per cent is pure silver?

20. A power boat whose speed in still water is unknown makes a trip of unknown distance in 75 min and the return trip in 1 hr and 40 min. The rate of the stream is 5 mph. Find the rate of the boat in still water and the distance.

21. One half the distance from A to B is level; the other half is part uphill and part downhill. A messenger can travel 6 mph uphill, 12 mph on the level, and 18 mph downhill. If it takes him 2 hr and 40 min to go from A to B and 2 hr to return, what is the distance from A to B, and how much of it is uphill?

8 Determinants

42. DETERMINANTS OF THE SECOND ORDER

The square array of quantities enclosed within two vertical bars

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

is called a *determinant of the second order* and means, by definition, $A_1B_2 - A_2B_1$.

Thus, the symbolic form

$$\begin{vmatrix} 8 & 5 \\ -7 & -6 \end{vmatrix}$$

means $8(-6) - (-7)5 = -13$.

Similarly, the symbolic form

$$\begin{vmatrix} (a+b) & c \\ 1 & 5 \end{vmatrix}$$

means $5a + 5b - c$.

The solution of the system of equations

$$A_1x + B_1y = C_1 \tag{1}$$

$$A_2x + B_2y = C_2, \tag{2}$$

when $A_1B_2 - A_2B_1 \neq 0$, leads to the system

$$A_1B_2x + B_1B_2y = C_1B_2 \tag{3}$$

$$A_2B_1x + B_1B_2y = C_2B_1, \tag{4}$$

if each member of (1) is multiplied by B_2 and each member of (2) is multiplied by B_1. When the members of (4) are subtracted from the corresponding members of (3), the value of x is found to be

$$x = \frac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1}.$$

It also may be determined that

$$y = \frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1}.$$

These two results may be displayed in convenient form through the use of determinants; in fact, they may be written

$$x = \frac{\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}.$$

This method for the solution of a system of two linear equations may be applied in almost mechanical fashion. For instance, let us consider the system

$$5x - 3y = 5,$$

$$8x + 9y = 11.$$

The desired values of x and y may be written down immediately in symbolic form as follows:

$$x = \frac{\begin{vmatrix} 5 & -3 \\ 11 & 9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 8 & 9 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} 5 & 5 \\ 8 & 11 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 8 & 9 \end{vmatrix}}.$$

The evaluation of these forms leads to

$$x = \frac{(5)(9) - (11)(-3)}{(5)(9) - (8)(-3)} = \frac{26}{23},$$

$$y = \frac{(5)(11) - (8)(5)}{(5)(9) - (8)(-3)} = \frac{5}{23}.$$

Attention is directed to the form of the general solution involving the determinants. It may be seen that the expressions for x and y have the same determinant as a denominator, namely, the determinant

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

which is made up of the coefficients of x and y in their natural order as they appear in the given equations. It is also seen that the numerator of the expression for x is obtained from the determinant of the denominator by replacing A_1, A_2 (the coefficients of x) by C_1, C_2. Similarly, the numerator of y is obtained from the determinant of the denominator by replacing B_1, B_2 (the coefficients of y) by C_1, C_2. These observations should assist in setting up the desired determinants.

We note that when the denominator determinant equals zero, the system of equations can not be independent. In this case one should consider the numerators of the expressions for x and y. If neither or only one of these determinants is zero, the pair of equations is inconsistent;

but if both of these determinants are zero, the pair of equations is consistent and dependent.

Thus, for the equations

$$3x + 5y = 4$$

$$3x + 5y = 11$$

the required denominator is

$$\begin{vmatrix} 3 & 5 \\ 3 & 5 \end{vmatrix} = 0.$$

The numerator of the value of x is

$$\begin{vmatrix} 4 & 5 \\ 11 & 5 \end{vmatrix} = 20 - 55 = -35 \neq 0,$$

and the numerator of the value of y is

$$\begin{vmatrix} 3 & 4 \\ 3 & 11 \end{vmatrix} = 33 - 12 = 21 \neq 0.$$

Hence, the equations are inconsistent.

For the equations

$$3x + 0 \cdot y = 4$$

$$5x + 0 \cdot y = 11$$

the required denominator is

$$\begin{vmatrix} 3 & 0 \\ 5 & 0 \end{vmatrix} = 0.$$

The numerator of the value of x is

$$\begin{vmatrix} 4 & 0 \\ 11 & 0 \end{vmatrix} = 0,$$

but the numerator of the value of y is

$$\begin{vmatrix} 3 & 4 \\ 5 & 11 \end{vmatrix} = 13 \neq 0.$$

So these equations are also inconsistent.

For the system,

$$3x + 6y = 10$$

$$6x + 12y = 20$$

both the numerator and denominator in the value of x and of y are zero. It follows that the equations are consistent, but dependent.

EXERCISES 31

Solve the first twelve problems of Exercises 28 by the determinant method.

43. DETERMINANTS OF THE THIRD ORDER

The square array

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

is called a *determinant of the third order* and means, by definition,

$$A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2.$$

Thus, in the determinant

$$\begin{vmatrix} 8 & 5 & 6 \\ 7 & 9 & 5 \\ 6 & 4 & 2 \end{vmatrix}$$

we have

$$\begin{array}{lll} A_1 = 8 & B_1 = 5 & C_1 = 6 \\ A_2 = 7 & B_2 = 9 & C_2 = 5 \\ A_3 = 6 & B_3 = 4 & C_3 = 2. \end{array}$$

Hence, the value of the determinant is

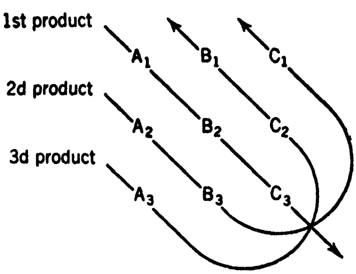
$$(8)(9)(2) + (7)(4)(6) + (6)(5)(5) - (6)(9)(6) - (7)(5)(2) - (8)(4)(5) = -92.$$

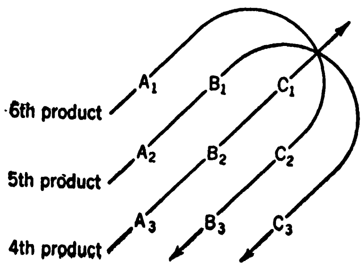
However, it is easier to remember the evaluation of

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

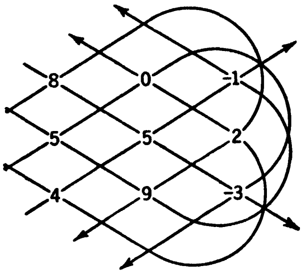
as the sum of the first, second, and third products in the following diagram minus the sum of the fourth, fifth, and sixth product of the next diagram; that is,

$$(A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2) - (A_3B_2C_1 + A_2B_1C_3 + A_1B_3C_2).$$





Thus, the determinant involving the following array of numbers,



has the value

$$\begin{aligned}&[(8)(5)(-3) + (5)(9)(-1) + (4)(2)(0)]\\ &\qquad -[(4)(5)(-1) + (5)(0)(-3) + (8)(2)(9)]\\ &\qquad\qquad = (-120 - 45 + 0) - (-20 + 0 + 144) = -289.\end{aligned}$$

44. SOLUTION OF SYSTEMS OF THREE FIRST-DEGREE EQUATIONS BY USE OF DETERMINANTS

By using methods that have already been discussed, the solution of the system of equations

$$A_1x + B_1y + C_1z = D_1 \tag{1}$$

$$A_2x + B_2y + C_2z = D_2 \tag{2}$$

$$A_3x + B_3y + C_3z = D_3 \tag{3}$$

is found to be

$$x = \frac{D_1B_2C_3 + D_2B_3C_1 + D_3B_1C_2 - D_3B_2C_1 - D_2B_1C_3 - D_1B_3C_2}{A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2},$$

$$y = \frac{A_1D_2C_3 + A_2D_3C_1 + A_3D_1C_2 - A_3D_2C_1 - A_2D_1C_3 - A_1D_3C_2}{A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2},$$

$$z = \frac{A_1B_2D_3 + A_2B_3D_1 + A_3B_1D_2 - A_3B_2D_1 - A_2B_1D_3 - A_1B_3D_2}{A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2}.$$

If we compare the numerator and denominator of these values of x, y, and z with the development of a determinant of the third order, and if we designate

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \text{ by } \Delta \text{ and assume } \Delta \neq 0,$$

the solution may be written

$$x = \frac{\begin{vmatrix} D_1 & B_1 & C_1 \\ D_2 & B_2 & C_2 \\ D_3 & B_3 & C_3 \end{vmatrix}}{\Delta}; y = \frac{\begin{vmatrix} A_1 & D_1 & C_1 \\ A_2 & D_2 & C_2 \\ A_3 & D_3 & C_3 \end{vmatrix}}{\Delta}; z = \frac{\begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix}}{\Delta}.$$

The rules for the formation of the determinants for the common denominator of the solution for x, y, z, and the different numerators of the solution are exactly as given previously for the solution of two equations in two unknowns.

This same method applies to the solution of n first-degree equations in the same n unknowns, when the determinant of the common denominator of the solution is not equal to zero. It should be emphasized at this point, however, that no general method of expanding a determinant of any order has yet been discussed.

Illustration: Solve the following system:

$$\begin{aligned} x - 3y \phantom{{}- 3z} &= 7 \\ 2x + y - 3z &= 8 \\ 5x - y + 9z &= 14. \end{aligned}$$

It should be noted that in the first equation the unknown z does not occur; hence, the coefficient of z in that equation is 0. The required solution is

$$x = \frac{\begin{vmatrix} 7 & -3 & 0 \\ 8 & 1 & -3 \\ 14 & -1 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & -3 \\ 5 & -1 & 9 \end{vmatrix}} = \frac{384}{105} = \frac{128}{35};$$

$$y = \frac{\begin{vmatrix} 1 & 7 & 0 \\ 2 & 8 & -3 \\ 5 & 14 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & -3 \\ 5 & -1 & 9 \end{vmatrix}} = \frac{-117}{105} = -\frac{39}{35};$$

$$z = \frac{\begin{vmatrix} 1 & -3 & 7 \\ 2 & 1 & 8 \\ 5 & -1 & 14 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & -3 \\ 5 & -1 & 9 \end{vmatrix}} = \frac{-63}{105} = -\frac{21}{35}.$$

EXERCISES 32

Solve the following systems of equations by determinants, and evaluate the determinants.

1. $3x + 2y - 4z = 15$
   $5x - 3y + 2z = 28$
   $-x + 3y + 4z = 24$

2. $4x + 6y - 3z = 17$
   $x + 7y + z = 35$
   $5x + 13y + 4z = 82$

3. $4x - 7y + z = 16$
   $3x + y - 2z = 10$
   $5x - 6y - 3z = 10$

4. $x + y = 4$
   $y + z = 5$
   $x + z = 7$

5. $3x - z = 0$
   $x + 5y = 6$
   $y + z = 7$

6. $\frac{2}{x} + \frac{3}{y} - \frac{4}{z} = 20$
   $\frac{1}{x} - \frac{2}{y} + \frac{3}{z} = 30$
   $\frac{3}{x} - \frac{4}{y} - \frac{5}{z} = 5$

7. $\frac{5}{x} - \frac{7}{y} + \frac{1}{z} = \frac{1}{2}$
   $\frac{3}{x} + \frac{2}{y} = \frac{2}{5}$
   $\frac{4}{y} - \frac{5}{z} = \frac{1}{3}$

8. Three men, A, B, and C, were solicited to give money for a certain charity. A agreed to give half as much as B and C combined. B said he would give \$1000 more than A and C combined. The solicitor finally raised \$9000 from the three men. How much did each give?

9. In the theory of electricity it is a fundamental principle that the reciprocal of the total resistance of any number of conductors connected in parallel is the sum of the reciprocals of the individual resistances. The total resistance of three resistances connected in parallel is 4 ohms. Moreover, the greatest resistance is twice as many ohms as the smallest resistance and is $1\frac{1}{2}$ times as many ohms as the third resistance. What is the magnitude of each resistance in ohms?

45. SOME PROPERTIES OF DETERMINANTS

A general analysis of the meaning and significance of determinants cannot be undertaken in an elementary text. Suffice it to say, the general subject of determinants has been the object of much research, and many interesting properties have been discovered. For convenience in calculation we shall record in the following paragraphs a few elementary properties of determinants.

Property 1. *The value of a determinant is not changed when corresponding rows and columns are interchanged.* Thus,

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Property 2. *Interchanging any two rows (or columns) of a determinant changes the sign of the determinant.* Thus,

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = - \begin{vmatrix} C_1 & B_1 & A_1 & D_1 \\ C_2 & B_2 & A_2 & D_2 \\ C_3 & B_3 & A_3 & D_3 \\ C_4 & B_4 & A_4 & D_4 \end{vmatrix}.$$

Property 3. *If two rows (or columns) of a determinant are identical, the determinant is equal to zero.*

This may be proved as follows: An interchange of two identical columns (or rows) obviously does not change the value of the determinant. But according to Property (2), if the value of the original determinant is Δ, then the interchange of the two identical columns (or rows) yields

$$\Delta = -\Delta \quad \text{or} \quad 2\Delta = 0;$$

so

$$\Delta = 0.$$

Property 4. *If each element in any row (or column) is multiplied by the same factor, the determinant is multiplied by that factor.* Thus,

$$\begin{vmatrix} MA_1 & MB_1 & MC_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = M \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Property 5. *The value of any determinant is not changed if each element of any row (or column) multiplied by any factor* M *is added to the corresponding element of any other row (or column).* Thus,

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 + MB_1 & B_1 & C_1 \\ A_2 + MB_2 & B_2 & C_2 \\ A_3 + MB_3 & B_3 & C_3 \end{vmatrix}.$$

This property follows from the fact that

$$\begin{vmatrix} A_1 + MB_1 & B_1 & C_1 \\ A_2 + MB_2 & B_2 & C_2 \\ A_3 + MB_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} + M \begin{vmatrix} B_1 & B_1 & C_1 \\ B_2 & B_2 & C_2 \\ B_3 & B_3 & C_3 \end{vmatrix}.$$

The last determinant is zero by Property (3); hence, Property (5) is true.

If we select the proper value for M, the application of this property enables us to replace the original determinant by another determinant in which one or more of the elements are zeros. The expert in the use of determinants may use this device repeatedly for the purpose of simplifying a determinant before obtaining its evaluation.

Property 6. *If in a determinant of any order* n, *such as*

$$\begin{vmatrix} A_1 & B_1 \cdots L_1 \\ A_2 & B_2 \cdots L_2 \\ \cdots & \cdots\cdots \\ A_n & B_n \cdots L_n \end{vmatrix},$$

we exclude the row and the column containing A_1, *the determinant of order* $(n-1)$, *which remains, is called the minor of* A_1 *and may be designated by* $M(A_1)$. If we exclude the row and the column containing L_1, the determinant of order $(n-1)$ which remains is called the minor of L_1 and is designated by $M(L_1)$. Similarly a determinant of order $(n-1)$ may be designated as the minor for each element of the determinant. We now state, without proof, that

$$\Delta = A_1[M(A_1)] - A_2[M(A_2)] + \cdots + (-1)^{n+1}A_n[M(A_n)],$$

or

$$\Delta = A_1[M(A_1)] - B_1[M(B_1)] + \cdots + (-1)^{n+1}L_1[M(L_1)].$$

It is observed that, as these products are written in order with respect to the elements down the left column or across the top row, the signs preceding the products alternate.

Similarly, we may expand the determinant by minors relative to the elements of any column (or row) by multiplying the elements of any column (or row) by their corresponding minors and prefixing a plus or minus sign according as the sum of the number of the column and the number of the row of the element is even or odd. Thus, for example,

$$\Delta = -B_1[M(B_1)] + B_2[M(B_2)] - \cdots \pm B_n[M(B_n)],$$

$$\Delta = +C_1[M(C_1)] - C_2[M(C_2)] + \cdots \pm C_n[M(C_n)].$$

If, as an illustration, the determinant under consideration is

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix};$$

then

$$\Delta = A_1\begin{vmatrix} B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix} - B_1\begin{vmatrix} A_2 & C_2 & D_2 \\ A_3 & C_3 & D_3 \\ A_4 & C_4 & D_4 \end{vmatrix}$$

$$+ C_1\begin{vmatrix} A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \\ A_4 & B_4 & D_4 \end{vmatrix} - D_1\begin{vmatrix} A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \\ A_4 & B_4 & C_4 \end{vmatrix}.$$

By virtue of this very important property, a determinant of the nth order may be replaced by the algebraic sum of n determinants of order $(n-1)$; each of these latter determinants may be replaced by the sum of $(n-1)$ determinants of order $(n-2)$; and so on. This operation provides us with a general method for the evaluation of a determinant of any order.

Illustration: Evaluate the determinant

$$\begin{vmatrix} 2 & 3 & -4 & 6 \\ 3 & 5 & -1 & 7 \\ 2 & 1 & 0 & 3 \\ -1 & 0 & 4 & 2 \end{vmatrix}.$$

The evaluation of this determinant by the use of minors would involve the expansion of four determinants of the third order. However, if we apply Property (5), we may select various values for M so that a row or a column of the derived determinant may have as many as three zeros as elements.

Thus, if we replace column 1 (numbered from left to right) by elements obtained by adding -2 times the elements of column 2 to the respective elements of column 1, we have

$$\begin{vmatrix} -4 & 3 & -4 & 6 \\ -7 & 5 & -1 & 7 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 4 & 2 \end{vmatrix}.$$

If we then replace column 4 by elements obtained by adding -3 times the elements of column 2 to the respective elements of column 4, we obtain

$$\begin{vmatrix} -4 & 3 & -4 & -3 \\ -7 & 5 & -1 & -8 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 2 \end{vmatrix}.$$

Developing this determinant by the use of minors with respect to the third row, we have in this case only the one determinant

$$-1\begin{vmatrix} -4 & -4 & -3 \\ -7 & -1 & -8 \\ -1 & 4 & 2 \end{vmatrix} = -1(8 + 84 - 32 + 3 - 56 - 128)$$

$$= 121.$$

As an alternate method, starting with the original determinant, we can obtain a new determinant containing three zeros in column 1 by adding 2 times the elements in row 4 (numbered from top to bottom) to the elements in row 1; 3 times the elements in row 4 to the elements in row 2; and 2 times the elements in row 4 to the elements in row 3. This gives

$$\begin{vmatrix} 0 & 3 & 4 & 10 \\ 0 & 5 & 11 & 13 \\ 0 & 1 & 8 & 7 \\ -1 & 0 & 4 & 2 \end{vmatrix}.$$

Writing the minors of this determinant with respect to column 1, we

have in this case only the one determinant

$$1\begin{vmatrix} 3 & 4 & 10 \\ 5 & 11 & 13 \\ 1 & 8 & 7 \end{vmatrix} = 231 + 400 + 52 - 110 - 312 - 140,$$
$$= 121.$$

This method serves as a check upon the solution obtained by the previous method.

EXERCISES 33

1. Show that $\begin{vmatrix} 4 & 3 & 2 & 1 \\ 8 & 8 & 7 & 2 \\ 16 & 2 & 8 & 4 \\ 12 & 6 & 3 & 3 \end{vmatrix} = 0.$

HINT: Use Properties 3 and 4.

2. Show that $\begin{vmatrix} 2 & 6 & 10 & 2 \\ 3 & 6 & 15 & 3 \\ 8 & 7 & 7 & 1 \\ 9 & 1 & 3 & 2 \end{vmatrix} = 6\begin{vmatrix} 1 & 3 & 5 & 1 \\ 1 & 2 & 5 & 1 \\ 8 & 7 & 7 & 1 \\ 9 & 1 & 3 & 2 \end{vmatrix}.$

3. Show that $\begin{vmatrix} 1 & 1 & 1 & 4 \\ 0 & 2 & -1 & 0 \\ 2 & -3 & 0 & 2 \\ 2 & 4 & 2 & 11 \end{vmatrix} = -15.$

4. By use of Property (5), obtain an equivalent determinant in which some of the elements of a row, or column, are zero. Evaluate the resulting determinant by the method of minors:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ -3 & 4 & -2 & -1 \\ 6 & 3 & 9 & -12 \\ 5 & 2 & -3 & -10 \end{vmatrix}$$

5. Evaluate the following determinant:

$$\begin{vmatrix} 5 & 2 & -3 & 4 \\ 2 & -3 & 5 & 5 \\ 4 & 2 & -7 & -3 \\ -2 & 8 & 0 & 2 \end{vmatrix}$$

6. Evaluate the following determinant:

$$\begin{vmatrix} 3 & 2 & 6 & 0 \\ -3 & 1 & -3 & 2 \\ 2 & 7 & 4 & 8 \\ 5 & 10 & -15 & 25 \end{vmatrix}$$

7. Evaluate the following determinant:

$$\begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & 2 & 3 & 4 \\ \frac{5}{6} & -3 & -2 & -1 \\ 2 & -\frac{1}{2} & 0 & -\frac{1}{3} \end{vmatrix}$$

8. Expand the determinant that follows and solve the resulting equation for x; check your result:

$$\begin{vmatrix} 2 & x & 3 & 1 \\ 1 & x & 2 & 3 \\ 2 & 4 & 7 & 6 \\ 3 & 0 & 1 & -1 \end{vmatrix} = 0$$

9. Expand the determinant and solve the resulting equation for x; check your solution.

$$\begin{vmatrix} x & -1 & 1 & 0 \\ 3 & -2x & 2 & 2 \\ 4 & 0 & -1 & 3 \\ 0 & x-1 & 2 & -1 \end{vmatrix} = 6$$

Solve each of the following systems of four equations by the use of determinants.

10.
$$\begin{aligned} 2x - 3y - 5z + w &= 17 \\ 3x - 4y + 2z - 2w &= 8 \\ x + y - 2z + 3w &= 15 \\ -5x + 6w &= -40 \end{aligned}$$

11.
$$\begin{aligned} x + y - 2z &= -6 \\ y - 3z + w &= -17 \\ 2x - 5w &= 20 \\ 3x - 2y &= 21 \end{aligned}$$

12.
$$\begin{aligned} 13x - 7y - 2z &= 15 \\ 2x + 5z - 2w &= 8 \\ 6x - 4z + 5w &= 6 \\ 3y - 7w &= -5 \end{aligned}$$

9 Exponents and Radicals

46. THE FUNDAMENTAL LAWS OF POSITIVE INTEGRAL EXPONENTS

When n is a positive integer, a^n is defined as the product of n factors, each equal to a. Thus, a^3 is an abbreviation for the product $a \cdot a \cdot a$. By definition, $a^1 = a$. The number a is called the *base* and the number n the *exponent.*

Since $a^2 = a \cdot a$ and $a^3 = a \cdot a \cdot a$, it follows that $a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5$. In general, $a^m \cdot a^n + a^{m+n}$, if the exponents are positive and integral. In a somewhat similar manner, by returning to the definition of a positive integral exponent, it is easy to demonstrate all the following fundamental laws involving the use of positive integral exponents.

I $\qquad a^m \cdot a^n = a^{m+n}.$

II $\qquad a^m \div a^n = a^{m-n}$ (when n is less than m)

$$= \frac{1}{a^{n-m}} \quad \text{(when } n \text{ is greater than } m\text{).}$$

III $\qquad (a^m)^n = a^{mn}.$

IV $\qquad (ab)^m = a^m b^m.$

V $\qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0).$

47. ZERO EXPONENTS

In order that we may have a meaning for a^0, we shall require that Law I of Section 46 shall hold for all exponents. Consequently,

$$a^n a^0 = a^{n+0} = a^n.$$

If $a \neq 0$, we may solve the previous equation for a^0 and obtain

$$a^0 = \frac{a^n}{a^n} = 1.$$

It appears, therefore, that a^0 must be defined as 1.

48. NEGATIVE EXPONENTS

Again requiring that Law I of Section 46 shall hold, whatever a^{-n} may mean, it follows that

$$a^n a^{-n} = a^0 = 1.$$

Hence, if $a \neq 0$,

$$a^{-n} = \frac{1}{a^n}.$$

49. FRACTIONAL EXPONENTS

The meaning to be associated with a fractional exponent may be determined by means similar to those employed in finding meanings for zero or negative exponents. Assuming that Law I of Section 46 holds, we would have, for example,

$$a^{1/2}a^{1/2} = a^{1/2+1/2} = a^1 = a,$$

and

$$a^{1/3}a^{1/3}a^{1/3} = a^{1/3+1/3+1/3} = a^1 = a.$$

From this it is seen that $a^{1/2}$ may be regarded as one of two equal factors of a, and $a^{1/3}$ is one of three equal factors of a. Thus, $a^{1/2}$ is defined as a square root of a, and $a^{1/3}$ is a cube root of a.

If a is positive, we know that it has two square roots, one positive and one negative. Therefore, $a^{1/2}$ is still ambiguous in meaning; hence, we shall limit $a^{1/2}$ to mean the positive square root of a and write $a^{1/2} = \sqrt{a}$, where the plus sign before the radical denoting square root is always implied. If we wish to express the negative square root of a, where a is positive, we must write $-a^{1/2}$ or $-\sqrt{a}$.

If a is negative, its square roots are not real numbers, since the product of two equal real numbers is always positive. We shall consider this case later in the text.

If a is positive, it has one real cube root and that is positive; hence, we write

$$a^{1/3} = \sqrt[3]{a}.$$

If a is negative, it has one real cube root and that is negative; but we still write

$$a^{1/3} = \sqrt[3]{a}.$$

Similarly, $a^{1/n}$ where a is positive and n is an even positive integer means the real positive nth root of a. We write $a^{1/n} = \sqrt[n]{a}$, where the plus sign before the radical is implied. If we wish to express the real negative nth root of a, we write $-\sqrt[n]{a}$. If in $a^{1/n}$, n is an odd positive integer, then $a^{1/n} = \sqrt[n]{a}$ is positive if a is positive, but $a^{1/n} = \sqrt[n]{a}$ is negative if a is negative.

In general, the real positive nth root of a, where a is positive and n is even, is called the *principal* n*th root of* a.

Thus, $(3^{1/2})^4 = 3^2 = 9$; but, on the other hand, $(3^4)^{1/2} = 9$ only if we restrict the left member to being the principal square root of 3^4.

The exponential form $a^{m/n}$, where m and n are positive integers, is defined to mean $\sqrt[n]{a^m}$. Moreover, under the restrictions implied in this treatment, $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$. Thus,

$$(-8)^{2/3} = (\sqrt[3]{-8})^2 = 4,$$

and $$(-8)^{2/3} = \sqrt[3]{(-8)^2} - \sqrt[3]{64} = 4.$$

Also, $$(-8)^{5/3} = (\sqrt[3]{-8})^5 = -32,$$

and $$(-8)^{5/3} = \sqrt[3]{(-8)^5} = -32.$$

Meanings have thus been associated with zero, negative, and fractional exponents by assuming that the first of the five fundamental laws for positive integral exponents holds. If only positive bases are considered, it is now possible to prove that all powers with rational exponents follow the remaining four laws. It is even possible to use irrational exponents in a way consistent with these laws. However, because of the difficulty involved in some of these demonstrations, we shall assume without proof the following fundamental principle.

Positive bases with zero, negative, and fractional exponents obey the five fundamental laws for positive bases with positive integral exponents.

The student should be careful to note that the meanings for zero and fractional exponents are essentially definitions, since they are based upon the requirement that Law I of Section 46 holds.

The following illustrations demonstrate the application of the laws of exponents to numerical examples.

Illustration 1: Simplify $4^{1/2} \cdot 16^{-3/4} \cdot 64^{-1/3}$.

After recalling the meaning of a negative exponent, we have

$$4^{1/2} \cdot \frac{1}{16^{3/4}} \cdot \frac{1}{64^{1/3}}.$$

Since, however, $4^{1/2} = \sqrt{4} = 2$, $16^{3/4} = (\sqrt[4]{16})^3 = 8$, and $64^{1/3} = \sqrt[3]{64} = 4$, we may further simplify the product to

$$2 \cdot \tfrac{1}{8} \cdot \tfrac{1}{4} = \tfrac{1}{16}.$$

Illustration 2: Write the following algebraic expression without negative exponents and in a simple form:

$$\frac{3a^{-1}b^2c^{-5}}{7a^{-2}b^{-1}c^3}.$$

We have

$$\frac{3\cdot\dfrac{1}{a}\cdot b^2\cdot\dfrac{1}{c^5}}{7\cdot\dfrac{1}{a^2}\cdot\dfrac{1}{b}\cdot c^3}=\frac{\dfrac{3b^2}{ac^5}}{\dfrac{7c^3}{a^2b}}=\frac{3ab^3}{7c^8}.$$

Illustration 3: Write the following expression without negative exponents and in a simple form:

$$5x^{-2}-(3x)^{-1}.$$

We have

$$\frac{5}{x^2}-\frac{1}{3x}\quad\text{or}\quad\frac{15-x}{3x^2}$$

EXERCISES 34

Write each of the following expressions without the use of fractional or negative exponents, and reduce the result to a simple form:

1. $\dfrac{3}{2^{-1/2}}$
2. $x^{2/3}y^{-1/3}$
3. $8^{-1/3}$
4. $(-\frac{4}{5})^{-4/5}$
5. $(-32)^{3/5}$
6. $(\frac{125}{27})^{-2/3}$
7. $5^{1/4}x^{2/3}y^{7/3}z^{-3/4}$
8. $(0.36)^{-1/2}$
9. $(\frac{1}{27})^{2/3}$
10. $\dfrac{2x^{-3}y^2z^{-4}}{5x^4y^{-5}z^{-5}}$
11. $7x^{-4}$
12. $a^2\cdot b^3\cdot c^3\cdot b^{-1}$

Perform the indicated operations and express your results in a simple form without the use of negative exponents.

13. $\dfrac{2+3}{2^{-1}+5^{-1}}$
14. $\dfrac{2-2^{-2}}{2+2^2}$
15. $5x^{-2}-10x^{-3}$
16. $\dfrac{1}{x^{-3}+y^{-3}}$
17. $a^{-1}+2a^{-2}+3a^{-3}$
18. $\dfrac{1}{a^{-1}-2a^{-2}-3a^{-3}}$
19. $(64a^{-9}b^6c^{12})^{-2/3}$
20. $\dfrac{a^{-1}b^2x^{-2}-2a^{-1}b^2y^3}{a^{-5}x^{-3}-2a^{-5}y^3}$
21. $\dfrac{1}{2}-\dfrac{3^{-1}}{2^{-1}}$
22. $a^{1/2}\cdot b^{-2}\cdot a\cdot b^{5/2}$
23. $(a^{-2/3}\cdot b^{-3}a\cdot b^{1/2})^{-2}$
24. $\left(\dfrac{ab^{-3}}{a^{-2}b^2}\right)^{-3}$
25. $(9a^{2/3}\cdot b^{-5/6})^{-3/2}$
26. $(-27a^{-3/2}\cdot b\cdot c^{-9/4})^{-1/3}$
27. $(a^0-b^{-1})^{-2}$
28. $(a^{-1/2}-b^{-1/2})(a^{-1/2}+b^{-1/2})$
29. $\dfrac{1}{a^{-1}+b^{-2}}\div(a-b^2)$
30. $\dfrac{4a^{-2}-9b^{-2}}{3a+2b}$

Multiply the following:

31. $a^{1/2} + a^{1/4} + b^{1/2}$ by $a^{1/4} - b^{1/2}$

32. $x^5 + y^5$ by $x^{5/2} - y^{5/2}$

33. $m^{8/3} + m^{4/3} + n^{4/3} + n^{8/3}$ by $m^{4/3} - n^{4/3}$

50. RADICALS

We have assumed that the student is familiar with the fact that the symbol $\sqrt[n]{\ }$ (n, a positive integer) is called a *radical sign*. A bar is usually written over the number affected by the radical sign; this bar is a *vinculum*, a sign of aggregation.

We reserve the name of radical for $\sqrt[n]{a}$ (when n is a positive integer and a is a real number) if $\sqrt[n]{a}$ cannot be reduced to a rational real number. Thus, from our definition, $\sqrt{4}$ and $\sqrt[3]{\frac{8}{27}}$ are not radicals.

Special attention is called to the fact that since the radical sign denoting square root calls for the positive square root of a positive number,

$$\begin{aligned}\sqrt{(x-1)^2} &= x - 1, \quad \text{if} \quad x > 1,\\ &= 1 - x, \quad \text{if} \quad x < 1;\end{aligned}$$

and

$$\begin{aligned}\sqrt{\frac{(x-2)^2}{x^4}} &= \frac{x-2}{x^2}, \quad \text{if} \quad x > 2,\\ &= \frac{2-x}{x^2}, \quad \text{if} \quad x < 2.\end{aligned}$$

In such an expression as $b\sqrt[n]{a}$, where $\sqrt[n]{a}$ is a radical and b is any constant, a is called the *radicand;* n the *index;* and b the *coefficient of the radical.*

A radical is said to be in its simplest form when

(1) The radicand is an integer;

(2) The radicand contains no factors raised to powers equal to, or greater than, the index of the radical;

(3) The radicand is not a power whose index has a factor in common with the index of the radical.

Thus, the radicals $\sqrt{\frac{2}{3}}$, $\sqrt[3]{a^4}$, and $\sqrt[6]{a^4}$ are not in their simplest form because they do not meet the requirements (1), (2), and (3), respectively.

51. SIMPLIFICATION OF RADICALS

Certain radicals can be simplified by means of one or more of the following reductions:

(1) *Reduction of a Fractional Radicand to the Integral Form.* **This** reduction can always be performed as follows: If the radicand is not a single fraction in its lowest terms, put it into that form. Then, if the radical is of index p, make the denominator of the radicand a perfect pth power by multiplying the numerator and the denominator of the fraction by a properly chosen number. The original radical is equal to the pth

root of the resulting numerator divided by the pth root of the resulting denominator, which is rational. Thus, for example,

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}.$$

(2) *The Removal of Factors from the Radicand.* This reduction can be made only when the radicand contains factors to powers equal to, or greater than, the index of the radical. The following examples illustrate the usual procedure.

EXAMPLES. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$

$$\sqrt[3]{108} = \sqrt[3]{27 \cdot 4} = \sqrt[3]{27} \cdot \sqrt[3]{4} = 3\sqrt[3]{4}.$$

$$\sqrt[3]{(a-b)^4} = \sqrt[3]{(a-b)^3} \cdot \sqrt[3]{a-b} = (a-b)\sqrt[3]{(a-b)}.$$

(3) *The Lowering of the Index of the Radical.* When the radicand is a power whose index has a factor in common with the index of the radical, the radical is equal to another radical of lower index.

EXAMPLE: $\sqrt[6]{a^4} = a^{4/6} = a^{2/3} = \sqrt[3]{a^2}.$

In some problems it is desirable to introduce factors into the radicand or to increase the index of the radical.

EXAMPLES: $4\sqrt[3]{2} = \sqrt[3]{64} \cdot \sqrt[3]{2} = \sqrt[3]{128}.$

$$\sqrt{5} = 5^{1/2} = 5^{3/6} = \sqrt[6]{125}.$$

52. ADDITION AND SUBTRACTION OF RADICALS

Definition: Two or more radicals are said to be similar if, when simplified, according to the meaning of this term as given in Section 50, they have the same index and the same radicand.

For example, $2\sqrt[3]{5}$ and $3\sqrt[3]{5}$ are similar, as are also $\sqrt{a^3b}$ and $3\sqrt{ab}$.

An expression involving two or more radicals can sometimes be simplified by first simplifying each radical and then combining the similar radicals in the way illustrated in the following examples:

EXAMPLES:

$$2\sqrt{98} - 3\sqrt{50} + \sqrt{72} = 14\sqrt{2} - 15\sqrt{2} + 6\sqrt{2} = 5\sqrt{2}.$$

$$\begin{aligned} 2\sqrt{98} - 50\sqrt{3} + \sqrt{32} - \sqrt{108} &= 14\sqrt{2} - 50\sqrt{3} + 4\sqrt{2} - 6\sqrt{3} \\ &= 18\sqrt{2} - 56\sqrt{3}. \end{aligned}$$

It should be observed that the sum or difference of two dissimilar radicals cannot be expressed as a single radical.

53. MULTIPLICATION OF RADICALS

The product of two radicals with a common index is a radical with the same index whose coefficient and radicand are equal, respectively, to the products of the coefficients and of the radicands of the factors. This follows directly from Law IV of Section 46. Thus,

$$a\sqrt[n]{b} \cdot c\sqrt[n]{d} = ac\sqrt[n]{bd}.$$

If n is even, it is assumed here that b and d are both positive. *This law does not hold when both* b *and* d *are negative.*

If the radicals do not have the same index, they should first be changed to equal radicals with a common index. Thus,

$$\sqrt[3]{5} \cdot \sqrt{6} = \sqrt[6]{5^2} \cdot \sqrt[6]{6^3} = \sqrt[6]{25} \cdot \sqrt[6]{216} = \sqrt[6]{5400}.$$

The product of such expressions as $\sqrt{a} + \sqrt{b} - \sqrt{c}$ and $\sqrt{a} - \sqrt{b} + \sqrt{c}$ can be found by applying the usual rules for the multiplication of polynomials in connection with the principles just stated in case a, b, and c are positive. Thus,

$$\begin{aligned}
&(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c}) \\
&= [\sqrt{a} + (\sqrt{b} - \sqrt{c})][\sqrt{a} - (\sqrt{b} - \sqrt{c})] \\
&= a - (\sqrt{b} - \sqrt{c})^2 = a - (b - 2\sqrt{bc} + c) \\
&= a - b - c + 2\sqrt{bc}.
\end{aligned}$$

It is best, in general, to simplify all the radicals in a problem before attempting to perform any operations upon them.

54. RATIONALIZING THE DENOMINATOR

Certain fractions whose denominators are irrational can be changed into equivalent fractions with rational denominators.

EXAMPLES: $$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

$$\frac{a}{\sqrt{a} + \sqrt{b}} = \frac{a}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{a(\sqrt{a} - \sqrt{b})}{a - b}.$$

Such a change can always be made when the denominator of the fraction is a single radical or is the sum of two terms, one of which is a square root and the other either a square root or rational.

In computing the numerical value of a fraction with an irrational denominator, it is best to rationalize the denominator whenever it is possible to do so.

EXERCISES 35

Simplify the following:

1. $\sqrt{18}$

2. $\sqrt{45}$

3. $\sqrt[3]{\frac{2}{3}}$

4. $\sqrt[4]{\frac{5}{6}}$

5. $\sqrt[3]{\frac{16}{9}}$

6. $\sqrt{\frac{1}{9} + 16}$

7. $\sqrt{\frac{1}{6}}$

8. $\dfrac{2}{\sqrt{3}}$

9. $\sqrt[4]{64}$

10. $\sqrt[3]{8(a+b)^6}$

11. $\sqrt{2x^2 - 12x + 18}$

12. $\sqrt{\dfrac{x-1}{x^3}}$

13. $\sqrt{(a^2 - b^2)^3}$

14. Which is the greater, $\sqrt[3]{9}$ or $\sqrt{5}$?

Solution:

$$\sqrt[3]{9} = 9^{2/6} = \sqrt[6]{81}.$$

$$\sqrt{5} = 5^{3/6} = \sqrt[6]{125}.$$

But, $$\sqrt[6]{125} > \sqrt[6]{81};$$

hence, $$\sqrt{5} > \sqrt[3]{9}.$$

15. Which is the greater, $\sqrt{10}$ or $\sqrt[3]{28}$? $\sqrt{3}$ or $\sqrt[3]{6}$? $\sqrt{19}$ or $\sqrt[3]{65}$?

Simplify the following:

16. $\sqrt{2} + \sqrt{8} + 3\sqrt{18} - \sqrt{50} + \sqrt[4]{4}$

17. $\sqrt{\frac{1}{7}} + \sqrt{\frac{9}{7}} - 3\sqrt{\frac{81}{7}} - \sqrt{\frac{16}{7}}$

18. $b\sqrt{a} + c\sqrt{a} - d\sqrt{a^3}$

19. $3\sqrt{a-b} + \sqrt{9a - 9b} - \sqrt{c^2a - c^2b} - \dfrac{\sqrt{a-b}}{2}$

20. $3\sqrt[3]{16} - 2\sqrt[3]{\frac{1}{4}} + \sqrt[3]{250} - \sqrt[6]{4}$

21. $\sqrt{10} \cdot \sqrt{2}$

22. $\sqrt{2} \cdot \sqrt[3]{2}$

23. $\sqrt{5}(\sqrt{2} - \sqrt{5})$

24. $(\sqrt[3]{2} - 2\sqrt[3]{3} - \sqrt[3]{9})\sqrt[3]{3}$

25. $(\sqrt{3} - \sqrt{2})^2$

26. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

27. $(3\sqrt{3} - 2)(\sqrt{3} + \sqrt{5})$

28. $(\sqrt{x-y} - 3\sqrt{x+y})(\sqrt{x-y} + 2\sqrt{x+y})$

29. $(\sqrt[3]{2} - \sqrt[3]{5})^2$

30. Is $1 - \sqrt{3}$ a root of the equation $x^2 - x - 1 = 0$?

31. Is $\dfrac{-3 + \sqrt{5}}{2}$ a root of the equation $x^2 + 3x + 1 = 0$?

Reduce each of the following fractions to an equivalent fraction with a rational denominator:

32. $\dfrac{x}{\sqrt{a-x} - \sqrt{a+x}}$

33. $\dfrac{4}{\sqrt{5} - 2}$

34. $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$

35. $\dfrac{7 - \sqrt{5}}{7 + \sqrt{5}}$

36. $\dfrac{\sqrt[4]{9} - \sqrt{\frac{8}{5}}}{\sqrt{5}}$

37. $\dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} + \sqrt{a - x}}$

38. $\dfrac{x}{x + \sqrt{x^2 - 1}}$

39. Change $\dfrac{bx - b\sqrt{x^2 + a^2}}{a}$ to an equivalent fraction free of radicals in the numerator.

40. Find the value of $\dfrac{5 - 4\sqrt{5}}{\sqrt{5} + 2}$ with a precision of three decimal places.

41. Find the value of $\dfrac{2\sqrt{3} - 5}{\sqrt{3} - \sqrt{2}}$ with a precision of three decimal places.

55. COMPLEX NUMBERS

So far in our discussion of radicals we have confined ourselves to the real-number system. Since all real numbers have positive squares, it is evident that the conditional equation $x^2 = -3$ does not have solutions in terms of real numbers. In other words, if we confine ourselves to real numbers, the equation $x^2 = -3$ cannot be solved. However, if we define a new number symbolized by $\sqrt{-3}$ as a number whose square is -3, the equation has the two solutions $+\sqrt{-3}$ and $-\sqrt{-3}$. Similarly, we define $\sqrt{-a}$, where $a > 0$, as a number whose square is $-a$.

If $a > 0$, $\sqrt{-a}$ may be written $\sqrt{a}\sqrt{-1}$. Thus,

$$\sqrt{-32} = \sqrt{32}\sqrt{-1} = 4\sqrt{2}\sqrt{-1}.$$

The positive square root of -1, that is, $+\sqrt{-1}$, is usually denoted by the symbol i. Thus,

$$2 + \sqrt{-4} = 2 + 2i,$$

and

$$a + \sqrt{-(a^2 + x^2)} = a + i\sqrt{a^2 + x^2},$$

where a and x are real numbers.

The square roots of negative numbers are called pure imaginary numbers. Thus, $\sqrt{-9}$, $\sqrt{-5}$, $\sqrt{a}$, where $a < 0$, are pure imaginary numbers.

A binomial $a + bi$, where a and b are real numbers, is called a *complex number*. If $b = 0$, the complex number is a real number. If $a = 0$ and $b \neq 0$, the complex number is a pure imaginary.

For the symbol $i = \sqrt{-1}$ we note the following fundamental relationships: $i^2 = -1$; $i^3 = i \cdot i^2 = -i$; $i^4 = i^2 \cdot i^2 = +1$.

With these relationships in mind, the operations on pure imaginary

numbers or complex numbers in general are performed like the operations on real numbers.

Thus, $\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$. We thus note that $\sqrt[n]{b} \cdot \sqrt[n]{d}$, where n is even and b and d are both negative, does not equal $\sqrt[n]{bd}$. It is desirable to introduce the symbol i as soon as possible in dealing with imaginary numbers; this facilitates the use of the algebraic operations in an acceptable manner.

Illustration 1: Add $3 + \sqrt{-4}$, $5 - \sqrt{-25}$, $1 + \sqrt{-64}$.

By employing the symbol i these numbers may be written $3 + 2i$, $5 - 5i$, and $1 + 8i$. Hence, the sum is $9 + 5i$.

Illustration 2: $\sqrt{-2} \cdot \sqrt{-5} = \sqrt{2}i \cdot \sqrt{5}i = \sqrt{10}i^2 = -\sqrt{10}$.

Illustration 3: Change $\dfrac{2 - \sqrt{-3}}{5 + \sqrt{-3}}$ to the form $a + bi$.

$$\frac{2 - \sqrt{-3}}{5 + \sqrt{-3}} = \frac{2 - \sqrt{3}i}{5 + \sqrt{3}i} = \frac{(2 - \sqrt{3}i)(5 - \sqrt{3}i)}{(5 + \sqrt{3}i)(5 - \sqrt{3}i)}$$
$$= \frac{10 - 7\sqrt{3}i + 3i^2}{25 - 3i^2} = \frac{7 - 7\sqrt{3}i}{28}$$
$$= \frac{1 - \sqrt{3}i}{4} = \frac{1}{4} - \frac{\sqrt{3}}{4}i.$$

EXERCISES 36

Perform the indicated operations and write each result in the form $a + bi$:

1. $(2 + \sqrt{-12}) + (3 - \sqrt{-32})$
2. $(5 - \sqrt{-27}) - (7 - \sqrt{-12})$
3. $(5 - \sqrt{-12} + 7\sqrt{-18})\sqrt{-3}$
4. $(5 - 3i)(2 + i)$
5. $(2 - \sqrt{-5})(7 - \sqrt{-15})$
6. $(a + \sqrt{-b})(a - \sqrt{-b})$, where a and b are positive real numbers
7. $\dfrac{6 + 10i - \sqrt{-3}}{\sqrt{-3}}$
8. Express $\dfrac{2 - i}{3 + 2i}$ in the form $a + bi$.
9. Express $\dfrac{a - \sqrt{-c}}{a + \sqrt{-b}}$ in the form $a + bi$.
10. Expand $(1 - \sqrt{-3})^8$ by the binomial theorem; simplify and express the result in the form $a + bi$.

10 Quadratic Functions and Equations

56. THE STANDARD FORM OF A QUADRATIC FUNCTION

Any expression of the form

$$ax^2 + bx + c, \quad a \neq 0,$$

where a, b, and c are constants and x is a variable, is called a *quadratic function in the standard form.*

Thus, the function

$$7 - 3x^2 + 5x + 7x^2$$

may be reduced to the standard form

$$4x^2 + 5x - 7,$$

where $$a = 4, \quad b = 5, \quad c = -7.$$

Similarly, the function

$$2qx - 3 + 5x^2 - px$$

may be written $$5x^2 + (2q - p)x - 3,$$

where $$a = 5, \quad b = 2q - p, \quad c = -3.$$

57. GRAPHICAL REPRESENTATION OF THE FUNCTION $y = ax^2 + bx + c$

We may obtain a value of the function $y = ax^2 + bx + c$ corresponding to any given value of the variable x. As in the case of linear functions, we may tabulate the corresponding pairs of values of x and y and locate the associated points relative to a set of axes in the plane.

Illustration 1: Given $y = 3x^2 - 2x + 5$. Let us assign to x the values $-3, -2, -1, 0, 1, 2, 3$ and tabulate corresponding values of variable and function and locate the points in the plane (note Figure 17).

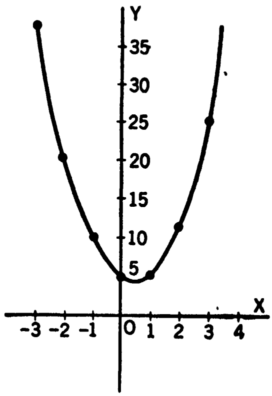


FIG. 17

$$y = 3x^2 - 2x + 5$$

x	y
-3	38
-2	21
-1	10
0	5
1	6
2	13
3	26

It is apparent that these points do not lie on a straight line. The curve of the quadratic function, on which these points lie, is called a *parabola*. In general, if we plot points sufficiently near each other, the smooth curve through these successive points is the required graph.

Illustration 2: Given $y = -3x^2 + 8x - 1$. We may tabulate corresponding values of x and y and locate the associated points (note Figure 18).

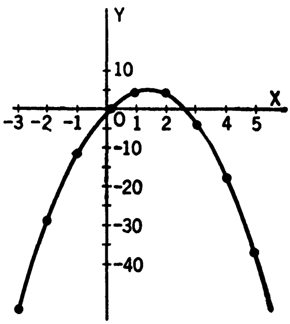


FIG. 18

$$y = -3x^2 + 8x - 1$$

x	y
-3	-52
-2	-29
-1	-12
0	-1
1	4
2	3
3	-4
4	-17
5	-36

The points corresponding to the pairs of numbers, the variable and its associated quadratic function $ax^2 + bx + c$, will always lie on a curve similar to either Figure 17 or Figure 18.

58. VERTEX OF A QUADRATIC FUNCTION

The function

$$y = 3x^2 - 2x + 5$$

may be written

$$y - 5 = 3(x^2 - \tfrac{2}{3}x).$$

The expression within parentheses may be made a perfect square by adding the square of one half the coefficient of x; this number is $\frac{1}{9}$. The addition of $\frac{1}{9}$ to the quantity within the parentheses is equivalent to the addition of $\frac{1}{3}$ to the right member, in view of the factor 3 that precedes the parentheses. Thus, it is also necessary to add $\frac{1}{3}$ to the left member, thereby giving

$$y - \tfrac{14}{3} = 3(x^2 - \tfrac{2}{3}x + \tfrac{1}{9}),$$

or

$$y - \tfrac{14}{3} = 3(x - \tfrac{1}{3})^2.$$

Since $3(x - \frac{1}{3})^2$ is positive or zero, it is seen that $y = \frac{14}{3}$ is the smallest possible value of y and that $y = \frac{14}{3}$ when $3(x - \frac{1}{3})^2 = 0$, that is, when $x = \frac{1}{3}$.

The point corresponding to $x = \frac{1}{3}$ and $y = \frac{14}{3}$ is called the *vertex of the parabola.*

59. MAXIMUM OR MINIMUM VALUES OF A QUADRATIC FUNCTION

The ordinate that corresponds to $x = \frac{1}{3}$, in considering the function of Section 58, is smaller than that of any neighboring points. The value of y corresponding to $x = \frac{1}{3}$ is defined, therefore, as a minimum value of the function, and the curve is said to have a minimum for $x = \frac{1}{3}$.

The function

$$y = -3x^2 + 8x - 1$$

may be written

$$y + 1 = -3(x^2 - \tfrac{8}{3}x).$$

If the quantity within the parentheses is increased by $\frac{16}{9}$, which may be accomplished by adding $-\frac{16}{3}$ to each member, we have

$$y - \tfrac{13}{3} = -3(x^2 - \tfrac{8}{3}x + \tfrac{16}{9}),$$

or

$$y - \tfrac{13}{3} = -3(x - \tfrac{4}{3})^2.$$

Since $-3(x - \frac{4}{3})^2$ is negative or zero, it is seen that $y = \frac{13}{3}$ is the largest value of y and that $y = \frac{13}{3}$ when $-3(x - \frac{4}{3})^2 = 0$, that is, when $x = \frac{4}{3}$.

The point corresponding to $x = \frac{4}{3}$ and $y = \frac{13}{3}$ is the vertex of the parabola. The ordinate which corresponds to $x = \frac{13}{3}$ is greater than that of any neighboring points. Consequently, the value of y corresponding to $x = \frac{13}{3}$ is defined as a maximum value of the function, and the curve is said to have a maximum for $x = \frac{4}{3}$.

For the standard quadratic form we have

$$y = ax^2 + bx + c, \quad a \neq 0,$$

and, hence,

$$y - c = a\left(x^2 + \frac{b}{a}x\right).$$

If we add $b^2/4a$ to both members of the equation, the right member becomes

$$a\left(x + \frac{b}{2a}\right)^2.$$

Thus,
$$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2.$$

If $a > 0$, it is seen that $y = c - \dfrac{b^2}{4a}$ is the smallest value of y, and that $y = c - \dfrac{b^2}{4a}$ when $x = -\dfrac{b}{2a}$.

The point corresponding to $x = -\dfrac{b}{2a}$ and $y = c - \dfrac{b^2}{4a}$ is the vertex of the parabola representing the quadratic function. Moreover, under the condition that $a > 0$, the curve of $y = ax^2 + bx + c$ is said to have a *minimum* at the vertex. In this case the entire curve lies above the line $y = c - \dfrac{b^2}{4a}$.

If $a < 0$, it is seen that $y = c - \dfrac{b^2}{4a}$ is the greatest value of y and that $y = c - \dfrac{b^2}{4a}$ when $x = -\dfrac{b}{2a}$. This time, when $a < 0$, the curve $y = ax^2 + bx + c$ is said to have a *maximum* at the vertex. In this case the entire curve lies below the line $y = c - \dfrac{b^2}{4a}$.

EXERCISES 37

1. Graph a few points of each of the following functions and connect the points by a smooth curve:

(a) $y = x^2 - 6x + 5$ (b) $y = -x^2 + 6x + 1$
(c) $y = 3x^2 + x + 2$

2. Draw the graph of each of the following equations and determine the coordinates of the vertex of each curve:

(a) $y = x^2 - 2x - 5$ (b) $y = -3x^2 - 2x + 2$
(c) $y = 5x^2 - 8x + 2$

3. Find the coordinates of the vertex and the maximum or minimum value of each of the following functions. Determine the coordinates of a few points for each curve and sketch the curve.

(a) $y = 3x^2 - 8x - 3$ (b) $y = 3x^2 - 8x$
(c) $y = -2x^2 + 3x - 5$ (d) $y = 7 - 3x - 2x^2$
(e) $y = 2x^2 - 3$ (f) $y = 2x^2 - 10x + 12$

4. Find the coordinates of the vertex and draw the curve of each of the following equations:

(a) $x = 12 - 5y^2$ (b) $x = 2y^2 - 3y - 5$
(c) $x = 3 - 3y - 5y^2$

5. Find the dimensions of the rectangle of largest area, whose perimeter is 60 ft.

HINT: Let x = number of feet in each of the two equal sides, and let A = area of rectangle, which is to be a maximum. Therefore, $60 - 2x$ is the sum of the lengths of the other two sides. So $\dfrac{60 - 2x}{2}$ or $30 - x$ is the length of each of the other sides. Hence, $A = x(30 - x)$.

We must now answer the question: For what value of x will A have the greatest value?

6. A rope of 60 ft is to be used to fence three sides of a rectangle of which the fourth side is a fence. Find the dimensions of the largest rectangle.

HINT: The area A is to be a maximum. Let x = number of feet in each of the equal sides formed by the rope. Therefore, $60 - 2x$ equals the length of the third side. So $A = x(60 - 2x)$.

7. What is the least value of the function $y = 2x^2 - 7x + 3$?

8. Divide a into two parts such that their product is a maximum.

9. Find the number that exceeds its square by the greatest possible quantity.

60. QUADRATIC EQUATIONS

An equation of the form

$$ax^2 + bx + c = 0, \quad a \neq 0 \quad \text{and} \quad a, b, c \text{ constants}$$

is called a *quadratic equation in one unknown*. Many important problems may be solved through the use of quadratic equations in one unknown.

Illustration 1: A and B start on a journey of 36 miles. A travels 2 mph faster than B and arrives 3 hr before him. Find the rate of each.

An equation based upon an equality in terms of time may be derived as follows:

Let $\quad x$ = number of miles per hour traveled by A.

Then, $\quad x - 2$ = number of miles per hour traveled by B,

$\dfrac{36}{x}$ = number of hours traveled by A,

$\dfrac{36}{x - 2}$ = number of hours traveled by B.

But since A's time is 3 hr less than B's, we have

$$\frac{36}{x} + 3 = \frac{36}{x - 2},$$

which may be simplified to

$$36(x - 2) + 3x(x - 2) = 36x,$$

or

$$x^2 - 2x - 24 = 0.$$

This equation is of the form $ax^2 + bx + c = 0$.

Illustration 2: A cistern can be filled by two pipes in 36 min. If the smaller pipe takes 15 min more than the larger pipe to fill the cistern, in what time will it be filled by the larger pipe?

An equation may be derived as follows:

Let x = time in minutes required for the big pipe to fill the cistern. Then, $x + 15$ = time in minutes required for the smaller pipe to fill the cistern.

$$\frac{1}{x} = \text{amount of cistern filled in 1 min by large pipe,}$$

$$\frac{1}{x+15} = \text{amount of cistern filled in 1 min by smaller pipe.}$$

But, $$\frac{1}{36} = \text{amount of cistern filled in 1 min by both pipes.}$$

Therefore, $$\frac{1}{x} + \frac{1}{x+15} = \frac{1}{36},$$

or $$x^2 - 57x - 540 = 0.$$

This equation is also of the form $ax^2 + bx + c = 0$.

We shall now consider how to solve such equations.

61. SOLUTION OF QUADRATIC EQUATIONS BY FACTORING

Quadratic equations may always be solved either by a method known as *completing the square* or by the formula which is derived by completing the square of

$$ax^2 + bx + c = 0.$$

However, the nonzero members of certain special quadratic equations are readily factorable, and hence their solution may be obtained in a simple manner by the solution of two linear equations. Such a special case is considered in the illustration that follows.

Illustration: Solve the equation $x^2 - 7x + 12 = 0$.

The function $x^2 - 7x + 12$ is factorable into $(x - 3)(x - 4)$. Hence,

$$(x - 3)(x - 4) = 0.$$

This product is zero if and only if one of the factors is zero. But the roots thus obtained are values of x which cause $(x - 3)(x - 4)$ or $x^2 - 7x + 12$ to be equal to zero. Hence, $x - 3 = 0$ gives the root $x = 3$, and $x - 4 = 0$ gives the root $x = 4$.

We may check these roots in the given equation. Thus,

$$3^2 - 7(3) + 12 = 9 - 21 + 12 = 0,$$

and $$4^2 - 7(4) + 12 = 16 - 28 + 12 = 0.$$

62. GENERAL SOLUTION OF QUADRATIC EQUATIONS

The general quadratic equation may be solved as follows:

Given $$ax^2 + bx + c = 0. \tag{1}$$

Hence, $$x^2 + \frac{b}{a}x = -\frac{c}{a}, \tag{2}$$

and $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}, \tag{3}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{4}$$

Consequently, $$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}, \tag{5}$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{6}$$

The last form is equivalent to the two solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation may now be solved either by "completing the square," as illustrated by the derivation just completed, or by the use of Formula (6).

Illustration 1: Solve the equation $3x^2 + 4x - 7 = 0$.

Solution by Factoring: The equation is of the simple type which may be solved by factoring, since $3x^2 + 4x - 7$ is factorable into $(x - 1)(3x + 7)$. Hence, we have

$$(x - 1)(3x + 7) = 0,$$

and $x = 1$, $x = -\frac{7}{3}$ are the required roots.

Solution by "Completing the Square": The equation

$$3x^2 + 4x - 7 = 0 \tag{1}$$

may be written $$x^2 + \tfrac{4}{3}x = \tfrac{7}{3}, \tag{2}$$

or $$x^2 + \tfrac{4}{3}x + \tfrac{4}{9} = \tfrac{4}{9} + \tfrac{7}{3}. \tag{3}$$

Consequently, $$(x + \tfrac{2}{3})^2 = \tfrac{25}{9}, \tag{4}$$

or $$x + \tfrac{2}{3} = \pm\tfrac{5}{3}. \tag{5}$$

So, $$x = -\tfrac{2}{3} + \tfrac{5}{3} = 1, \tag{6}$$

and $$x = -\tfrac{2}{3} - \tfrac{5}{3} = -\tfrac{7}{3}.$$

Solution by Formula: If we compare the coefficients of $3x^2+4x-7=0$ with those of $ax^2 + bx + c = 0$, we see that $a = 3$, $b = 4$, $c = -7$. Hence, the roots by the direct application of the formula are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-7)}}{(2)(3)},$$

or
$$x = \frac{-4 \pm \sqrt{100}}{6}.$$

Hence, $\quad x = 1, \quad x = -\frac{7}{3}.$

Illustration 2: Solve the equation $x^2 - x - 3 = 0$. This equation is not readily factorable. Solving by completing the square, we have

$$x^2 - x - 3 = 0, \tag{1}$$

$$x^2 - x = 3, \tag{2}$$

$$x^2 - x + \tfrac{1}{4} = \tfrac{1}{4} + 3, \tag{3}$$

$$(x - \tfrac{1}{2})^2 = \tfrac{13}{4}, \tag{4}$$

$$(x - \tfrac{1}{2}) = \pm\sqrt{\tfrac{13}{4}} = \pm\frac{\sqrt{13}}{2}. \tag{5}$$

Therefore, $\quad x = \dfrac{1 + \sqrt{13}}{2} \quad$ and $\quad x = \dfrac{1 - \sqrt{13}}{2}.$

These solutions, correct to the nearest ten-thousandth, are

$$x = 2.3028 \qquad \text{and} \qquad x = -1.3028.$$

EXERCISES 38

Solve the following equations by factoring the left members.

1. $x^2 - 7x + 12 = 0$
2. $3x^2 - 7x + 2 = 0$
3. $8x^2 - 6x + 1 = 0$
4. $21x^2 - 41x + 10 = 0$
5. $21x^2 + 29x - 10 = 0$
6. $24x^2 + 33x - 30 = 0$
7. $6x^2 + 17x + 5 = 0$
8. $6x^2 + 11x - 35 = 0$
9. $8x^2 + 14x - 15 = 0$
10. $6x^2 - x - 15 = 0$
11. $2x^2 - 5ax + 2a^2 = 0$
12. $a^2x^2 + (c - b)ax - bc = 0$
13. $2p^2x^2 - 5apx + 2a^2 = 0$
14. $c^2x^2 - 2cdx + d^2 = 0$
15. $a^2x^2 - 4abx + 4b^2 = 0$
16. Solve the preceding 15 exercises by completing the square.

Solve the following equations by completing the square:

17. $x^2 - 2x - 1 = 0$
18. $3x^2 - 5x - 1 = 0$
19. $4x^2 - 7x + 1 = 0$
20. $5x^2 + 7x + 1 = 0$
21. $6x^2 - 7x + 1 = 0$
22. $3x^2 - 7x - 2 = 0$

23. $8x^2 - 10x + 1 = 0$

24. $5x^2 - 11x + 5 = 0$

25. $p^2x^2 - 2px - 5 = 0$

26. $px^2 + qx + r = 0$

27. $x^2 + px + q = 0$

28. Solve Exercises 17 to 24, inclusive, by use of the quadratic formula.

Solve each of the following equations:

29. $(2x - 3)^2 - (x + 2)^2 + 7 = 0$

30. $\dfrac{x}{x + 2} - \dfrac{x - 1}{x - 2} = 5$

31. $\dfrac{1}{x} - \dfrac{3}{x - 2} + \dfrac{1}{4}$

32. $\dfrac{2x(x - 1)}{5} - \dfrac{x(x + 4)}{3} - \dfrac{x^2 - 1}{2} = 0$

33. $\dfrac{1}{\frac{x}{2} - 1} + \dfrac{2}{\frac{x}{3} - 2} + \dfrac{3}{\frac{x}{4} - 3} = 0$

34. $\dfrac{6}{x} - 2 = \dfrac{5}{x - 3}$

35. $\dfrac{1}{2(x - 1)} - \dfrac{2}{(x - 1)^2} - \dfrac{3}{5} = 0$

36. $0.02(x - 1) - 0.05(x - 2)x + 0.06(x - 3)(x - 2) = 0$

37. $\dfrac{x}{x - 5} - \dfrac{2x - 3}{x} = \dfrac{9}{2}$

38. $\dfrac{x^2 - 3}{x} - \dfrac{7x - 5}{2} = 6(5 - 2x)$

39. $\dfrac{2}{2 - x} - \dfrac{x - 5}{2 + x} = \dfrac{16\frac{3}{4}}{4 - x^2}$

40. $\dfrac{5}{2x^2 - 7x + 6} + \dfrac{3(1 - x)}{9 - 9x + 2x^2} = \dfrac{7}{x^2 - 5x + 6}$

41. $\frac{3}{4}(x - \frac{1}{2}) - \frac{2}{3} \cdot \dfrac{1}{x + \frac{2}{3}} = 7\frac{7}{8}x$

63. IRRATIONAL EQUATIONS

Such equations as

$$\sqrt{x} + \sqrt{x + 6} = 3, \qquad \sqrt{1 - x} = 2 - x, \qquad \text{and} \qquad 27x^{3/2} - 4 = 26x^{3/4}$$

are frequently called *irrational equations.* An equation of this type may usually be solved by the method employed in the following illustrations.

Illustration 1: Solve the equation

$$\sqrt{x} + \sqrt{x + 6} = 3.$$

This may be rewritten in the form

$$\sqrt{x + 6} = 3 - \sqrt{x},$$

the radical $\sqrt{x}$ being subtracted from each member of the given equation in order to have one radical upon each side. Let us now square each member, thereby obtaining

$$x + 6 = 9 - 6\sqrt{x} + x,$$

or
$$\sqrt{x} = \tfrac{1}{2}.$$

After squaring both sides again, we have

$$x = \tfrac{1}{4}.$$

It is readily observed that this root satisfies the given equation.

The original equation and the equation obtained after squaring each member may not be equivalent. It is demonstrable that squaring each member of an equation does not cause a loss of roots; unfortunately, however, the new equation thus obtained may have roots that are not solutions of the original equation. Consequently, it is always necessary to check all suspected roots obtained by such a process; those that do not satisfy the given equation are frequently characterized as extraneous.

Illustration 2: Solve the equation

$$\sqrt{x+9} = 5\sqrt{x} - 3.$$

After squaring each member, we have

$$x + 9 = 25x - 30\sqrt{x} + 9,$$

or
$$5\sqrt{x} = 4x.$$

After squaring a second time, there results

$$16x^2 - 25x = 0,$$

or
$$x(16x - 25) = 0.$$

Hence, the suspected roots of the given equation are

$$x = 0 \qquad \text{and} \qquad x = \tfrac{25}{16}.$$

However, $x = 0$ does not satisfy the original equation and is not a solution of that equation. The only root is $\frac{25}{16}$.

Illustration 3: If we consider the equation

$$\tfrac{29}{16} - x = \sqrt{x - 2},$$

we see from the right member that, for real values of $\sqrt{x-2}$, x must be greater than 2; moreover, since $\sqrt{x-2}$ is positive, the left member must be positive and, hence, x must be less than $\frac{29}{16}$. It is impossible to reconcile these two conditions upon x; thus, it is seen that the equation has no real roots.

However, if we square both members and simplify, we obtain

$$256x^2 - 1{,}184x + 1{,}353 = 0,$$

whose roots are

$$x = \tfrac{41}{16} \qquad \text{and} \qquad x = \tfrac{33}{16}.$$

From the previous considerations, however, we know that these values are not roots of the original equation. This fact is also discovered when we substitute the suspected roots. Hence, the original equation has no roots whatsoever.

It is readily observed by substitution that $x = \frac{41}{16}$ and $x = \frac{33}{16}$ are roots of the equation

$$\tfrac{29}{16} - x = -\sqrt{x - 2}.$$

It is due to the fact that squaring the members of

$$\tfrac{29}{16} - x = -\sqrt{x - 2}$$

and

$$\tfrac{29}{16} - x = \sqrt{x - 2}$$

produces the same equation, namely,

$$256x^2 - 1{,}184x + 1{,}353 = 0$$

that we obtained the suspected roots $x = \frac{41}{16}$ and $x = \frac{33}{16}$.

Illustration 4: Let us consider the irrational equation

$$x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12. \tag{1}$$

If we add 3 to each member of the equation, we have

$$(x^2 - 5x + 3) - 2\sqrt{x^2 - 5x + 3} - 15 = 0. \tag{2}$$

An equation of this type is said to be in quadratic form, since the substitution of y for $\sqrt{x^2 - 5x + 3}$ results in the quadratic equation $y^2 - 2y - 15 = 0$, or $(y - 5)(y + 3) = 0$. Hence, $y = 5$ and $y = -3$.

We note, however, that $y = -3$ cannot equal $\sqrt{x^2 - 5x + 3}$, in view of the fact that the radical sign implies a positive value.

Since the only possibility is $y = 5$, it follows that

$$x^2 - 5x + 3 = 25,$$

or

$$x^2 - 5x - 22 = 0,$$

and

$$x = \frac{5 \pm \sqrt{113}}{2}.$$

The values $x = \dfrac{5 \pm \sqrt{113}}{2}$ are the roots of the original equation.

These illustrations are sufficient to show that an equation resulting from squaring the members of a given irrational equation is not necessarily equivalent to the given equation. Consequently, we must test all suspected roots of an irrational equation and must retain as solutions only those which satisfy the original irrational equation. All other suspected roots are said to be *extraneous*.

EXERCISES 39

Solve the following equations:

1. $\sqrt{x} - \sqrt{x-5} = \sqrt{5}$

2. $\sqrt{2x+1} + \sqrt{x-3} = 2\sqrt{x}$

3. $\dfrac{\sqrt{x} + \sqrt{x-3}}{\sqrt{x} - \sqrt{x-3}} = \dfrac{3}{x-3}$

4. $\sqrt{x+5} + \sqrt{x} = \dfrac{10}{\sqrt{x}}$

5. $2\sqrt{x} = \dfrac{12 - 6\sqrt{x}}{2\sqrt{x} - 3}$

6. $\sqrt[3]{2x+3} + 4 = 7$

7. $\sqrt{x-a} = \sqrt{x} - \frac{1}{2}\sqrt{a};\ a \neq 0$

8. $2\sqrt{x-a} + 3\sqrt{2x} = \dfrac{7a + 5x}{\sqrt{x-a}}$

9. $\dfrac{x-a}{\sqrt{x}} = \dfrac{\sqrt{x}}{a};\ a \neq 0$

10. $x = \sqrt{a^2 + x\sqrt{b^2 + x^2}} - a;\ a \neq 0$

11. $x + 16 - 7\sqrt{x+16} = 10 - 4\sqrt{x+16}$

12. $\dfrac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$

13. $\dfrac{6\sqrt{x} - 21}{3\sqrt{x} - 14} = \dfrac{8\sqrt{x} - 11}{4\sqrt{x} - 13}$

14. $\sqrt{x+16} + \sqrt{x} = 1$

15. $\sqrt{x+12} + 2\sqrt{x-12} = \sqrt{8x-7}$

16. $x + \sqrt{9 + 2x\sqrt{x-7}} = 3$

17. $\sqrt{5x-4} - 3\sqrt{5x} + 6 = 0$

18. $2\sqrt{x+2} + \dfrac{2}{\sqrt{x+2}} = \sqrt{x+3}$

19. $\sqrt{x^2 - x + 1} + \sqrt{x^2 + x + 1} = 3$

20. $\dfrac{5}{\sqrt{x-5}} + \sqrt{x-5} = 9$

21. $2x^2 - 10x + 12 - 2\sqrt{x^2 - 5x + 8} = 0$

22. $\sqrt{10 - x^2 - x} = 8 - x^2 - x$

23. $3x^2 - 4x + \sqrt{3x^2 - 4x - 6} = 18$

24. $27x^{3/2} - 4 = 26x^{3/4}$

HINT: Let $x^{3/4} = y$.

25. $2x^{-1/2} - 9x^{-1/4} = -4$

26. $8x^{3/2n} - 8x^{-3/2n} = 63$

27. $3x^{1/2n} - x^{1/n} - 2 = 0$

28. $20x^{-2/5} - x^{-4/5} = 64$

29. $3x^{3/4} - 4x^{3/8} = 160$

MISCELLANEOUS EXERCISES 40

1. A man bought two farms for $2500 each. The larger contained 3 acres more than the smaller, but he paid $5 more per acre for the smaller than for the larger. How many acres were there in each farm?

2. The combined area of two squares is 980 sq ft, and a side of one square is 18 ft longer than the side of the other. What is the size of each square?

3. *A* and *B* start together on a 40-mile trip. *A* travels 2 mph faster than *B* and arrives 3 hr earlier. Find the rate of each.

4. A cistern can be filled by two pipes in 1 hr. The smaller pipe takes

$\frac{1}{2}$ hr more than the larger one to fill the cistern. Find the time it takes each to fill the cistern.

5. Find two numbers whose difference is 9 and whose sum multiplied by the greater is 266.

6. A certain farm is in the shape of a rectangle with its length twice its width. If it is enlarged two rods on each side its area is increased by 496 sq rods. Find the area of the farm in acres.

7. A square is surrounded by a border whose width lacks 1 ft of being one fourth the length of a side of the square. The area of the border is $\frac{24}{25}$ of the area of the square. Find the width of the border and the length of a side of the square.

8. The corners of a square, the length of whose side is S, are cut off in such a way that a regular octagon (eight sides) remains. What is the length of a leg of the triangle cut off?

9. In 1938, a man traveled 400 miles by train, and after a 3-hr visit, he returned by airplane. The airplane traveled 80 mph faster than the train, and the total elapsed time for the entire journey was 16 hr and 20 min. Find the rate of the train and of the airplane.

10. A man drove his car 60 miles in a certain interval of time. Had the time been $\frac{1}{2}$ hr less, the rate would have been 20 mph greater. Find the time and rate.

11. Members of an automobile party in the mountains 90 miles from a railroad wished to make a certain train. They traveled the first 60 miles at a certain average rate; then they realized that they must increase their average speed $2\frac{1}{2}$ mph to make the train. If they had continued to drive at the rate they were going during the first part of their journey, they would have been 10 min late. Assuming that the party reached the station just at train time, find the total time that it took to drive the 90 miles.

12. If, from a height of a ft, a body is thrown vertically downward with an initial velocity of b ft per sec, its height at the end of t sec is given by the formula $h = a - bt - 16t^2$.

(*a*) If a body is thrown vertically downward from a height of 400 ft with an initial velocity of 32 ft per sec, when will it be at a height of 16 ft from the ground?

(*b*) When will it reach the ground?

64. DISCUSSION OF THE ROOTS OF THE QUADRATIC EQUATION

We have observed that the roots of the general quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is called the *discriminant* of the quadratic function and is designated by D.

(*a*) If for given values of a, b, and c, D is a negative number, the roots involve imaginary numbers.

(*b*) If for given numerical values of a, b, and c, D is zero, the two roots are equal. The value of the two equal roots is, of course, $-\dfrac{b}{2a}$.

The converse of this statement is also true; that is, if the two values of x are equal, D must equal zero.

This may readily be shown as follows:

By hypothesis, $$\frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Therefore, $$\sqrt{b^2 - 4ac} = -\sqrt{b^2 - 4ac},$$

and $$b^2 - 4ac = 0.$$

(c) If for given numerical values of a, b, and c, D is a positive number, not zero, it is evident that the roots are real, and, because of the demonstration under (b), it follows that in this case the roots are not equal.

The conclusions of (a), (b), and (c) may be restated in mathematical symbols as follows:

For (a) $D < 0$, both roots involve imaginary numbers;
(b) $D = 0$, both roots have the same real value;
(c) $D > 0$, both roots are real but have different values.

If D is a perfect square of some rational number, both roots will be real and rational.

Obviously, whenever the curve of a quadratic function does not cut the x axis, there are no real roots of the equation. This situation, therefore, corresponds to case (a) just discussed.

If the graph of the quadratic function under consideration merely touches (is tangent to) the x axis, the two points of intersection between the x axis and the curve become identical, and we have case (b).

Case (c) is obviously of greatest interest in the graphical solution of the quadratic equation; then the graph of the quadratic function intersects the x axis in two distinct points. If we consider, for example, the equation $-3x^2 + 8x - 1 = 0$, we know that since $D > 0$, that is, $64 - 12$ is positive, the roots are real and unequal.

Solving the equation by formula, we find that

$$x = \frac{-8 + \sqrt{52}}{-6} = \frac{-8 + 7.2111}{-6} = 0.13, \quad \text{approximately,}$$

and $$x = \frac{-8 - \sqrt{52}}{-6} = \frac{-8 - 7.2111}{-6} = 2.53, \quad \text{approximately.}$$

Hence, these are the x coordinates of the points on the x axis where the curve cuts the x axis.

Graphical methods for the solution of quadratics are not necessary, since these equations are so readily solvable by algebraic methods. However, we call the student's attention to the desirability of graphical methods,

to be studied later, for the solution of equations which may not be solved readily by algebraic methods.

65. SUM AND PRODUCT OF ROOTS

It is often inconvenient to check roots by substituting them for x in the given equation. Hence, a knowledge of what the sum and product of the roots should be in terms of a, b, and c is of great aid as a check.

If we designate one root by x_1 and the other root by x_2, we have

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Therefore,
$$x_1 + x_2 = \frac{-2b}{2a} = \frac{-b}{a}.$$

Also,
$$(x_1)(x_2) = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$
$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Thus, it is known immediately that the sum of the two roots of the equation

$$2x^2 - 3x + 7 = 0$$

is
$$x_1 + x_2 = \frac{-b}{a} = \frac{3}{2},$$

and that their product is

$$(x_1)(x_2) = \frac{c}{a} = \frac{7}{2}.$$

EXERCISES 41

Show from an examination of the discriminant that the roots of

1. $x^2 + x + 5 = 0$ are not real;
2. $9x^2 - 12x + 4 = 0$ are real and equal;
3. $5x^2 - 8x + 1 = 0$ are real and unequal.
4. Solve each of the above equations, and check the roots by use of the formulas for the sum and the product of the roots.
5. What must be the value of q in order that $qx^2 - 3x + 6 = 0$ shall have real and equal roots?
6. What must be the value of q in order that $5x^2 - 6x + q = 0$ shall have real and equal roots? For what values of q will the roots be real and unequal?
7. What must be the values of q in order that $3x^2 - 2qx + 12 = 0$ shall have real and equal roots? For what values of q will the roots not be real?
8. For what values of q will the roots of $3x^2 - qx + 10$ be real? Not real?

9. Write the sum and product of the roots of $5x^2 - x - 1 = 0$ without solving the equation.

10. By use of the formulas in Section 65 determine whether or not $\dfrac{1 + \sqrt{14}}{3}$ and $\dfrac{1 - \sqrt{14}}{3}$ are roots of the equation $3x^2 - 2x + 5 = 0$.

11 Systems Involving Quadratic Equations

66. ONE LINEAR AND ONE QUADRATIC EQUATION

This section is concerned with systems of equations consisting of one linear equation and one equation which involves at least one second-degree term, and no term higher than the second degree. We recall that such terms as $3x^2$, $5y^2$, $7xy$ are second-degree terms, the degree being the sum of the exponents of the unknowns in the term.

Thus, the pairs of equations

$$\left.\begin{aligned} 3x - 5y &= 1 \\ xy + y &= 3 \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} 2x + 3y &= 5 \\ x^2 + 3y^2 &= 6 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} 8x - y &= 5 \\ 7x^2 + 3xy - 5y^2 &= 6 \end{aligned}\right\} \tag{3}$$

are examples of the systems to be considered.

Such pairs of equations may always be solved by solving the linear equation for one of the unknowns in terms of the other and substituting the result in the second-degree equation.

Illustration 1: From the first equation of system (1), we have

$$x = \frac{5y + 1}{3},$$

and, hence,
$$\frac{(5y + 1)y}{3} + y = 3,$$

or
$$5y^2 + 4y - 9 = 0.$$

The roots of this latter equation are

$$y = \frac{-4 \pm \sqrt{16 + 180}}{10} = 1 \quad \text{and} \quad -\frac{9}{5}.$$

By substituting these values of y in $x = \dfrac{5y + 1}{3}$, we find the corre-

sponding values of x to be 2 and $-\frac{8}{3}$. So the solutions properly paired are (2, 1) and $(-\frac{8}{3}, -\frac{8}{3})$.

NOTE ON PAIRING SOLUTIONS. Whenever a system of two equations is solved, it is essential that the values of x and y that satisfy both equations be indicated as corresponding pairs. A corresponding pair of x and y values constitutes a solution of the system of equations.

EXERCISES 42

Solve the following systems of equations, and check:

1. $2x^2 + y^2 = 33$
   $x - y = -3$
2. $3x + 4y = 24$
   $x^2 + y^2 = 25$
3. $x^2 - xy + y^2 = 3$
   $2x + 3y = 8$
4. $x - y = b$
   $xy = a^2$
5. $\frac{6}{x} + \frac{4}{y} = 1$
   $4x - 3y + 16 = 0$
6. $\frac{2}{x} - \frac{1}{y} = \frac{4}{3}$
   $\frac{3}{x^2} + \frac{2}{y^2} = \frac{35}{36}$
   HINT: Let $\frac{1}{x} = u$, $\frac{1}{y} = v$.
7. $3x^2 - 3xy - y^2 - 4x - 8y + 3 = 0$
   $3x - y - 8 = 0$
8. $\frac{1}{y} - \frac{3}{x} = 1$
   $\frac{7}{xy} - \frac{1}{y^2} = 12$
9. $x^2 + 5x = 4y^2$
   $3x - 4y = 24$
10. $x^2 + xy + 2 = 0$
    $2x + y - 1 = 0$

67. TWO QUADRATIC EQUATIONS REDUCIBLE TO THE PREVIOUS CASE

If we consider the two second-degree equations

$$xy + x = 20 \tag{1}$$

$$xy - y = 12, \tag{2}$$

we note that each equation is of the second degree. Yet the solution of these equations is reducible to the solution of a system of the type already considered.

Thus, subtracting the members of Equation (2) from those of (1), we have

$$x + y = 8. \tag{3}$$

Thus, from (3), $y = 8 - x$. If we substitute $8 - x$ for y in (1), we have

$$x(8 - x) + x = 20,$$

or

$$x^2 - 9x + 20 = 0.$$

Consequently, $x = 5$ and $x = 4$.

From the equation $y = 8 - x$ we find that the values of y corresponding to $x = 5$ and $x = 4$, respectively, are $y = 3$ and $y = 4$. Thus, the values properly paired are (5, 3) and (4, 4).

If the solutions of the system composed of (3) and (1) satisfy (2), they are solutions of the system comprising (1) and (2). Moreover, it is demonstrable that the two systems are equivalent.

68. EQUATIONS HOMOGENEOUS WITH RESPECT TO THE UNKNOWNS

If the terms containing the unknowns in an equation are all of the same degree, the equation is said to be *homogeneous with respect to the unknowns.*

Illustration: Solve the following system of homogeneous equations:

$$x^2 + y^2 = 25 \tag{1}$$

$$xy - y^2 = -4. \tag{2}$$

First Method: We shall make the *homogeneous substitution*

$$y = vx.$$

Therefore, we obtain the equations

$$x^2 + v^2x^2 = 25 \tag{3}$$

and

$$vx^2 - v^2x^2 = -4. \tag{4}$$

But from (3) we have

$$x^2 = \frac{25}{v^2 + 1},$$

and from (4) we have

$$x^2 = \frac{-4}{v - v^2};$$

and, hence,

$$\frac{25}{v^2 + 1} = \frac{-4}{v - v^2}.$$

Upon simplification, this becomes

$$21v^2 - 25v - 4 = 0.$$

Hence,

$$v = \tfrac{4}{3} \quad \text{and} \quad -\tfrac{1}{7}.$$

Substituting $v = \frac{4}{3}$ in

$$x^2 = \frac{25}{v^2 + 1},$$

we have

$$x = \pm 3.$$

Since $v = \frac{4}{3}$, then corresponding to $x = +3$ the value of y obtained from the homogeneous substitution is $+4$, and corresponding to $x = -3$ the value of y is -4.

Substituting $v = -\frac{1}{7}$ in

$$x^2 = \frac{25}{v^2 + 1},$$

we have

$$x = \pm\frac{7}{\sqrt{2}} = \pm\frac{7}{2}\sqrt{2}.$$

Consequently, the value of y corresponding to $x = \frac{7\sqrt{2}}{2}$ is $-\frac{\sqrt{2}}{2}$, and the value of y corresponding to $x = -\frac{7\sqrt{2}}{2}$ is $+\frac{\sqrt{2}}{2}$. The solutions, therefore, are

$$(3, 4), \quad (-3, -4), \quad \left(7\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \quad \text{and} \quad \left(-7\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

SECOND METHOD: If we multiply the members of Equation (1) by 4, and those of (2) by 25, and add, we have

$$4x^2 + 25xy - 21y^2 = 0, \tag{5}$$

or

$$(4x - 3y)(x + 7y) = 0. \tag{6}$$

This last equation is equivalent to the two linear equations, $4x - 3y = 0$ and $x + 7y = 0$.

The given system (1) and (2) may be replaced by either of the two sets of systems that follow:

$$\begin{aligned} x^2 + y^2 &= 25 \\ x + 7y &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x^2 + y^2 &= 25 \\ 4x - 3y &= 0 \end{aligned}$$

or

$$\begin{aligned} xy - y^2 &= -4 \\ x + 7y &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} xy - y^2 &= -4 \\ 4x - 3y &= 0. \end{aligned}$$

Each of the systems in the first set, for example, is of the type previously discussed, and the solutions are, respectively, $\left(7\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-7\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and (3, 4), (−3, −4).

EXERCISES 43

Solve the following systems of equations:

1. $x^2 + 3xy = 28$
   $x^2 + y^2 = 20$
2. $x^2 + xy + y^2 = 6$
   $x^2 + y^2 = 12$
3. $x^2 - xy + y^2 = 21$
   $y^2 - 2xy + 15 = 0$
4. $x^2 + xy + 2y^2 = 74$
   $2x^2 + 2xy + y^2 = 73$

5. $x^2 - 4y^2 = 20$
$xy = 12$

6. $x^2 - 12xy + 119 = 0$
$y^2 - 2xy + 24 = 0$

7. $x^2 - 3xy + 4y^2 = 58$
$2x^2 - 7xy = 50$

8. $3xy - 9y^2 + 2 = 0$
$4x^2 + 5xy + 6y^2 = 3$

9. $x^2 - 4xy = 15$
$x^2 - 10xy + 8y^2 = 2$

10. $2x^2 - 3xy + y^2 = 2$
$3x^2 - 2y^2 + 15 = 0$

69. EQUATIONS SYMMETRICAL WITH RESPECT TO UNKNOWNS

If an interchange of x and y in all the terms of an equation does not alter the equation, the equation is said to be symmetrical with respect to x and y.

Illustration: Solve the system

$$4(x + y) + 3xy = 0 \tag{1}$$

$$x^2 + x + y + y^2 = 20. \tag{2}$$

Let us make the *symmetric substitution*, namely,

$$x = u + v,$$

and

$$y = u - v.$$

Therefore, Equation (1) becomes

$$8u + 3u^2 - 3v^2 = 0, \tag{3}$$

and Equation (2) becomes

$$u^2 + v^2 + u = 10. \tag{4}$$

After multiplying the members of (4) by 3, we have

$$3u^2 + 3v^2 + 3u = 30. \tag{5}$$

Adding the corresponding members of (3) and (5), we have

$$6u^2 + 11u - 30 = 0.$$

Hence,

$$u = \frac{-11 \pm\sqrt{121 + 720}}{12};$$

that is,

$$u = \tfrac{3}{2} \quad \text{and} \quad -\tfrac{10}{3}.$$

Substituting $u = \frac{3}{2}$ in Equation (3), we have $v^2 = \frac{25}{4}$, and, hence,

$$v = \tfrac{5}{2} \quad \text{and} \quad -\tfrac{5}{2}.$$

Therefore,

$$x = \tfrac{3}{2} + \tfrac{5}{2} = 4,$$

$$y = \tfrac{3}{2} - \tfrac{5}{2} = -1.$$

Also

$$x = \tfrac{3}{2} - \tfrac{5}{2} = -1,$$

and

$$y = \tfrac{3}{2} + \tfrac{5}{2} = 4.$$

As would be expected, since the equations are symmetrical, the x and y values are interchangeable. Substituting $u = -\frac{10}{3}$ in (3), we have $v^2 = \frac{20}{9}$; hence,

$$v = \pm \tfrac{2}{3}\sqrt{5}.$$

Therefore,
$$x = \frac{-10}{3} + \tfrac{2}{3}\sqrt{5},$$

$$y = \frac{-10}{3} - \tfrac{2}{3}\sqrt{5};$$

and
$$x = \frac{-10}{3} - \tfrac{2}{3}\sqrt{5},$$

$$y = \frac{-10}{3} + \tfrac{2}{3}\sqrt{5}.$$

It is left as an exercise for the student to check these values.

EXERCISES 44

Solve the following systems of equations:

1. $x^2 + y^2 - x - y = 78$
   $xy + x + y = 39$
2. $xy - (x + y) - 1 = 0$
   $xy = 2$
3. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{2}$
   $\dfrac{1}{x} + \dfrac{1}{y} = 4$
4. $x^2 - xy + y^2 = 12$
   $x^2 + xy + y^2 = 4$
5. $xy = 3(x + y)$
   $x^2 + xy + y^2 = 28$
6. $6(x + y) = 5xy$
   $x + y + x^2 + y^2 = 18$
7. $45(x + y) + 4xy = 0$
   $x^2 + y^2 - 2x - 2y = 98$
8. $x^2 + y^2 + x + y = 26$
   $xy + 10 = 3(x + y)$
9. $x^2 + y^2 - 3(x + y) = 46$
   $5(x + y) - xy = 41$
10. $2(x^2 + y^2) - 5(x + y) = 1$
    $x^2 + y^2 - xy = 7$

70. SPECIAL SYSTEMS OF QUADRATICS

Illustration 1: Let us consider the solution of the following system of two equations of the second degree:

$$x^2 - 2y = 3 \tag{1}$$

$$y^2 + xy = 5. \tag{2}$$

If we solve Equation (1) for y, we have

$$y = \frac{x^2 - 3}{2}.$$

Substituting this expression for y in Equation (2), we have

$$\left(\frac{x^2 - 3}{2}\right)^2 + \frac{x(x^2 - 3)}{2} = 5,$$

which, upon simplification, becomes

$$x^4 + 2x^3 - 6x^2 - 11 = 0. \tag{3}$$

Equation (3) is of the fourth degree in one unknown. Methods for solving equations higher than the second degree have not yet been considered in this course, and hence, a system of simultaneous quadratics requiring for their solution methods not yet presented will not be considered at this time.

Illustration 2: Solve the system

$$x^2 + y^2 = 25 \tag{1}$$

$$xy = 12. \tag{2}$$

This system involves equations that are homogeneous and symmetric; so they may be solved by methods given previously.

We may, however, solve this system as follows: Multiplying the members of Equation (2) by 2, we have

$$2xy = 24. \tag{3}$$

Therefore, adding (3) and (1),

$$x^2 + 2xy + y^2 = 49. \tag{4}$$

Hence, after extracting the square roots of each member of (4), we obtain

$$x + y = 7, \tag{5}$$

and

$$x + y = -7. \tag{6}$$

Subtracting the members of (3) from those of (1), we have

$$x^2 - 2xy + y^2 = 1, \tag{7}$$

which yields the two equations,

$$x - y = 1, \tag{8}$$

and

$$x - y = -1. \tag{9}$$

Thus, the given pair of quadratic equations may be replaced by the four linear pairs:

$$\begin{aligned} x + y &= 7 \qquad & x + y &= -7 \\ x - y &= 1 \qquad & x - y &= -1 \end{aligned}$$

$$\begin{aligned} x + y &= -7 \qquad & x + y &= 7 \\ x - y &= 1 \qquad & x - y &= -1 \end{aligned}$$

The respective solutions are (4, 3), (−4, −3), (−3, −4), (3, 4).

Illustration 3: Solve the system

$$x^3 + y^3 = 152 \tag{1}$$

$$x^2 - xy + y^2 = 19. \tag{2}$$

These equations may be solved by methods already discussed. However, if we divide the members of (1) by the corresponding members of (2), we obtain

$$x + y = 8, \tag{3}$$

which is linear.

The given system may now be replaced by the equivalent system

$$x + y = 8$$
$$x^2 - xy + y^2 = 19.$$

The solutions are (3, 5) and (5, 3).

It is highly desirable that the student use his ingenuity in devising special methods for the solution of the various special systems that may come up for consideration.

EXERCISES 45

Solve the following systems of equations:

1. $xy - (x + y - 1) = 0$
$xy = 2$

2. $x^2 - xy + y^2 = 19$
$xy = 15$

3. $x^2 - xy = 54$
$xy - y^2 = 18$

4. $x^2 + y^2 = 5$
$x^2 - xy + y^2 = 3$

5. $x^3 - 8y^3 = 224$
$x - 2y = 8$

6. $3(x^3 - y^3) = 13xy$
$x - y = 1$

7. $x^2 - 2xy - 3y^2 = 0$
$x^2 + 2y^2 = 12$

8. $6x^2 + 5xy + y^2 = 0$
$y^2 - x - y = 32$

9. $x^2y + xy^2 = 6$
$x^2y - xy^2 = 5$

10. $x^2 + 3xy + 2y^2 - (x + y) = 0$
$2x^2 - 3xy = 5$

11. $x^3 + 8y^3 = 72$
$x + 2y = 6$

12. $x^2 + xy - 6y^2 = 12y$
$2x^2 - 5xy + 2y^2 = 18x$

13. $x^2 + 4xy + 3y^2 = -2$
$x^2 + 2xy - 3y^2 = 32$

14. $x^2 - 3xy + 2y^2 = 6x$
$x^2 - y^2 = -5y$

MISCELLANEOUS EXERCISES 46

Solve the following systems of equations:

1. $x^2 + 3xy - 2y^2 = 26$
$x^2 + 3xy - y^2 = 51$

2. $x^2 - y^2 = 27$
$xy = 18$

3. $xy^2 + x - 10y = 0$
$xy^2 - x - 6y = 0$

4. $x^2 + y^2 = 36$
$xy = 12$

5. $xy - x^2 + y^2 = -91$
$x^2 + y^2 + xy = 225$

6. $x^2 + y^2 = 20$
$x^2 - y^2 = 12$

7. $x^2 + xy - y^2 = 5$
$3x^2 - 2xy - 2y^2 = 6$

8. $x^2 + y^2 - xy = 7$
$x^2 - y^2 - xy = -1$

9. $x^2 + y^2 - 6x - 8y = 56$
$x^2 + y^2 = 25$

10. $y^2 = 8x$
$x^2 + y^2 = 64$

11. $9x^2 + 4y^2 = 36$
$y^2 = 4x$

12. $y = \dfrac{3}{x^2 + 2}$
$y = x^2$

13. $y = \dfrac{8a^3}{x^2 + 4a^2}$
$x^2 = 4ay$

14. $y = x^3$
$y = 2x - x^2$

15. $x^2 + y^2 = 25$
$y - 7x + 25 = 0$

16. $6x^2 + 7xy - 3y^2 = 0$
$3x^2 - 5xy + 8y^2 = 60$

17. $\dfrac{1}{x^2} - \dfrac{1}{y^2} = \dfrac{5}{36}$
$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$

18. Find the legs of a right triangle whose hypotenuse is 64 in. and whose area is 50 sq in.

19. The perimeter of a rectangle is 64 in. and the area is 50 sq in. Find the length of the sides.

20. Two men start at the same time to meet each other from towns which are 25 miles apart. One takes 18 min longer than the other to walk a mile, and they meet in 5 hr. How fast does each walk?

21. In 1938, two airplanes left New York simultaneously for St. Louis, which is 1170 miles distant; one went 20 mph faster than the other and arrived 2 hr and 15 min sooner. Find the rate of each airplane.

22. What pairs of numbers have the same number for their sum, their product, and the difference of their squares?

23. A man divides a tract of land into city lots. He sells the lots all at the same price and realizes \$4800. If the number of lots had been one less and the price per lot \$8 more, he would have received the same amount of money. How many lots were there, and what was the price per lot?

24. Psychologists assert that the rectangle most pleasing to the human eye is that in which the sum of the two dimensions is to the longer as the longer is to the shorter. If the area of a page of this book remains unchanged, what should be its dimensions?

71. GRAPHICAL REPRESENTATION OF CERTAIN QUADRATIC EQUATIONS

It is desirable that the student familiarize himself with the curves corresponding to certain quadratic equations that are often needed in practice.

The Parabola. It has already been indicated that curves corresponding to the equation $y = ax^2 + bx + c$ will be of a form similar to that of Figure 19 if a is negative or of Figure 20 if a is positive.

The equation $y = ax^2$ is but a special case of $y = ax^2 + bx + c$; its corresponding curve, however, will be symmetric with respect to the y axis, taking the form of Figure 21 if a has a positive numerical value or that of Figure 22 if a has a negative numerical value.

As one might expect, the equation $y^2 = ax$ will have as its correspond-

ing graph the form of Figure 23 if a has a positive numerical value, and it will have as its corresponding graph the form of Figure 24 if a has a negative value.

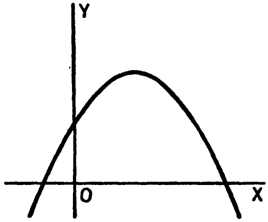


FIG. 19

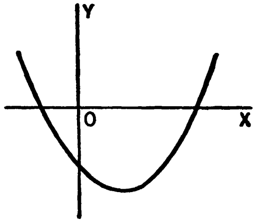


FIG. 20

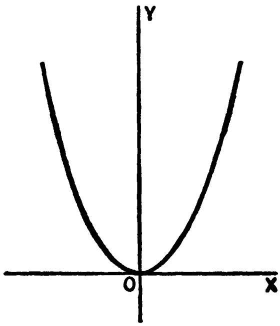


FIG. 21

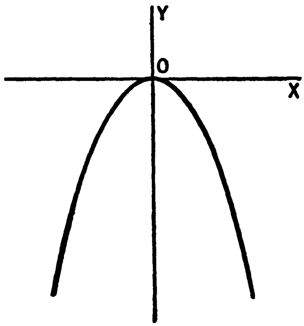


FIG. 22

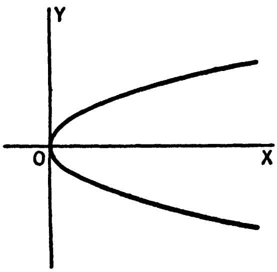


FIG. 23

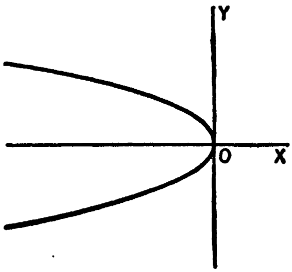


FIG. 24

The Circle. If we consider a point (x, y) moving so that it is always r units from a fixed point (a, b), it will trace a circle with (a, b) as center and with r as radius.

By the use of the Pythagorean theorem and by reference to Figure 25,

the relation between x and y is readily found to be

$$(x - a)^2 + (y - b)^2 = r^2. \tag{1}$$

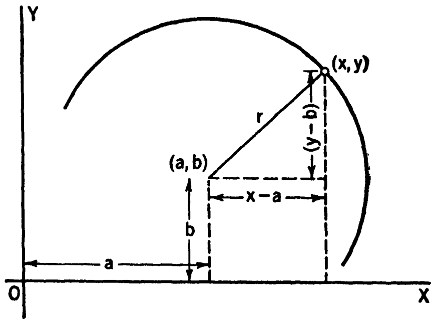


Fig. 25

Thus, as an example, the equation of the circle with its center at (2, −5) and having a radius of 7 is

$$(x - 2)^2 + (y + 5)^2 = 7^2.$$

If, in Equation (1), $a = 0$ and $b = 0$, the circle has its center at the origin, and its equation is

$$x^2 + y^2 = r^2. \tag{2}$$

Equations that can be put in the forms (1) or (2) can readily be graphed by drawing the corresponding circle. Thus,

$$x^2 + y^2 - 6x + 8y = 24$$

may be written

$$(x^2 - 6x) + (y^2 + 8y) = 24.$$

After completing the squares in the parentheses, we obtain

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 24 + 9 + 16,$$

or $$(x - 3)^2 + (y + 4)^2 = 7^2.$$

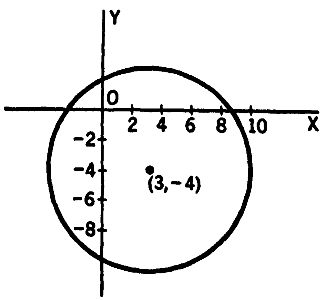


Fig. 26

Comparing the form of this equation with (1), we see that it represents a circle with center at (3, −4) and radius 7. Its graph appears as Figure 26.

EXERCISES 47

Sketch the curves corresponding to the following equations:

1. $y^2 = 8x$
2. $y^2 = -8x$
3. $y = 4x^2$
4. $y = -4x^2$
5. $3x^2 = 7y$
6. $y = x^2 - 5x + 6$
7. $x^2 + y^2 = 25$
8. $3x^2 + 3y^2 = 18$
9. $x^2 + y^2 - 6x - 9y = 61$
10. $x^2 + y^2 - 2x + 14y = 50$
11. $4x^2 + 4x + 4y^2 - 12y - 15 = 0$
12. $x^2 + 5x + y^2 = 14$
13. $x^2 + y^2 - 7y - 13 = 0$
14. $x^2 + y^2 + 4x - 9y = 0$

The Ellipse. An equation that can be put in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

has as its corresponding graph an ellipse whose center is at (0, 0) and whose axes of symmetry* coincide with the x and y axes. If $a = b$, the curve is a circle. The general form of the ellipse with center at the origin and with axes of symmetry that coincide with the x and y axes is shown in Figure 27.

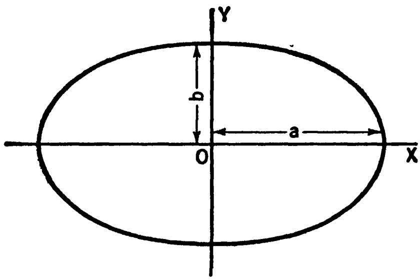


FIG. 27

If the values of a and b are known, the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be sketched with little trouble.

Illustration: Study the graphical representation of the following equation:

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

If we solve the given equation for x, we have

$$x = \pm\tfrac{3}{4}\sqrt{16 - y^2}, \tag{4}$$

* The concept of symmetry will be discussed in detail in the third part of the text.

and if we solve for y, we have

$$y = \pm\tfrac{4}{3}\sqrt{9 - x^2}. \tag{5}$$

Equation (4) shows that we cannot substitute any numerical value for y greater than 4 or less than -4 if x is to be a real number. This is important since our attention in this text is restricted to real graphs. Similarly, Equation (5) shows that we cannot substitute any numerical value for x greater than 3 or less than -3 if y is to be a real number.

We may form a table of values from either (4) or (5) from which a smooth curve is graphed. The resulting curve is shown in Figure 28. As implied in Figure 27, the graph intersects the x axis when $x = \pm a$ and intersects the y axis when $y = \pm b$.

x	y
0	± 4
± 1	± 3.76
± 2	± 2.97
± 3	0

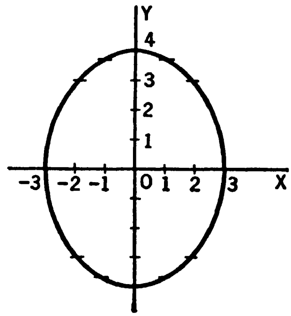


Fig. 28

The equation of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{6}$$

represents an ellipse with its center at (h, k) and with its axes of symmetry parallel to the x and y axes.

This can be seen as follows: If we let

$$x - h = x' \qquad \text{and} \qquad y - k = y',$$

Equation (6) becomes

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

which represents an ellipse, relative to new axes x' and y' that are parallel to and at a distance of h and k, respectively, from the x and y axes.

Illustration: The equation

$$4x^2 + 9y^2 - 8x + 36y + 4 = 0$$

may be written

$$4(x^2 - 2x) + 9(y^2 + 4y) = -4.$$

After completing the squares within the parentheses, we have

$$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = -4 + 4 + 36,$$

or

$$4(x - 1)^2 + 9(y + 2)^2 = 36,$$

which may be written

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1.$$

Comparing this equation with the form of (6), we see that it represents an ellipse with its center at (1, −2) and its axes of symmetry parallel to the x and y axes.

The Hyperbola. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{7}$$

has a graph of the general form shown in Figure 29.

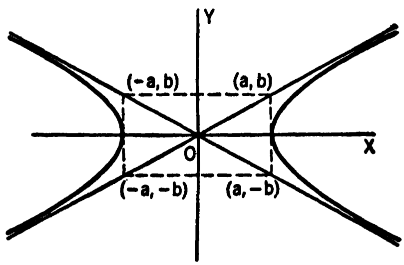


FIG. 29

The equation

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has as its graph the form displayed in Figure 30.

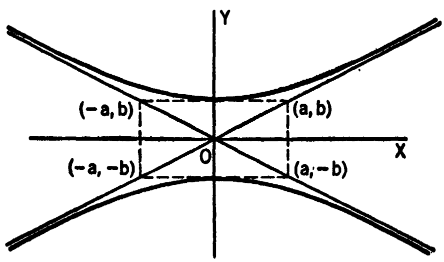


FIG. 30

The guide lines for drawing these curves, as will be noted from a study of the two figures, are the diagonals through (a, b) and $(-a, -b)$ and through $(-a, +b)$ and $(a, -b)$.

If the values of a and b are known, the graph of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

may be constructed.

Illustration: Graph the equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1.$$

If we solve the given equation for x, we have

$$x = \pm\tfrac{3}{4}\sqrt{y^2 + 16}; \tag{8}$$

and if we solve the given equation for y, we have

$$y = \pm\tfrac{4}{3}\sqrt{x^2 - 9}. \tag{9}$$

Equation (8) shows that we may substitute any numerical value, positive or negative, for y. Equation (9) shows that we cannot substitute any value for x between -3 and 3 if y is to be a real number.

We may form a table of values from either (8) or (9) from which the curve can be graphed. The resulting curve is shown in Figure 31.

x	y
± 3	0
± 4	± 3.52
± 5	± 5.33
± 6	± 6.92
± 7	± 8.42
± 8	± 9.88
± 9	± 11.31
± 10	± 12.72

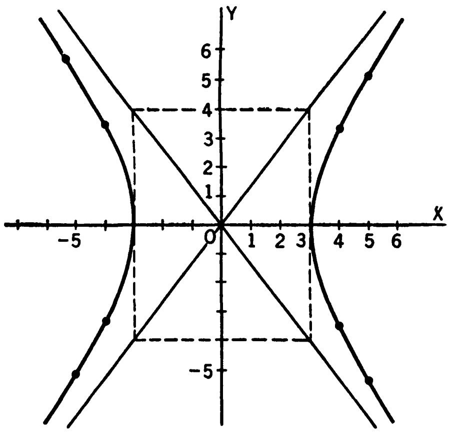


FIG. 31

Similarly the graph of the equation

$$\frac{-x^2}{9} + \frac{y^2}{16} = 1$$

is shown in Figure 32.

If we solve the given equation for x, we have

$$x = \pm\tfrac{3}{4}\sqrt{y^2 - 16}; \tag{10}$$

and if we solve the given equation for y, we have

$$y = \pm\tfrac{4}{3}\sqrt{x^2 + 9}. \tag{11}$$

Equation (10) shows that we cannot substitute any numerical value for y between -4 and 4 if x is to be a real number. Equation (11) shows that we may substitute for x any numerical value, positive or negative. We may form a table from either (10) or (11) from which the curve is to be graphed.

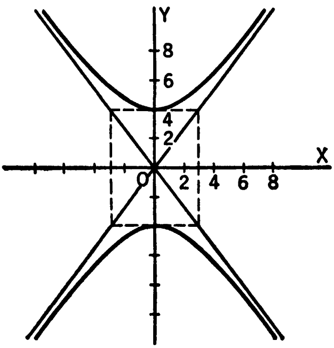


Fig. 32

x	y
0	± 4
± 2.25	± 5
± 3.35	± 6
± 4.31	± 7
± 5.19	± 8
± 6.09	± 9
± 6.87	± 10

Equations of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{12}$$

and

$$-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{13}$$

also represent hyperbolas. The center of each hyperbola is at (h, k), and the axes of symmetry are parallel to the x and y axes.

The student should explain the above statement by the method used in the investigation of the corresponding equation of the ellipse.

Illustration: The equation

$$9x^2 - 16y^2 + 36x + 96y = 252$$

may be written $\quad 9(x^2 + 4x) - 16(y^2 - 6y) = 252.$

Completing the squares within the parentheses, we have

$$9(x^2 + 4x + 4) - 16(y^2 - 6y + 9) = 252 + 36 - 144,$$

or

$$9(x + 2)^2 - 16(y - 3)^2 = 144,$$

which may be written in the form

$$\frac{(x+2)^2}{16} - \frac{(y-3)^2}{9} = 1.$$

Comparing this equation with the form of (12), we see that it represents a hyperbola with center at the point $(-2, 3)$.

The Hyperbola: $xy = c$. The equation

$$xy = c \tag{14}$$

will have as its graph a curve of the form displayed in Figure 33 if c has a positive value and a curve of the form shown in Figure 34 if c has a negative value.

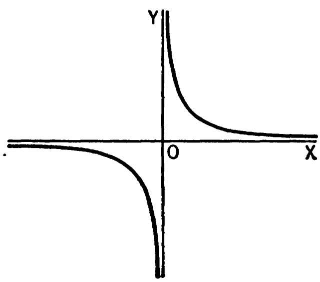


FIG. 33

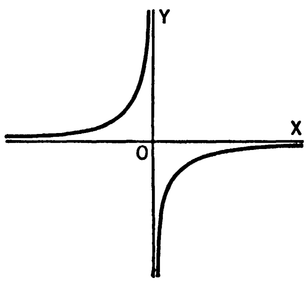


FIG. 34

An equation of the form

$$(x - h)(y - k) = c \tag{15}$$

represents a hyperbola similar to that of $xy = c$, but with its center at the point (h, k).

Illustration: The equation

$$4xy - 8x + 6y = 9$$

becomes

$$xy - 2x + \tfrac{3}{2}y = \tfrac{9}{4}$$

when we divide by 4.

After subtracting 3 from each member, we may write

$$(x + \tfrac{3}{2})(y - 2) = \tfrac{9}{4} - 3,$$

and, consequently,

$$(x + \tfrac{3}{2})(y - 2) = -\tfrac{3}{4}.$$

This represents a hyperbola similar to that shown in Figure 34, but with its center at $(-\frac{3}{2}, 2)$. Note Figure 35, in which auxiliary axes have been drawn through the point $(-\frac{3}{2}, 2)$.

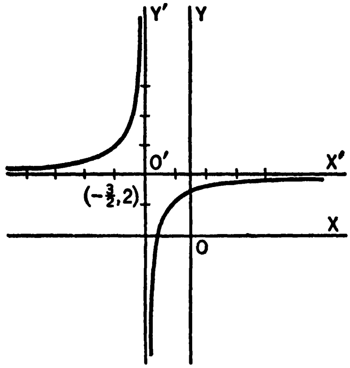


FIG. 35

A detailed investigation of these various curves is undertaken in the third part of this text. Many statements just given without proof will be justified at that point in the book.

EXERCISES 48

Identify the type of each of the following curves, and draw a rough sketch of each one:

1. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

2. $\dfrac{(x-3)^2}{25} + \dfrac{(y+1)^2}{9} = 1$

3. $x^2 - 4x + 4y^2 = 0$

4. $9x^2 + 16y^2 = 144$

5. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

6. $-\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

7. $\dfrac{(x-5)^2}{9} - \dfrac{(y-1)^2}{4} = 1$

8. $x^2 - 4x - y^2 = 0$

9. $xy + 4 = 0$

10. $3xy = 10$

11. $3x^2 + 25y^2 + 12x - 50y = 38$

12. $16x^2 + 9y^2 - 96x + 18y + 9 = 0$

13. $16x^2 - 9y^2 + 32x + 36y = 164$

14. $9x^2 - 25y^2 + 54x + 50y = 369$

15. $3xy + 5x - 6y = 10$

16. A right triangle of legs x and y is inscribed in a circle of diameter 6. Draw the graphical representation of the equation which must exist relating x and y.

17. If the temperature of a given amount of gas remains constant, Boyle's law states that the volume of the gas varies inversely as the pressure. Display graphically this relation between pressure and volume.

18. A point (x, y) moves so that its distance from the point $(2, 0)$ equals its distance to the line $x = -2$. Obtain the equation of the curve generated by such a moving point; graph it.

72. GRAPHICAL SOLUTION OF SYSTEMS

If we have a system of two equations in two unknowns x and y, we may draw their corresponding curves relative to the same axes. The values of the coordinates (x, y) corresponding to the points of intersection of the two curves are the real and finite solutions of the given system of equations.

It is thus evident that an acquaintance with the curves corresponding to given equations is quite helpful. However, it is not easy to know, by means of a few points on the curve, the general appearance of many curves corresponding to equations of higher degree. It requires a knowledge of the calculus to get the necessary information relative to a careful graphing of curves corresponding to higher-degree equations. For the present we therefore limit ourselves to such curves as have been discussed in the previous sections.

Illustration 1: Solve graphically the following system:

$$x^2 - 6x + y^2 - 8y = 24 \tag{1}$$

$$y = 3x. \tag{2}$$

It is readily determined that Equation (1) may be graphed as a circle with its center at (3, 4) and with $r = 7$. Equation (2) leads to a straight line passing through the origin. The two curves have been drawn upon the same axis system in Figure 36.

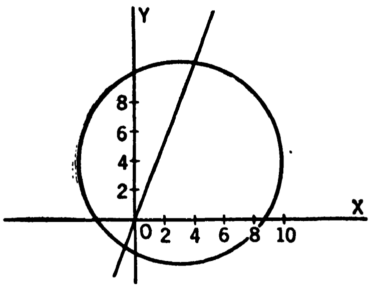


Fig. 36

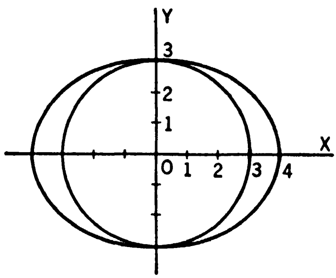


Fig. 37

The student should draw this figure to scale and obtain the solutions from the graph by noting the coordinates of the points of intersection of the two curves. These solutions may be checked by solving algebraically.

Illustration 2: Solve graphically the following system:

$$9x^2 + 16y^2 = 144 \tag{1}$$

$$x^2 + y^2 = 9. \tag{2}$$

Figure 37 shows the graphs of Equations (1) and (2).

The student should solve the system graphically by noting the points of intersection of the two curves, then check by an algebraic solution.

EXERCISES 49

Solve the following systems graphically, and then check by solving algebraically:

1. $x^2 + y^2 = 25$
$y = 4x$

2. $x^2 + y^2 + 6x = 0$
$y = 5x$

3. $4x^2 + 9y^2 = 36$
$3x + 2y = 6$

4. $4x^2 - 9y^2 = 36$
$x + y = 1$

5. $y = 4x^2$
$x = 4y^2$

6. $x^2 - y^2 = 1$
$x^2 + y^2 = 25$

7. $y^2 - x^2 = 5$
$4x^2 + 9y^2 = 36$

8. $x^2 + y^2 - 8x - 6y = 24$
$2y + 3x - 7 = 0$

9. $y = 3x^2 + 6x$
$x = 3y^2 - 5$

10. $4x^2 + 9y^2 = 36$
$y^2 = 4x$

12 Integral Rational Functions

73. CALCULATING THE VALUES OF AN INTEGRAL RATIONAL FUNCTION

The value of the integral rational function

$$f(x) = A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n,$$

$A_0 \neq 0$, when $x = a$, is evidently

$$f(a) = A_0a^n + A_1a^{n-1} + A_2a^{n-2} + \cdots + A_{n-1}a + A_n. \tag{1}$$

If we let

$$\left.\begin{aligned} A_0a + A_1 &= B_1 \\ B_1a + A_2 &= B_2 \\ B_2a + A_3 &= B_3 \\ B_3a + A_4 &= B_4 \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ B_{n-2}a + A_{n-1} &= B_{n-1} \\ B_{n-1}a + A_n &= B_n \end{aligned}\right\} \tag{2}$$

it is readily seen that

$$\begin{aligned} B_1 &= A_0a + A_1 \\ B_2 &= A_0a^2 + A_1a + A_2 \\ B_3 &= A_0a^3 + A_1a^2 + A_2a + A_3 \\ B_4 &= A_0a^4 + A_1a^3 + A_2a^2 + A_3a + A_4 \\ \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \\ B_{n-1} &= A_0a^{n-1} + A_1a^{n-2} + A_2a^{n-3} + \cdots + A_{n-2}a + A_{n-1} \\ B_n &= A_0a^n + A_1a^{n-1} + A_2a^{n-2} + \cdots + A_{n-1}a + A_n \end{aligned}$$

Thus,

$$B_n = f(a). \tag{3}$$

The Equations (2) therefore give us the following method of calculating $f(a)$:

We write a to the right of the coefficients of the terms of $f(x)$ as follows:

$$A_0 + A_1 + A_2 + \cdots + A_{n-1} + A_n \,\underline{|a.}$$

We multiply A_0 by a and add to A_1. This gives B_1. We multiply B_1 by a

and add to A_2. This gives B_2. We multiply B_2 by a and add to A_3. This gives B_3. We continue this process until we multiply B_{n-1} by a and add to A_n. This gives $B_n = f(a)$. The entire process can be arranged conveniently in the following manner:

$$\begin{array}{rrrrrrl} A_0 + A_1 & + A_2 & + \cdots & + A_{n-1} & + A_n & \underline{|a} \\ & A_0a & + B_1a & + \cdots + B_{n-2}a & + B_{n-1}a & \\ \hline A_0 + B_1 & + B_2 & + \cdots & + B_{n-1} & + B_n & \end{array}$$

Illustration 1: The method of this article for finding $f(a)$ is often, but not always, much briefer than the actual substitution of a for x in $f(x)$.

Thus, to find $f(2)$ when

$$f(x) = 8x^5 + 9x^4 - 3x^3 + 2x^2 + 7x + 1,$$

we proceed by the method of this section as follows:

$$\begin{array}{rrrrrrl} 8 & + \ 9 & - \ 3 & + \ 2 & + \ 7 & + \ 1 & \underline{|2} \\ & + 16 & + 50 & + 94 & + 192 & + 398 & \\ \hline 8 & + 25 & + 47 & + 96 & + 199 & + 399 & \end{array}$$

Hence, $$f(2) = 399.$$

By the method of substitution, we have

$$8(2)^5 + 9(2)^4 - 3(2)^3 + 2(2)^2 + 7(2) + 1$$
$$= 256 + 144 - 24 + 8 + 14 + 1 = 399$$

Illustration 2: Find $f(-3)$ if $f(x) = 5x^4 + 7x^2 - 9$.

To find $f(-3)$, we write -3 to the right of the coefficients of the terms of $f(x)$, inserting zeros for the coefficients of the missing powers of x; then we proceed as before.

$$\begin{array}{rrrrrl} 5 & + \ 0 & + \ 7 & + \ 0 & - \ 9 & \underline{|-3} \\ & - 15 & + 45 & - 156 & + 468 & \\ \hline 5 & - 15 & + 52 & - 156 & + 459 & \end{array}$$

Hence, $$f(-3) = 459.$$

EXERCISES 50

1. Given $f(x) = x^4 - 5x^3 + 7x^2 + x - 1$, find $f(1), f(2), f(3), f(-1), f(0)$.
2. Given $f(x) = 7x^4 - 8x^2 + x - 5$, find $f(2), f(3), f(-2), f(-4)$.
3. Given $f(x) = 5x^5 - 4x^4 + 3x^3 - 2x^2 + x$, find $f(-1), f(2), f(5), f(10)$.
4. Given $f(x) = 2x^6 - 3x^4 + 5x - 6$, find $f(2), f(-1), f(-2)$.
5. Given $f(x) = x^5 - 2x^4 + 5x^3 - x^2 + 7x + 3$, find $f(2), f(-1), f(3), f(-3)$.

74. SYNTHETIC DIVISION

If we let

$$Q(x) = A_0x^{n-1} + B_1x^{n-1} + B_2x^{n-2} + \cdots + B_{n-1},$$

and multiply both members by $x - a$, we have

$$(x - a)Q(x) = A_0x^n + (B_1 - A_0a)x^{n-1} + (B_2 - B_1a)x^{n-2} + \cdots + (B_{n-1} - B_{n-2}a)x - B_{n-1}a.$$

After referring to relations (2) of Section 73, we note that

$$(x - a)Q(x) = A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n - B_n,$$

and by (1) and (3) (Section 73),

$$(x - a)Q(x) = f(x) - f(a).$$

Hence,
$$f(x) = (x - a)Q(x) + f(a).$$

The last equation states specifically that if $f(x)$ is divided by $(x - a)$, we obtain $Q(x)$ as the quotient and $f(a)$ as the remainder.

Since the method of Section 73 determines $B_1, B_2, \cdots, B_{n-1}$, as well as $B_n = f(a)$, we have a short method of finding the quotient as well as the remainder when $f(x)$ is divided by $x - a$. This method is known as *synthetic division.* It is to be noted that synthetic division can be used only if the divisor is a linear expression of the form x − a.

Illustration: Find the quotient and the remainder when

$$f(x) = 3x^3 - 5x^2 + 7x - 6$$

is divided by $x - 2$.

Solution:

$$\begin{array}{l|l} 3 - 5 + 7 - \ \ 6 & \underline{2} \\ \quad + 6 + 2 + 18 & \\ \hline 3 + 1 + 9 + 12 & \end{array}$$

The numbers 3, 1, and 9 are the coefficients of the powers of x in the quotient $Q(x)$, and $12 = f(2)$ is the remainder R. Hence,

$$Q(x) = 3x^2 + x + 9 \qquad \text{and} \qquad R = 12.$$

75. THE REMAINDER THEOREM

If a rational integral function of x is divided by $x - a$, the remainder is $f(a)$. This is a restatement of a fact established in the previous section.

76. THE FACTOR THEOREM

If a is a root of the equation

$$A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n = 0 \qquad (A_0 \neq 0),$$

then $x - a$ is a factor of the polynomial

$$A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n;$$

and, conversely, if $x - a$ is a factor of the polynomial, then a is a root of the equation formed by equating the polynomial to zero.

This may be proved by means of the remainder theorem. Thus, let

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n,$$

and recall that

$$f(x) = f(a) + (x - a)Q(x).$$

Since a is a root of $f(x) = 0$, it is a zero of $f(x)$; that is, $f(a) = 0$. Hence, $f(x) = (x - a)Q(x)$, which shows that $(x - a)$ is a factor of $f(x)$.

Conversely, since $x - a$ is a factor of $f(x)$, this means that the remainder, when $f(x)$ is divided by $x - a$, is zero. But since $f(a)$ equals the remainder, $f(a) = 0$, which implies that a is a root of $f(x) = 0$.

EXERCISES 51

1. Is 2 a zero of $x^3 + 6x^2 + 11x - 6$?
2. Is $(x - 1)$ a factor of $x^{99} - 1$?
3. What is the constant remainder when $2x^{17} - 3$ is divided by $x - 1$?
4. What is the constant remainder when $2x^{33} - 3x^{17} + 1$ is divided by $x + 1$?
5. Without actual division, show that $x - 4$ is a factor of $x^3 - 6x^2 + 6x + 8$.
6. By synthetic division find $f(4)$ when $f(x) = x^3 - 7x^2 + 3x + 14$.
7. Find $f(-2)$ by synthetic division when $f(x) = x^3 - 7x^2 + 3x + 14$.
8. Find the quotient and the constant remainder
   (a) if $x^3 + 7x^2 - 8x + 3$ is divided by $x - 3$;
   (b) if $x^3 + 2x^2 + 13x - 1$ is divided by $x - 4$;
   (c) if $2x^3 + 3x^2 - 7x + 6$ is divided by $x + 2$;
   (d) if $x^4 + 5x^2 - 9x - 7$ is divided by $x + 5$;
   (e) if $2x^3 - 9x^2 + 10x - 3$ is divided by $x - \frac{1}{2}$.

By the use of synthetic division determine the linear factors of the following five polynomials:

9. $x^3 - 2x^2 - 5x + 6$

Solution: By trial it is discovered that $f(-2)$, $f(1)$, and $f(3)$ are all zero. Thus, the factors are $(x + 2)$, $(x - 1)$, and $(x - 3)$.

10. $x^3 + 4x^2 - 4x - 16$
11. $x^3 + 2x^2 - x - 2$
12. $x^3 + x^2 - 14x - 24$
13. $x^4 - 10x^3 + 35x^2 - 50x + 24$
14. Determine the roots of the equations formed by equating each of the previous five polynomials to zero.

77. THE FUNDAMENTAL THEOREM OF ALGEBRA

Every integral rational function $f(x)$ has at least one zero. This theorem will be assumed in view of the fact that the various proofs that are available are all too difficult for an elementary text.

78. FACTORS OF A POLYNOMIAL

The polynominal

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$$

may be written as

$$f(x) = A_0(x - r_1)(x - r_2) \cdots (x - r_n),$$

where $r_1, r_2, \cdots, r_n$, are the zeros of $f(x)$. This may be proved as follows:

By the preceding theorem $f(x)$ has at least one zero, say r_1 and, hence, by the factor theorem $x - r_1$ is a factor of $f(x)$.

Hence, $f(x) = (x - r_1) F(x)$, where $F(x)$ is the polynomial of degree $n - 1$ obtained on dividing $f(x)$ by $x - r_1$. Similarly, by the theorem of Section 77, $F(x)$ has at least one zero, say r_2 (where r_2 need not necessarily differ from r_1), and, hence, $x - r_2$ is a factor of $F(x)$, so that $f(x)$ may be written as

$$f(x) = (x - r_1)(x - r_2)G(x),$$

where $G(x)$ is the polynomial of degree $n - 2$, obtained on dividing $F(x)$ by $x - r_2$. But upon dividing $f(x)$ by $x - r_1$, and then the quotient by $x - r_2$, and then the next quotient by $x - r_3$, where r_3 is a zero of $G(x)$, this reasoning may be continued until the final quotient becomes a constant. This final constant must be A_0, since for every division the first term of the quotient has A_0 as its coefficient. Hence,

$$f(x) = A_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

It is evident from this form of $f(x)$ that $x = r_1$, $x = r_2, \cdots, x = r_n$ are all zeros of $f(x)$, since they cause $f(x)$ to reduce to zero.

79. NUMBER OF ROOTS

Theorem: The integral rational equation $f(x) = 0$, of degree n, has n, and only n, roots. These n roots are not necessarily all distinct.

It has been shown that

$$f(x) = A_0(x - r_1)(x - r_2) \cdots (x - r_n),$$

where $r_1, r_2, \cdots, r_n$ are the zeros of $f(x)$. Clearly $f(x)$ becomes zero when x is put equal to any one of the r's. If k of the linear factors $x - r$ are equal, we say that the equation $f(x) = 0$ has k equal roots. With this understanding there are at least n zeros of the function; that is, there are at least n roots of the equation $f(x) = 0$.

Moreover, if s is any number different from every one of the r's, then

$$f(s) = A_0(s - r_1)(s - r_2) \cdots (s - r_n).$$

A product cannot be zero unless at least one of the factors is zero. Since no one of the factors can be zero, $f(s)$ cannot be zero. Therefore, $f(x)$ has only n zeros; that is to say, $f(x) = 0$ has only n roots.

As an illustration, there are three, and only three, zeros of the function

$$F(x) = x^3 - 3x^2 - 4x + 12 = (x - 2)(x + 2)(x - 3).$$

They are 2, -2, and 3. Thus, these values are the roots of $F(x) = 0$.

80. A TRANSFORMATION OF $A_0x^n + \cdots + A_n = 0$ $(A_0 \neq 0)$

The equation of nth degree, namely,

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n = 0 \qquad (A_0 \neq 0), \tag{1}$$

where the coefficients are rational numbers, may be divided by A_0, since $A_0 \neq 0$, and we have the equivalent equation,

$$x^n + \frac{A_1}{A_0}x^{n-1} + \frac{A_2}{A_0}x^{n-2} + \cdots + \frac{A_n}{A_0} = 0.$$

The coefficients of this equation are rational but not necessarily integers.

If we let $y = kx$, then we have

$$\frac{y^n}{k^n} + \frac{A_1}{A_0}\frac{y^{n-1}}{k^{n-1}} + \frac{A_2}{A_0}\frac{y^{n-2}}{k^{n-2}} + \cdots + \frac{A_n}{A_0} = 0,$$

or

$$y^n + k\frac{A_1}{A_0}y^{n-1} + k^2\frac{A_2}{A_0}y^{n-2} + \cdots + k^n\frac{A_n}{A_0} = 0.$$

We may choose k so that all the coefficients of the last equation are integers. Of course, there is an endless number of such values of k; however, the smallest integral value of k that will convert the coefficients to integers will be preferable for purposes of later calculations.

We then have the equation

$$y^n + C_1y^{n-1} + C_2y^{n-2} + \cdots + C_n = 0, \tag{2}$$

wherein the coefficient of the highest power is 1, and the other coefficients are integers. We have thus transformed equation (1) to this new and equivalent form by means of the transformation $y = kx$, the k being properly chosen.

The equation of form (2) will be referred to as the standard integral rational equation. If the roots of (2) are $r_1, \cdots, r_n$, the roots of (1) are $\frac{r_1}{k}, \frac{r_2}{k}, \cdots, \frac{r_n}{k}$. Thus, the roots of (1) may be found by finding the roots of (2) and dividing each root by k.

81. RATIONAL ROOTS OF THE STANDARD INTEGRAL RATIONAL EQUATION

Theorem 1. The equation

$$y^n + C_1y^{n-1} + C_2y^{n-2} + \cdots + C_{n-1}y + C_n = 0, \tag{1}$$

where $C_1, C_2, \cdots, C_n$ are integers, cannot have a rational fraction other than an integer as a root.

Let us assume that $y = p/q$ is a root of (1), where p and q are integers without a common divisor, and $q \neq 1$. Thus,

$$\frac{p^n}{q^n} + C_1 \frac{p^{n-1}}{q^{n-1}} + \cdots + C_n = 0.$$

If we multiply each member by q^{n-1}, we have

$$\frac{p^n}{q} + C_1p^{n-1} + \cdots + C_nq^{n-1} = 0,$$

or

$$\frac{p^n}{q} = -(C_1p^{n-1} + \cdots + C_nq^{n-1}). \tag{2}$$

Since p is not divisible by q, p^n is not divisible by q; hence, p^n/q is a fraction. Therefore, the left member of the equation cannot equal the right member, which is an integer.

Hence, the original assumption is impossible, and Equation (1) cannot have a rational root other than an integer.

Theorem 2. Any integral root of (1) is a divisor of C_n. Since

$$C_n = -y(y^{n-1} + C_1y^{n-2} + \cdots + C_{n-1}),$$

we see that any integer y_1 which satisfies Equation (1) satisfies the relation

$$C_n = -y(y^{n-1} + C_1y^{n-2} + \cdots + C_{n-1}).$$

Consequently, C_n is an integer that has y_1 as a factor.

Hence, to find the integral roots of an equation in the standard form, we need try only the positive and negative integers that are factors of C_n.

The rational roots of $A_0x^n + A_1x^{n-1} + \cdots + A_n = 0$ may be found by transforming the equation to the standard form, finding the integral roots of the standard equation, and then dividing each root by k.

82. DESCARTES'S RULE OF SIGNS

In the equation

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n = 0 \quad (A_0 \neq 0),$$

the coefficients other than A_0 may be either positive, negative, or zero. In any given equation the sign of each coefficient is known. Thus, in the equation $x^5 - 4x^4 + 3x^2 + 7x - 8 = 0$, the signs of the terms in their proper order are $+ - + + -$. If two successive terms differ in sign, there is said to be a change of sign. It follows that the function

$$x^5 - 4x^4 + 3x^2 + 7x - 8$$

has a change of sign between the first and second terms, between the second and third terms, and between the fourth and fifth terms. Hence, this function is said to have three changes of sign.

If we consider the given polynomial $f(x)$, wherein x is replaced by $-x$,

we have $f(-x) = -x^5 - 4x^4 + 3x^2 - 7x - 8$; the signs of the terms of $f(-x)$ are $- - + - -$. This function $f(-x)$ has a change of sign between the second and third terms and between the third and fourth terms. Hence, $f(-x)$ is said to have two changes of sign.

Now suppose that the equation

$$f(x) = A_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0,$$

has the roots, $x = r_1$, $x = r_2$, $\cdots$, $x = r_n$. Then,

$$f(-x) = A_0(-x - r_1)(-x - r_2) \cdots (-x - r_n) = 0$$

has the roots, $x = -r_1$, $x = -r_2, \cdots$, $x = -r_n$. Hence, it is evident that the roots of $f(x) = 0$ are the roots of $f(-x) = 0$ with their signs changed.

It is demonstrated in advanced algebra that the number of positive roots of $f(x) = 0$ does not exceed the number of changes of sign in $f(x)$. We thus have the following rule:

Descartes's Rule. The number of positive roots of the equation

$$f(x) = A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n = 0 \quad (A_0 \neq 0),$$

does not exceed the number of changes of sign in $f(x)$; and the number of negative roots does not exceed the number of changes of sign in $f(-x)$.

We shall now give two illustrations of the common procedure for finding the rational roots of equations with rational coefficients.

Illustration 1: Find the rational roots of

$$x^3 - 9x^2 + 23x - 15 = 0.$$

By Section 81 if this equation has any rational roots, they are integral factors of 15. The direct application of Descartes's rule of signs to the polynomial on the left tells us that the equation has at most three positive roots. Of course, since the equation is of third degree, there are only three roots of any kind.

Since $f(-x) = -x^3 - 9x^2 - 23x - 15$, there are no changes of sign, and, by Descartes's rule, the given equation has no negative roots. Consequently, to obtain the rational roots, we need try only the positive factors of 15, that is, 1, 3, 5, 15. This trial will be accomplished by the use of synthetic division.

$$\begin{array}{r|l} 1 - 9 + 23 - 15 & \underline{1} \\ +1 - 8 + 15 & \\ \hline 1 - 8 + 15 + 0 & \end{array}$$

Since $f(1) = 0$, 1 is a root of the given equation and $(x - 1)$ is a factor of the polynomial member.

Since $Q(x) = x^2 - 8x + 15$, the polynomial also has the factors $(x - 5)$ and $(x - 3)$.

Hence, 5 and 3 are the remaining roots. In this case all the roots are integers.

Illustration 2: Find the rational roots of

$$6x^4 - x^3 - 13x^2 + 2x + 2 = 0.$$

After dividing the members of the given equation by 6, we have

$$x^4 - \tfrac{1}{6}x^3 - \tfrac{13}{6}x^2 + \tfrac{2}{6}x + \tfrac{2}{6} = 0.$$

Let $y = kx$, then we have

$$\frac{y^4}{k^4} - \frac{1}{6}\frac{y^3}{k^3} - \frac{13}{6}\frac{y^2}{k^2} + \frac{1}{3}\frac{y}{k} + \frac{1}{3} = 0,$$

or

$$y^4 - \frac{k}{6}y^3 - k^2\frac{13}{6}y^2 + \frac{k^3}{3}y + \frac{k^4}{3} = 0.$$

If we choose $k = 6$, we have

$$y^4 - y^3 - 78y^2 + 72y + 432 = 0.$$

The rational roots of this equation in y, if there are any, are factors of 432. The factors that may be tried are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 9, ± 12, $\cdots$.

By Descartes's rule we know that the given equation can have at most two positive roots and at most two negative roots.

Let us try -2.

$$\begin{array}{rrrrr|l} 1 - & 1 - & 78 + & 72 + & 432 & \underline{-2} \\ & -2 + & 6 + & 144 - & 432 & \\ \hline 1 - & 3 - & 72 + & 216 + & 0 & \end{array}$$

Hence, $$y = -2.$$

The remaining roots of $y^4 - y^3 - 78y^2 + 72y + 432 = 0$ are the zeros of the quotient $y^3 - 3y^2 - 72y + 216$.

Of course, -2 may be a zero of this quotient, and thus -2 would be a multiple root of the original equation in y. So let us try -2 again.

$$\begin{array}{rrrr|l} 1 - & 3 - & 72 + & 216 & \underline{-2} \\ & -2 + & 10 + & 124 & \\ \hline 1 - & 5 - & 62 + & 340 & \end{array}$$

It is immediately evident that -2 is not a multiple root.

We shall now try some other integer that is a possible root; for instance, let us try 2.

$$\begin{array}{rrrr|l} 1 - & 3 - & 72 + & 216 & \underline{2} \\ & 2 - & 2 - & 148 & \\ \hline 1 - & 1 - & 74 + & 68 & \end{array}$$

Hence, 2 is not a root.

Let us next consider 3.

$$\begin{array}{r|l} 1 - 3 - 72 + 216 & \underline{3} \\ +3 + \ \ 0 - 216 & \\ \hline 1 + 0 - 72 + \quad 0 & \end{array}$$

Hence, $y = 3$ is a root.

The quotient obtained in the last synthetic division, when equated to zero, gives the equation

$$y^2 - 72 = 0;$$

so,

$$y = \pm 6\sqrt{2}.$$

Hence, the rational roots of the given equation in x are $-\frac{2}{6}$, $\frac{3}{6}$, that is, $x = -\frac{1}{3}$ and $x = \frac{1}{2}$.

The remaining roots are the irrational values $x = \pm\sqrt{2}$.

Since the given equation is of the fourth degree, we have found all the roots.

EXERCISES 52

Find the roots of the following equations:

1. $x^3 - x^2 - 14x + 24 = 0$
2. $x^3 - 7x^2 - 5x + 35 = 0$
3. $x^4 - 10x^3 + 20x^2 + 10x - 21 = 0$
4. $x^4 - 10x^3 + 24x^2 + 10x - 25 = 0$
5. $x^3 - 7x + 6 = 0$
6. $x^4 + 8x^3 + 23x^2 + 30x + 18 = 0$
7. $x^4 + x^3 - 9x^2 + 11x - 4 = 0$
8. $x^4 + x^3 - 4x^2 - 4x = 0$
9. $6x^3 + 13x^2 + 9x + 2 = 0$
10. $9x^3 - 27x^2 + 20x - 4 = 0$
11. $4x^3 - 8x^2 - x + 2 = 0$
12. $12x^3 - 23x^2 + 13x - 2 = 0$
13. $4x^3 - 4x^2 - 9x + 9 = 0$
14. $4x^3 + 8x^2 - 11x + 3 = 0$
15. $4x^4 + 8x^3 - 7x^2 - 21x - 9 = 0$
16. $14x^3 + 13x^2 - 4x - 3 = 0$
17. $5x^3 - 24x^2 - 9x + 20 = 0$
18. $6x^3 - 13x^2 + 4 = 0$
19. $3x^3 - 5x^2 - 6x + 10 = 0$
20. $9x^4 - 24x^3 - 25x^2 + 24x + 16 = 0$
21. The hypotenuse of a right triangle is 35 ft long, and its area is 294 sq ft. Determine the two legs.
22. The sum of the squares of the first n positive integers is given by the formula $n(n + 1)(2n + 1)/6$. How many terms would yield a sum of 2870?

83. RATIONAL APPROXIMATION OF THE IRRATIONAL ROOTS OF $A_0x^n + \cdots + A_n = 0$

Theorem: If $f(x)$ is a rational integral function, and if for any two real values of x, such as $x = a$ and $x = b$, $f(a)$ and $f(b)$ have opposite signs, at least one real root of the equation $f(x) = 0$ lies between a and b.

To be specific, suppose that $f(a)$ is negative and $f(b)$ is positive. It is proved in higher mathematics that the graph of $y = f(x)$ is a continuous

curve and that as x passes from $x = a$ to $x = b$, $f(x)$ passes through all real values between $f(a)$ and $f(b)$. Hence, at least one of the values of $f(x)$ is zero. The corresponding value of x is therefore a root of the equation $f(x) = 0$.

From the geometric point of view the theorem states that if the graph is below the x-axis at $x = a$, and above it at $x = b$, it must cross the axis at least once between $x = a$ and $x = b$; the situation is displayed in Figure 38 where point P denotes the point of crossing.

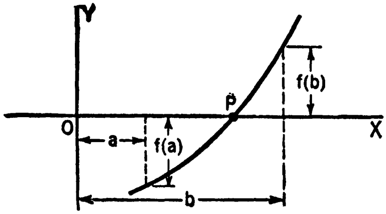


FIG. 38

If the real roots of an equation do not lie too close together, this theorem furnishes a convenient means of locating them, at least approximately, between two consecutive integers.

Illustration: Solve the equation $x^3 - 6x^2 + 6x + 5 = 0$.

This equation has no rational roots, for by Section 81 the only possible rational roots are ± 1 and ± 5. By trial it may be shown that ± 1 and ± 5 are not roots.

The following table of values may easily be computed either by direct substitution or by synthetic division. The graph indicated by the values thus obtained is shown in Figure 39.

$$y = f(x) = x^3 - 6x^2 + 6x + 5$$

x	y
-2	-39
-1	-8
0	5
1	6
2	1
3	-4
4	-3
5	10
6	41

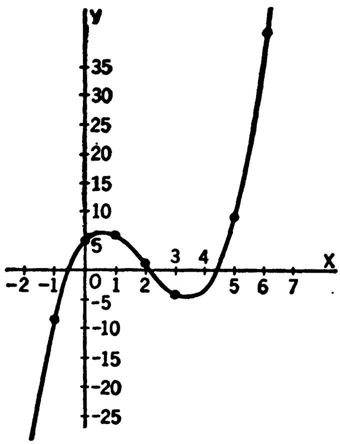


FIG. 39

The table of values shows that the polynomial function changes sign between $x = -1$ and $x = 0$, between $x = 2$ and $x = 3$, and between $x = 4$ and $x = 5$. We have thus located the desired roots of the equation be-

tween consecutive integers. If the graph is drawn carefully, we may estimate the values of the roots to sufficient degree of precision for many purposes.

To obtain a closer approximation to an irrational root as, for example, the root between 2 and 3, we proceed as follows:

From the tabulated values we have

x	$f(x)$
2	1
3	-4

By the method of interpolation, we substitute a straight line for the given function between $x = 2$ and $x = 3$, as shown in Figure 40.

If $(2 + d, 0)$ are the coordinates of the intersection of this line with the x axis, we have the following tabulation:

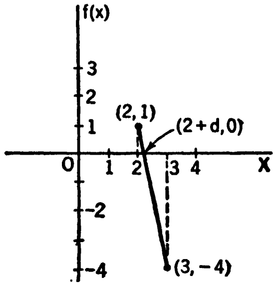


FIG. 40

x	$f(x)$
2	1
$2 + d$	0
3	-4

For the straight-line construction we are assuming that differences in values of x are proportional to differences in the corresponding values of the function; thus,

$$\frac{(2 + d) - 2}{3 - 2} = \frac{0 - 1}{-4 - 1},$$

or

$$d = \tfrac{1}{5} = 0.2.$$

The student may also show that $d = 0.2$ from the similar right triangles appearing in Figure 40.

Therefore, our first approximation to the root is 2.2, which is not necessarily a correct solution to the nearest first decimal.

By synthetic division we find that $f(2.2) = -0.192$; hence, $x = 2.2$ is larger than the required root. Similarly, we find that $f(2.1) = +0.401$; hence, $x = 2.1$ is smaller than the required root.

A second approximation may now be found by interpolation. Thus,

we have the tabulation

x	$f(x)$
2.1	+ 0.401
2.1 + d'	0
2.2	− 0.192

Hence, $$\frac{(2.1 + d') - 2.1}{2.2 - 2.1} = \frac{0 - 0.401}{-0.192 - 0.401}, \quad \text{or } d' = 0.067.$$

Therefore, our second approximation to the root is $2.1 + 0.067 = 2.167$ or, better, 2.17.

By synthetic division for $x = 2.16$, $f(x) = 0.0441$, and for $x = 2.17$, $f(x) = -0.0151$, which shows that our approximation is correct to three significant figures.

We may thus proceed to find the desired root to any degree of approximation.

EXERCISES 53

Find the irrational roots of each of the following equations to three significant figures:

1. $x^3 - 2x^2 - 7x - 10 = 0$
2. $x^4 - 3x^3 + x^2 + 3 = 0$
3. $x^3 - 3x + 7 = 0$
4. $x^3 + 4x^2 - 6 = 0$
5. $2x^3 - 3x^2 - 4x + 2 = 0$
6. $3x^3 + 5x^2 - 9 = 0$
7. $2x^4 - 9x - 3 = 0$
8. $x^3 - 5x^2 - 10x + 15 = 0$
9. $x^3 - 2.34x - 7.85 = 0$

84. HORNER'S METHOD

Another method of approximation of the irrational roots is known as Horner's method. This method is based on the operation of transforming a given equation into another whose roots are less by a fixed number than the roots of the original equation.

Suppose it is required to diminish the roots of the equation $f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0$ by the fixed number h. If x is replaced by $x' + h$, the new equation is

$$f(x' + h) = a_0(x' + h - r_1)(x' + h - r_2) \cdots (x' + h - r_n) = 0.$$

The roots of this equation obtained by setting the separate linear factors equal to zero are

$$x_1' = r_1 - h, \; x_2' = r_2 - h, \cdots, x_n' = r_n - h.$$

Thus, each root of the new equation is equal to the corresponding root of the old equation diminished by h. The required transformation is therefore accomplished by writing $x' + h$ for x, or, as it is ordinarily expressed,

by making the substitution

$$x = x' + h.$$

If h is a negative number, the transformation will increase all the roots of the original equation by $-h$.

Illustration: Find the approximate value of one root of the equation

$$x^4 - x^3 - 2x^2 - 3x - 1 = 0.$$

First Transformation: By the use of methods already discussed, we find that a root lies between $x = 2$ and $x = 3$.

If, therefore, the equation is transformed by using the substitution $x = x' + 2$, the corresponding root of the new equation will lie between 0 and 1. If $f(x)$ designates the polynomial of the given equation, we may write

$$f(x' + 2) = (x' + 2)^4 - (x' + 2)^3 - 2(x' + 2)^2 - 3(x' + 2) - 1$$

or $$f(x' + 2) = A_0(x')^4 + A_1(x')^3 + A_2(x')^2 + A_3x' + A_4,$$

where the A's are constants to be determined. Since $x = x' + 2$, $x' = x - 2$. Hence, if x' is replaced by $x - 2$, the resulting function will be identical with the original $f(x) = x^4 - x^3 - 2x^2 - 3x - 1$. Thus,

$$f(x) = A_0(x - 2)^4 + A_1(x - 2)^3 + A_2(x - 2)^2 + A_3(x - 2) + A_4.$$

Clearly, the remainder obtained by dividing the right member of this identity by $x - 2$ is A_4. But the remainder after dividing the left member of the identity, that is, of the original polynomial, by $x - 2$ is -7. Therefore, $A_4 = -7$.

Ignoring the remainder, the quotient obtained by dividing the left member by $x - 2$ is

$$Q_1(x) = x^3 + x^2 + 0x - 3.$$

The quotient, ignoring the remainder, obtained by dividing the right member by $x - 2$ is

$$Q_1(x) = A_0(x - 2)^3 + A_1(x - 2)^2 + A_2(x - 2) + A_3.$$

The coefficient A_3 may be found in exactly the same way that A_4 was found, namely, by dividing both sides of the new identity by $x - 2$. The process can be continued in this way until all the coefficients are found.

The successive quotients and remainders may be found by synthetic division, and the work may be arranged in almost mechanical fashion as follows:

$\begin{array}{r} 1 - 1 - 2 - 3 - 1 \ \lfloor 2 \\ 2 + 2 + 0 - 6 \\ \hline 1 + 1 + 0 - 3 - 7 \end{array}$	$A_4 = -7$ $Q_1(x) = x^3 + x^2 - 3$
$\begin{array}{r} 1 + 1 + 0 - 3 \ \lfloor 2 \\ 2 + 6 + 12 \\ \hline 1 + 3 + 6 + 9 \end{array}$	$A_3 = 9$ $Q_2(x) = x^2 + 3x + 6$
$\begin{array}{r} 1 + 3 + 6 \ \lfloor 2 \\ 2 + 10 \\ \hline 1 + 5 + 16 \end{array}$	$A_2 = 16$ $Q_3(x) = x + 5$
$\begin{array}{r} 1 + 5 \ \lfloor 2 \\ 2 \\ \hline 1 + 7 \end{array}$	$A_1 = 7$ $Q_4(x) = 1$
1	$A_0 = 1$

The transformed equation is, therefore,

$$x^4 + 7x^3 + 16x^2 + 9x - 7 = 0.$$

The previous work may be condensed as follows:

$$\begin{array}{l} 1 - 1 - \ 2 - \ 3 - 1 \ \lfloor 2 \\ \quad\ 2 + \ 2 + \ 0 - 6 \\ \hline 1 + 1 + \ 0 - \ 3 - \mathit{7} \\ \quad\ 2 + \ 6 + 12 \\ \hline 1 + 3 + \ 6 + \ \mathit{9} \\ \quad\ 2 + 10 \\ \hline 1 + 5 + \mathit{16} \\ \ \ + 2 \\ \hline 1 + \mathit{7} \\ \hline \mathit{1} \end{array}$$

Second Transformation: Since a root of the original equation

$$x^4 - x^3 - 2x^2 - 3x - 1 = 0$$

lies between 2 and 3, the corresponding root of

$$x^4 + 7x^3 + 16x^2 + 9x - 7 = 0$$

must lie 2 to the left of its original value, that is, between 0 and 1. Trial by synthetic division shows that the root lies between 0.4 and 0.5. If the roots of $x^4 + 7x^3 + 16x^2 + 9x - 7 = 0$ are diminished by 0.4, the corresponding root of the new equation will lie between 0 and 0.1.

The work is arranged as follows:

$$
\begin{array}{llllll}
1 + 7 & + 16 & + 9 & - 7 & & \underline{|0.4} \\
 + 0.4 + & 2.96 + & 7.584 + & 6.6336 & & \\
\hline
1 + 7.4 + & 18.96 + & 16.584 - & \mathit{0.3664} & & \\
 + 0.4 + & 3.12 + & 8.832 & & & \\
\hline
1 + 7.8 + & 22.08 + & \mathit{25.416} & & & \\
 + 0.4 + & 3.28 & & & & \\
\hline
1 + 8.2 + & \mathit{25.36} & & & & \\
 + 0.4 & & & & & \\
\hline
1 + \mathit{8.6} & & & & & \\
\hline
\mathit{1} & & & & &
\end{array}
$$

The new equation is, therefore,

$$x^4 + 8.6x^3 + 25.36x^2 + 25.416x - 0.3664 = 0.$$

Third Transformation: Since the root of the last equation lies between 0 and 0.1, the sum of the first three terms will be very small when the value of the root is substituted into the equation; so the root may be found approximately by neglecting the higher powers of x and solving the linear equation

$$25.416x - 0.3664 = 0.$$

Consequently,

$$x = \frac{0.3664}{25.416}, \text{ approximately.}$$

The value of this fraction is between 0.01 and 0.02.

After combining these results, it is apparent that the particular root under investigation in the given equation is $2 + 0.4 + 0.01 = 2.41$, approximately. This value is probably accurate enough for most purposes.

Of course, the equation

$$x^4 + 8.6x^3 + 25.36x^2 + 25.416x - 0.3664 = 0$$

could be transformed again, and a third decimal place of our root could be found. In fact, the process may be continued to any degree of approximation that may be required.

Negative Irrational Solutions. In order to find approximations to the negative, irrational roots of $f(x) = 0$, we may use Horner's method to find approximations to the positive irrational roots of $f(-x) = 0$. These with their signs changed are the roots sought.

85. SUMMARY FOR FINDING ROOTS

In order to find all the real roots of an equation $f(x) = 0$, in which

$f(x)$ is a polynomial with rational, numerical coefficients, proceed as follows:

(1) Find all the rational roots by the method described in Section 82. Each rational root of the given equation in x corresponds to an integral root of the transformed equation in y (Section 82). If $y = r_1, r_2, \cdots, r_s$, where $y = kx$, are the integral roots of the equation in y, then the equation in y is of the form

$$f(y) = (y - r_1)(y - r_2) \cdots (y - r_s)\phi(y) = 0,$$

where the degree of $\phi(y)$ is s less than the degree of $f(y)$. The polynomial $\phi(y)$ is the final quotient obtained after successively dividing the original polynomial in y by the factors $(y - r_1)$, $(y - r_2)$, $\cdots$, $(y - r_s)$.

(2) Approximations to the irrational roots of the original equation may be found by solving $\phi(y)$ as follows:

See if $\phi(y)$ has any positive roots, and determine by synthetic division two consecutive integers between which such a root lies.

Form a new equation whose roots are the roots of this equation each diminished by the smaller of these two integers (Section 84). The resulting equation has a root between 0 and 1. Find by synthetic division the two consecutive tenths between which the root lies.

Form a new equation whose roots are the roots of this equation each diminished by the smaller of these tenths. The resulting equation has a root between 0 and 0.1. Find by synthetic division two consecutive hundredths between which this root lies.

If the root is required to r decimal places, continue this process until $r + 1$ decimal places have been determined.

Add the amounts by which the root of the successive equations have been diminished. This sum is the root sought to the required degree of approximation.

After determining the irrational root to tenths or to hundredths, it is usually possible to determine the next decimal place by solving the linear equation resulting from ignoring all powers of y higher than the first in the last transformed equation.

If there are other positive, irrational roots, find each of them in the same way.

It may happen that more than one irrational root is contained between two consecutive integers. These roots may be located by means of the principle of Section 83, by choosing other values of y sufficiently close to each other between these two integers.

In order to find the negative irrational solutions, find the positive irrational roots of $\phi(-y) = 0$, and change the sign of each.

Since $y = kx$, the roots of $f(x) = 0$ are the roots just found for $\phi(y) = 0$, each divided by k.

EXERCISES 54

Find the values of the real roots of the following equations correct to two decimal places:

1. $2x^3 - 4x^2 - 10x + 3 = 0$
2. $5x^3 - 3x^2 - 6x + 3 = 0$
3. $4x^3 = 3x - \frac{3}{4}$
4. $2x^3 - 3x + \frac{1}{4} = 0$
5. $x^3 - 2x - 2 = 0$

6. Find the cube root of 3 to three decimal places. HINT: Find the approximate value of the real root of the equation $x^3 = 3$.

7. A prism with a square base and with a volume equal to 250 cu in. is inscribed in a sphere of radius 10 in. Find the altitude of the prism.

8. The volume of a right circular cylinder is 200 cu in., and its total surface is 200 sq in. Find its height and the radius of its base.

9. A right circular cylinder is inscribed in a right circular cone. If the altitude of the cone is 10 in. and the radius of its base 8 in., find the dimensions of the cylinder whose volume is one third that of the cone.

10. The weight of a sphere 2 ft in diameter is 85 lb. To what depth will this sphere sink when floated in a tank of water weighing 62.5 lb per cu ft? [The weight of water displaced is equal to the weight of the floating body. The volume of a spherical segment of one base equals $\pi\left(rh^2 - \frac{h^3}{3}\right)$, where r is the radius of the sphere and h is the height of the segment.]

11. A safe is to have the outside dimensions 4 ft by 4 ft by 6 ft. How thick may the metal walls be constructed if the inside capacity of the safe is to be at least 60 cu ft?

12. In trigonometry it is learned that the sine of a small angle x, where x is measured in radians, is given approximately by $x - \frac{x^3}{6}$. What is the approximate value of x in radians if the sine is $\frac{1}{2}$?

13. Solve each of the equations given in Exercises 53 by the use of Horner's method.

13 Logarithms

86. LOGARITHMS AS AN AID TO COMPUTATION

Laborious numerical computations involving products, quotients, powers, or roots frequently arise in connection with many problems. Many of these calculations can be performed quite simply by the use of "logarithms"; in fact, many problems that are practically impossible without the use of "logarithms" may be solved easily and quickly by their aid.

87. DEFINITION OF LOGARITHMS

In the equation

$$10^2 = 100$$

the number 10 is known as the *base* and 2 is called the *exponent.* In the language of logarithms, 2 is also known as the *logarithm of 100 when the base is 10.* This latter statement is generally written

$$\log_{10} 100 = 2.$$

In general, if $b^p = n$, where $b > 0$ and $b \neq 1$, then $p = \log_b n$. We may then state the following definition:

Definition: The logarithm of a positive number n to the positive base b is the exponent p which must be applied to b to produce n.

EXERCISES 55

1. Change each one of the following exponential statements to its corresponding logarithmic form:

(*a*) $2^3 = 8$ (*b*) $5^2 = 25$ (*c*) $2^{-1} = \frac{1}{2}$
(*d*) $3^0 = 1$ (*e*) $8^{2/3} = 4$

2. Find the logarithm to the base 3 of each of the following numbers: 27; $\frac{1}{9}$; 1; $\frac{1}{81}$; 3.

3. $\log_2 64 = ?$; $\log_5 25 = ?$; $\log_2 \frac{1}{16} = ?$; $\log_{10} 0.1 = ?$

4. Find x in each of the following: $\log_3 x = 4$; $\log_4 x = -4$; $\log_5 x = 3$.

5. Show that $\log_3 243 = \log_3 9 + \log_3 27$.

6. Show that $\log_{10} 1000 = \log_{10} 100{,}000 - \log_{10} 100$.

7. Find x in each of the following: $\log_x 27 = 3$; $\log_x \frac{1}{81} = -4$; $\log_5 x = 4$; $\log_5 x = -3$.

8. Find x if $\log_2 x = \log_2 32 + \log_2 \frac{1}{8}$.

9. Show that $6 \log_{10} 2 - 3 \log_{10} 3 + \frac{1}{2} \log_{10} 4 = \log_{10} \frac{128}{27}$.

10. Show that $3 \log_b c - 2 \log_b a - \frac{2}{3} \log_b d = \log_b \dfrac{c^3}{a^2\sqrt[3]{d^2}}$.

11. Show that $\frac{1}{2}(3 \log_b a + 5 \log_b c) - 4 \log_b \sqrt{d} = \log_b \dfrac{ac^2\sqrt{ac}}{d^2}$.

88. LAWS OF LOGARITHMS

In the following studies it is presumed that $M > 0$, $N > 0$, $b > 0$, and $b \neq 1$. These limitations are made in all our considerations of logarithms even though such restrictions are not essential to the analysis of all the properties of logarithms.

(I) Logarithm of 1

It is immediately apparent from the definition of logarithm that

$$\log_b 1 = 0.$$

(II) Logarithm of the Base

From the definition

$$\log_b b = 1.$$

Thus, $\log_{10} 10 = 1$; $\log_3 3 = 1$; and so on.

(III) Logarithm of MN

The logarithm of a product of two numbers to the same base is the sum of their logarithms.

Proof: Let $\log_b M = x$, then $b^x = M$.

Let $\log_b N = y$, then $b^y = N$.

Hence, $b^{x+y} = MN$ or $\log_b MN = x + y = \log_b M + \log_b N$.

(IV) Logarithm of $\dfrac{M}{N}$

The logarithm of a fraction is the logarithm of the numerator minus the logarithm of the denominator.

Proof: Let $\log_b M = x$, then $b^x = M$.

Let $\log_b N = y$, then $b^y = N$.

Hence, $b^{x-y} = \dfrac{M}{N}$ or $\log_b \dfrac{M}{N} = x - y = \log_b M - \log_b N$.

(V) Logarithm of M^p

The logarithm of M^p equals p times the logarithm of M.

Proof: Let $\log_b M = x$, then $b^x = M$.

Hence, $(b^x)^p = M^p$ or $b^{px} = M^p$.

This latter equation may be rewritten in the form

$$\log_b M^p = px = p \log_b M.$$

EXERCISES 56

1. Repeat the preceding demonstrations, using 5 as a base.

2. Repeat the preceding demonstrations, using 10 as a base.

3. Give a rough demonstration of the theorem that if $b > 1$, $\log_b M < 0$ if $0 < M < 1$, and $\log_b M > 0$ if $M > 1$. Explain what this theorem means if $b = 10$.

4. Make a graph of the equation $y = \log_2 x$, using only values for x of the type $2^{\pm p}$, p being an integer.

5. *Given:* $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\log_{10} 7 = 0.8451$; $\log_{10} 11 = 1.0414$; $\log_{10} 13 = 1.1139$. Find $\log_{10} 12$.

Solution: $\log_{10} 12 = \log (2^2)(3) = \log 2^2 + \log 3$, by Law III. But, $\log 2^2 = 2 \log 2$, by Law V. Therefore,

$$\log_{10} 12 = 2 \log 2 + \log 3 = 1.0791.$$

Find $\log_{10} 4$; $\log_{10} 5$; $\log_{10} 6$; $\log_{10} 8$; $\log_{10} 39$; $\log_{10} 55$; $\log_{10} \frac{1}{3}$; $\log_{10} \frac{1}{32}$.

89. SCIENTIFIC NOTATION

Any given positive number may be expressed as a number between 1 and 10* multiplied by 10 to some integral exponent, positive or negative. The factor of 10 to an exponent contains all the significant digits involved in the given number (see Section 9). Thus,

$$8635 = 8.635 \times 10^3,$$

$$86.35 = 8.635 \times 10^1,$$

$$0.008635 = 8.635 \times 10^{-3}.$$

The expression of any number in the above form is called the *scientific*, or *standard, notation*. It is readily observed that the exponent upon 10 is numerically equal to the number of places that the decimal point in the given number is displaced from the position after the first non-zero digit; the exponent is positive if the displacement is to the right; negative, if to the left.

EXERCISES 57

Express each of the following numbers in scientific notation:

7069; 1020.4; 0.7624; 0.003157; 0.0002756;
0.0082; 72.56; 0.0000007; 83000000

90. CHARACTERISTIC AND MANTISSA OF COMMON LOGARITHMS

In performing numerical computations, it is common to employ logarithms having the base 10. In fact, *the collection of logarithms to the base*

* To be more specific, $1 \leq N < 10$.

10 comprises the common system of logarithms. When 10 is used as a base, we propose to write merely log N, omitting the base. Thus, we have

$$\log 1 = 0; \quad \log 10 = 1; \quad \log 100 = 2; \quad \log 1000 = 3;$$

$$\log 0.1 = -1; \quad \log 0.01 = -2; \quad \log 0.001 = -3; \text{ and so on.}$$

To find the common logarithms of numbers that are not exact powers of 10, such as

$$8635; \quad 863.5; \quad 86.35; \quad 0.8635; \quad 0.08635;$$

we write these numbers in the scientific notation and then apply the laws of logarithms. Thus,

$$8635 = 8.635 \times 10^3,$$

$$863.5 = 8.635 \times 10^2,$$

$$86.35 = 8.635 \times 10^1,$$

$$0.8635 = 8.635 \times 10^{-1},$$

$$0.08635 = 8.635 \times 10^{-2}.$$

From a five-place table of logarithms, which will be explained in detail in the next section, we find that $\log 8.635 = 0.93626$, approximately. Combining this fact and the laws of logarithms, we have:

$$\log 8635 = \log 8.635 + 3 = 0.93626 + 3,$$

$$\log 863.5 = \log 8.635 + 2 = 0.93626 + 2,$$

$$\log 86.35 = \log 8.635 + 1 = 0.93626 + 1,$$

$$\log 0.8635 = \log 8.635 - 1 = 0.93626 - 1,$$

$$\log 0.08635 = \log 8.635 - 2 = 0.93626 - 2.$$

From these examples it will be noted (1) that the common logarithm of any number may be expressed as *a positive decimal fraction* plus or minus an integer, and (2) that the decimal portion of the logarithm will be independent of the position of the decimal point in any given sequence of digits in the number.

Definitions: The positive decimal portion of the common logarithm of a number is called the *mantissa.* The integral portion of the logarithm is called the *characteristic.*

If a number is written in the scientific notation, the integral exponent of 10 is the characteristic of the logarithm of the given number. Consequently, the characteristic of a logarithm may be obtained by the application of the rule previously explained for the determination of the exponent of 10 when writing a number in the scientific notation.

EXERCISES 58

1. Find the characteristic of the common logarithm of each of the following numbers:

65; 532; 87.3; 5.032; 0.1234; 0.02314; 26987000

2. If log 4.358 = 0.63929, what is log 435.8? log 0.4358? log 435,800? log 0.004358? log 43.58?

91. LOGARITHM TABLES

A table of common logarithms gives the approximate value of the logarithm of any number between 1 and 10. Consequently, the approximate mantissa of any number may be obtained by referring to a table of common logarithms. The numbers in these tables have been computed by the use of advanced methods. Every student should learn how to use a logarithm table accurately and rapidly.

Let us turn to the first page of the table entitled, "Five-Place Common Logarithms," given as Table I in the Appendix. First of all, it must be understood that the numbers in the table have not been completely written; many digits and all decimal points have been omitted. Thus, in the N column, the first number in the table is really 1.00; the second number down is 1.01; the next one is 1.02; and so on. The first numbers immediately under the next ten column headings in reality are 0.00000, 0.00043, 0.00087, 0.00130, and so on. In fact, 0.00 should be prefixed to all the readings in the upper part of the table until the asterisk is reached (the third number down in the 4 column); whereupon the prefix becomes 0.01 until the next asterisk is reached; and then 0.02 is prefixed until the next asterisk is reached, and so on. The sequence of prefixes started on the first page is continued to the other pages; so the reader should become well acquainted with the scheme employed.

Now we are ready to employ the table to obtain certain common logarithms. For example, let us obtain log 1.172. The first three digits of this number are to be found in the N column, and the fourth digit appears as a column heading. The problem, then, is to obtain the tabular reading in the 2 column to the right of 1.17 in the N column; this number is 0.06893, after 0.06 is prefixed to the 893 actually listed. Thus,

$$\log 1.172 = \text{approximately } 0.06893.$$

It is necessary to realize that virtually all the tabular readings are approximate, but the only approximation is in the fifth decimal place; in fact, all logarithms contained in a five-place table have been rounded off to the fifth decimal place. In the future, it will be our policy not to indicate the approximate nature of the readings. So, in looking up the logarithm of 2.623, we shall write

$$\log 2.623 = 0.41880;$$

likewise,

$$\log 3.77 = \log 3.770 = 0.57634.$$

To obtain log 328.5, the characteristic is +2. The mantissa, that is, log 3.285, is given in the table as 0.51654. Thus, log 328.5 = 0.51654 + 2, or 2.51654. Similarly, log 0.05404 = 0.73272 − 2.

EXERCISES 59

Determine the logarithm of each of the following numbers:

1. 4.643	**2.** 1.2000
3. 204.3	**4.** 9000
5. 88.98	**6.** 0.60600
7. 0.030830	**8.** 0.00067500
9. 1.0540	**10.** 3296
11. 53.74	**12.** 831,900
13. 0.0099	**14.** 247.6
15. 0.00006002	**16.** 86.09
17. 0.7	**18.** 0.06037
19. 6166	**20.** 7001

In the following examples, the given numbers are logarithms. By the use of the tables, find the numbers corresponding to the given logarithms.

We find from the table that log 8.03 = 0.90472. Since the characteristic of the given logarithm is +1, the decimal point must be shifted one place to the right. Hence, the desired number is 80.3; that is, log 80.3 = 1.90472.

21. 2.50799	**22.** 4.31197
23. 3.98091	**24.** 1.59770
25. 0.77235	**26.** 0.84516 − 1
27. 0.48144 − 3	**28.** 0.00000
29. 0.99961	**30.** 0.96242 − 2

92. SEVERAL WAYS OF WRITING THE CHARACTERISTIC

From the table of logarithms, we find log 256 = 0.40824 + 2. This logarithm may be written in various ways; thus,

$$\begin{aligned} \log 256 &= 2.40824 \\ &= 12.40824 - 10 \quad \text{By adding and subtracting 10,} \\ &= 22.40824 - 20 \quad \text{By adding and subtracting 20.} \end{aligned}$$

The advantages of the several forms will become apparent later.

As another illustration we have

$$\log 0.000256 = 0.40824 - 4.$$

The difference 0.40824 − 4 may be used without further change in form, or it may be written in various other ways. Thus,

$$0.40824 - 4 = -3.59176 \quad \text{By actually carrying out the operation.} \tag{1}$$

$$0.40824 - 4 = 6.40824 - 10 \text{ By adding and subtracting 6.} \quad (2)$$

$$0.40824 - 4 = 2.40824 - 6 \text{ By adding and subtracting 2.} \quad (3)$$

$$0.40824 - 4 = \bar{4}.40824. \quad (4)$$

Each method has its advantages. The first expresses the difference of two numbers as a negative number. This form is generally avoided in logarithmic work, since *the mantissas given in the tables are always positive numbers.*

The second form is much used by computers.

The third form has its advantages at times, as when it is necessary to divide 0.40824 − 4 by 6.

The fourth form is used by some computers for compactness. Generally, this form is avoided and will not be used in this work.

93. INTERPOLATION

In the use of a five-place table of logarithms there are two fundamental problems that present themselves:

1. Given a number; to find its logarithm.
2. Given a logarithm; to find the corresponding number.

We have had illustrations of both problems. No difficulty arises if the given number has only four significant figures or if the logarithm is one that is listed in the table. When these conditions are not fulfilled, the desired values are obtained by interpolation. This will now be explained with the aid of two examples.

EXAMPLE 1: Suppose we wish to find the logarithm of 1726.4. The characteristic is 3, but we cannot find the mantissa directly from the five-place table, since the number involves five significant digits. The logarithm of 1.7264, which is the desired mantissa, lies between the logarithm of 1.7260 and the logarithm of 1.7270. From the table, it is found that

$$\log 1.7260 = 0.23704$$

and

$$\log 1.7270 = 0.23729.$$

The problem of finding the logarithm of 1.7264 is based upon interpolation, which assumes that differences between logarithms are proportional to differences between the corresponding numbers. Let us write the numbers 1.7260, 1.7264, and 1.7270 in one column and the corresponding mantissas 0.23704, u (the unknown), and 0.23729 in another column.

N	Log N
1.7260	0.23704
1.7264	u
1.7270	0.23729

A change of 10 fourth-place units in the number corresponds to a change of 25 fifth-place units in the logarithm. A change of 4 fourth-place units in the number corresponds to a certain change c in the logarithm. Then c is the correction to the mantissa 0.23704; that is, c is the number of fifth-place units that must be added to 0.23704 to give u.

Now assuming that the change in the logarithm is proportional to the change in the number, and temporarily ignoring the decimal points, we may write the following proportion:

$$\frac{4}{10} = \frac{c}{25}.$$

Thus, $c = 10$ fifth-place units.

Since c in terms of fifth-place units is the correction to be added to 0.23704, corresponding to the change of 4 in the number, we have $\log 1.7264 = 0.23714$. Thus, $\log 1726.4 = 3.23714$.

The value of c may also be found in the auxiliary table, headed by *proportional parts,* on the same page on which the mantissas are located. Opposite 4 in the small table headed by 25 is the correction 10, the same value as previously obtained for c.

It should be noted that the change in the logarithm is not exactly proportional to the change in the number, and hence the method of interpolation does not give the exact value. Nevertheless it gives in general a result accurate to the number of significant figures expected from the table.

EXAMPLE 2: Suppose we need to find the number whose logarithm is 1.31720; or, to be more explicit, let us determine x if $\log x = 1.31720$.

The mantissa 0.31720 is not given in our table. However,

$$\log 2.0750 = 0.31702,$$

and

$$\log 2.0760 = 0.31723.$$

Again, let us construct a table, placing the numbers involved in one column and the mantissas of their logarithms in another.

N	Log N
2.0750	0.31702
x	0.31720
2.0760	0.31723

A change of 10 fourth-place units in the number corresponds to a change of 21 fifth-place units in the logarithm. The problem is to find how great a change in the number is demanded by a change of 18 fifth-place units in the mantissa. Let the change, or correction, in the number be represented by c fourth-place units.

Assuming that the change in the number is proportional to the change in the logarithm, and temporarily ignoring decimal points, we may write

$$\frac{c}{10} = \frac{18}{21},$$

so $c = 9$ fourth-place units. It is observed that c was rounded off to the nearest integral value; this is always done. Hence, the number having the logarithm 0.31720 will be taken as $2.0750 + 0.0009 = 2.0759$. Since, however, the given logarithm 1.31720 has the characteristic 1, the desired number is 20.759.

In this case also the value of c may be found in the auxiliary table headed by *proportional parts*. Looking in the small table under 21, we note that the closest number to the difference 18 is 18.9; this latter difference corresponds to a fifth digit of 9 in the number, the same value as obtained previously.

EXERCISES 60

Find the logarithms of the following numbers:

1. 12.734	**2.** 38.953
3. 941.71	**4.** 10.382
5. 200.46	**6.** 30.957
7. 0.0013246	**8.** 0.23667

Having given the following logarithms, find the corresponding numbers to five figures:

9. 4.84602	**10.** 2.48633
11. 1.65804	**12.** $0.32705 - 3$
13. 1.78156	**14.** 2.87207
15. $8.46512 - 10$	**16.** $9.38213 - 10$

94. COMPUTATIONS BY MEANS OF LOGARITHMS

We are now in a position to use logarithms in solving problems which involve multiplication, division, raising to powers, and extracting roots. The laws of logarithms explained in Section 88 will now have frequent use.

In computing with logarithms, the arrangement of the work is of great importance. We give several illustrations of a schematic device that is recommended.

Illustration 1: Find the value of $3.8^2 \times 54$.

Let x represent the desired number; then,

$$x = 3.8^2 \times 54,$$

and

$$\log x = 2 \log 3.8 + \log 54.$$

log 3.8	0.57978	2 log 3.8 (+)	1.15956
		log 54	1.73239
		log x	2.89195

$x = 779.74$

Illustration 2: Find the value of $4.7^3 \times 0.0083^2$.

Let $x = 4.7^3 \times (0.0083)^2$;

then, $\log x = 3 \log 4.7 + 2 \log 0.0083$.

log 4.7	0.67210	3 log 4.7 (+)	2.01630
log 0.0083	7.91908 − 10	2 log 0.0083	5.83816 − 10
		log x	7.85446 − 10

$x = 0.0071525$

Illustration 3: Find the value of $\sqrt[6]{705}$.

Let $x = \sqrt[6]{705}$;

then, $\log x = \frac{1}{6} \log 705$.

log 705	2.84819
$\log x = \frac{1}{6} \log 705$	0.47470

$x = 2.9833$

Illustration 4: Find the value of $(0.031426)^{1/3}$.

Let $x = (0.031426)^{1/3}$;

then, $\log x = \frac{1}{3} \log 0.031426$.

log 0.031426	28.49729 − 30*
$\log x = \frac{1}{3} \log 0.031426$	9.49910 − 10

$x = 0.31557$

Illustration 5: Find the value of $\dfrac{(6.2)^3}{(7.4)^4}$.

Let $x = \dfrac{(6.2)^3}{(7.4)^4}$;

* Note that the characteristic −2 is not divisible by 3 and, hence, is witten as 28 − 30.

then, $\log x = 3 \log 6.2 - 4 \log 7.4.$

log 6.2	0.70239	3 log 6.2 (−)	12.37717 − 10
log 7.4	0.86923	4 log 7.4	3.47692
		$\log x$	8.90025 − 10

$x = 0.079479$

In the above illustrations the given numbers were assumed to be accurate to at least five significant figures; hence, the answers were given to five significant figures, the limit of accuracy of a five-place table.

If the given numbers are approximate and any of them contain less than five significant digits, the answer should be rounded off to the number of significant figures equal to the number of significant figures in the number (among the given numbers) containing the least number of significant figures. Thus, answers to the above exercises would be 780; 0.0072; 2.98; 0.31557; and 0.079; respectively, if it is understood that all data are approximate.

EXERCISES 61

In the first 23 exercises that follow it is assumed that the given numbers are approximate and are correct only to the number of significant figures indicated. Find the value of each of the following:

1. 80.735×0.0013876
2. $\dfrac{74273}{0.00030243}$
3. $\sqrt[3]{54080}$
4. $\sqrt[5]{0.046932}$
5. $(1.045)^{25}$

Find the value of x in each of the following:

6. $x = (27^8)(0.0045^4)$
7. $x = \sqrt[3]{(50)(78.60)}$
8. $x = 65.30\sqrt{103}\sqrt[3]{2.68}$
9. $x = \sqrt[4]{0.0480}\sqrt[5]{403}$
10. $x = 137.20 \times \log 68893$
11. $x = \log 25 + \log 0.0042$
12. $x = \log 0.0083 - \log 36$
13. $x = \dfrac{\log 52172}{\log 47258}$
14. $x = \log\left(\dfrac{521}{472}\right)$
15. $x = \log\,(521 \cdot 472)$
16. $x = \log \sqrt[4]{72758}$
17. $x = (0.36944)^{3/4}(1.0346)^{2/3}$
18. $x = \dfrac{\sqrt[3]{(1624)(0.0471)}}{85.00}$
19. $x = \sqrt[3]{\dfrac{(110.57)^2(549.34)^{1/3}}{20045}}$

20. $x = \sqrt{\dfrac{(0.02691)^3(1.074)(9823)}{\sqrt{6800}(0.0005714)^{1/6}}}$ **21.** $x = \dfrac{(1.045)^{10} - 1}{0.0450}$

22. $x = (1.025)\dfrac{[(1.025)^{20} - 1]}{(1.025)^4 - 1}$ **23.** $x = 900\,(0.85)^{20}$

24. Given the formula $S = \dfrac{a(r^n - 1)}{r - 1}\cdot$ Find S to four significant digits if $a = 32$, $r = 8$, and $n = 10$. The data in this problem are not approximate.

25. In the formula in Exercise 24, if $S = 5000$, $r = 1.8$, and $n = 10$, find a.

26. The formula $S = P(1 + i)^n$ gives the amount of a sum of money at compound interest for n periods at rate i. If $P = \$2500$, $i = 5\%$, and $n = 30$, find S.

27. In the formula in Exercise 26, if $S = \$5000$, $i = 0.035$, and $n = 20$, find P.

28. In the formula in Exercise 26, if $S = \$5000$, $P = \$2000$, $i = 5\frac{1}{2}\%$, find n.

29. The stretch of a brass wire when a weight is hung at its free end is given by the relation $S = \dfrac{mgL}{\pi r^2 K}$, where m = the weight applied, $g = 980$, L = the length of the wire, r = its radius, and K = a constant. Find K if $m = 932.5$ gm, $L = 305$ cm, $r = 0.290$ cm, and $S = 0.082$.

30. The weight P in pounds which will crush a solid cylindrical cast-iron column is given by the formula $P = 98{,}900\,\dfrac{d^{3.55}}{L^{1.70}}$, where d is the diameter in inches and L the length in feet. What weight will crush a cast-iron column 8 ft long and $5\frac{1}{4}$ in. in diameter?

31. The weight W of 1 cu ft of saturated steam depends upon the pressure in the boiler, according to the formula $W = \dfrac{P^{0.94100}}{330.36}$, where P is the pressure in pounds per square inch. What is W if the pressure is 280 lb per sq in.?

32. Using the equation in Exercise 31, find the pressure required to make the steam weigh 0.75 lb per cu ft.

33. The diameter in inches of a connecting rod depends upon the diameter D of the engine cylinder, L the length of the connecting rod, and P the maximum steam pressure in pounds per square inch. According to Mark's formula, $d = 0.02758\sqrt{DL\sqrt{P}}$. What is d, when $D = 30$, $L = 75$, and $P = 150$?

34. The work in foot-pounds done during the adiabatic expansion of a gas from pressure p_1 to pressure p_2 is

$$W = 144\,\frac{p_1V_1}{K - 1}\left[1 - \left(\frac{p_2}{p_1}\right)^{(K-1)/K}\right],$$

where V_1 is the original volume of the gas and K is a constant. Find W when $K = 1.41$, $p_1 = 60$, $p_2 = 15$, and $V_1 = 3.5$.

95. COMPUTATION WITH NEGATIVE NUMBERS

Negative numbers have no real logarithms. This does not mean that logarithms cannot be used in computations involving negative numbers.

In such a problem, however, the sign of the result must be determined independently of the logarithmic work. In a problem involving negative numbers, except those involving even roots, the logarithmic work is carried out as though all the numbers were positive. The sign is prefixed at the conclusion of the computation after it has been determined according to algebraic principles.

Thus, to find the product of -62.5 and 83.2 by logarithms, we find the product of 62.5 and 83.2 by logarithms, and then give the result the negative sign. To find $\sqrt[3]{-2.96}$ we find $\sqrt[3]{2.96}$ by logarithms and then give the result the negative sign.

EXERCISES 62

Find the value of each of the following:

1. $\sqrt[3]{-7.4763}$

2. $\dfrac{(-62.837)^2(-5.3460)^{1/3}}{-71}$

3. $\dfrac{(-89.262)^{-2}(6.4545)}{-32.492}$

4. $\sqrt{41.227}\sqrt[5]{6.8264}$

5. $\dfrac{\sqrt[3]{(-2.0748)(0.83567)^2}}{74.359}$

6. $(-52.061)^3\sqrt{\dfrac{0.47363}{2.0974}}$

7. $x = \dfrac{(10^{0.73514})(-25)^{2/3}}{(-28)^{4/5}}$

96. SOLUTION OF EXPONENTIAL AND LOGARITHMIC EQUATIONS

There are many equations in which the unknown appears in the exponent; there are other equations that involve $\log x$. We cannot solve all equations of this type, but there are a great many that can be solved, at least approximately. The general principles involved will be illustrated by means of a few examples. We shall assume that if two positive numbers are equal, their real logarithms are equal.

Illustration 1: Find the value of x if it is known that $3^x = 72.9$. By applying the fifth law of logarithms after taking the logarithm of each member, we may write

$$x \log 3 = \log 72.9,$$

$$x = \frac{\log 72.9}{\log 3} = \frac{1.86273}{0.47712} = 3.90.$$

Illustration 2: Find the value of x if $15^x = 27 \times (9.3)^{2x}$. After taking the logarithm of each member, we have

$$x \log 15 = \log 27 + 2x \log 9.3,$$

from which we obtain

$$x \log 15 - 2x \log 9.3 = \log 27,$$

$$x(\log 15 - 2 \log 9.3) = \log 27,$$

$$x = \frac{\log 27}{\log 15 - 2 \log 9.3}.$$

The value of x may be found by performing the operations indicated. The work may be arranged conveniently as follows:

$$\begin{aligned} \log 27 &= 1.43136, \\ \log 15 &= 1.17609, \\ \log 9.3 &= 0.96848, \\ 2 \log 9.3 &= 1.93696. \end{aligned}$$

After substituting these numbers in the expression for x, we have

$$x = \frac{1.43136}{1.17609 - 1.93696} = \frac{1.43136}{-0.76087} = -1.88.$$

Illustration 3: Find the value of x if it is given that

$$4 + \log x = 6.50000 - \log 2x.$$

This equation may be rewritten in the form

$$\log 2x + \log x = 2.50000.$$

According to the third law of logarithms, this equation becomes

$$\begin{aligned} \log (2x)(x) = \log 2x^2 &= 2.50000, \\ 2x^2 &= 316.23, \\ x^2 &= 158.12, \\ x &= \pm 12.574. \end{aligned}$$

Only the positive value satisfies the original equation.

The student should analyze carefully the solution just given and be able to justify every operation.

Illustration 4: Given $7 (\log x)^2 + 20 (\log x) - 3 = 0$. Find the value of x. We have an equation in the quadratic form in which the unknown is the logarithm of x. We may then solve for $\log x$, using the quadratic formula. This gives

$$\log x = \frac{-20 \pm \sqrt{400 + 84}}{14},$$

or $$\log x = \frac{-20 \pm 22}{14}.$$

Hence, $$\log x = \tfrac{1}{7} \text{ or } -3.$$

Consequently, $x = 1.3895$

and $x = 0.001.$

Illustration 5: Solve $3^{-2x} = 0.25.$

After taking the logarithm of each member, there results

$$-2x \log 3 = \log 0.25,$$

$$-2x = \frac{\log 0.25}{\log 3} = \frac{0.39794 - 1}{0.47712} = \frac{-0.60206}{0.47712}.$$

Consequently, $$x = \frac{0.30103}{0.47712} = 0.63.$$

EXERCISES 63

The numbers in the following equations are to be treated as exact numbers; solve each equation for x:

1. $10^x = 24^2$
2. $15^{2x} = 1.73$
3. $2^x = 9(4^x)$
4. $x \log 3 = \log 7$
5. $2 (\log x)^2 + \log x - 1 = 0$
6. $25 = (1.05)^x$
7. $9 \log x = \log 27$
8. $9 \log x = 30$
9. $(\log x)^2 + \log x - 6 = 0$
10. $(\log x)^5 = 100$
11. $465 = 20(1 + x)^{20}$
12. $2500 = 1200(1.045)^x$
13. $2700 = 200 \dfrac{[(1.0225)^x - 1]}{0.0225}$
14. $350 = 1800(1 - x)^{10}$
15. $350 = 1800(0.70)^x$
16. $3.2^{(1-2x)} = 39$
17. $3^{x^2} = 563.4$
18. $e^x = 29.7$, where e is approximately 2.71828
19. $7^x = 0.697$
20. $(2.2)^{-x^2} = 0.723$

Equations of the following type are encountered in finding the rate of interest in certain investment problems. Solve for i to three significant figures in each of the following:

21. $(1 + i)^{10} = 1.6234$
22. $(1 + i)^{-35} = 0.42796$
23. $(1 + i)^{16} = 1.9372$

97. NATURAL LOGARITHMS

In most advanced mathematics and in much theoretical science the irrational number designated by e, approximately equal to 2.71828, is used as a base for a system of logarithms. We shall indicate how we may determine these logarithms, called *natural logarithms*, by means of a table of common logarithms.

To find $\log_e 763$, we first write

$$\log_e 763 = x. \tag{1}$$

Therefore, $$763 = e^x = 2.71828^x. \tag{2}$$

After taking the logarithm of each member to the base 10, we have

$$\log_{10} 763 = x \log_{10} 2.71828, \tag{3}$$

or

$$x = \frac{\log_{10} 763}{\log_{10} 2.71828}, \tag{4}$$

which becomes

$$x = \frac{2.88252}{0.43429} = 6.6373. \tag{5}$$

Since

$$\frac{1}{0.43429} = 2.30268,$$

the previous result could be obtained as the product $(2.88252)(2.30268)$.

It is apparent that, in general, the natural logarithm of any number equals the common logarithm of the given number multiplied by 2.30268.

It is often desirable to obtain the common logarithm of a number when the natural logarithm is known. Thus, if

$$\log_e x = 1.7830,$$

let us find

$$\log_{10} x.$$

We have

$$x = e^{1.7830}.$$

Hence,

$$\log_{10} x = 1.7830 \log_{10} e,$$

or

$$\log_{10} x = 1.7830(0.43429) = 0.77433.$$

In general, the logarithm of any number to the base 10 equals the natural logarithm of the given number multiplied by 0.43429.

EXERCISES 64

1. Find the natural logarithms of the integers from 2 to 10.
2. Find $\log_e 25$; $\log_e 250$; $\log_e 2500$; $\log_e (369)^5$.
3. Find the value of $e^{\log_e x}$, when $x = 50$. Generalize upon your result.
4. Find the value of $10^{\log_{10} x}$.
5. Find the natural logarithms of the following numbers: 0.0036; $\frac{2}{5}$; 10.3; 0.27; e; 6.782; 384; 9.643.
6. Evaluate each of the following:
   (a) $\log_e e^3$; (b) $\log_e \frac{7}{4}$; (c) $\log_e (4.3)(2.7)$; (d) $\log_e 12 - \log_e 3$.
7. Many collections of mathematical tables contain tabulations of natural logarithms. However, many tables of natural logarithms only list numbers from 1 to 10, inclusive. Suppose such a table gives $\log_e 2.5 = 0.91629$ and $\log_e 10 = 2.30259$. Using only this numerical information, determine $\log_e 25$; $\log_e 250$; $\log_e 2500$; $\log_e 0.25$; $\log_e 0.025$; $\log_e 6.25$.
8. The principles involved in obtaining logarithms to the base e, when a table of common logarithms is available, are likewise applicable to obtaining logarithms to any base.
   (a) Determine $\log_2 5$.
   (b) What is $\log_3 13.5$?
   (c) Evaluate $\log_{7.4} 63.9$.
   (d) Find x if $x = \log_3 0.86$.

14 Progressions

98. PROGRESSIONS

A sequence of numbers is a collection of numbers ordered in such a manner that there is a first number, a second number, and so on.

In general, we may symbolize a sequence whose terms have been written in some prescribed order by

$$a_1, a_2, a_3, \cdots, a_n,$$

where the subscript indicates the number of the term, and where the letter bearing the subscript indicates the numerical value of the term. Thus, a_k indicates the numerical value of the kth term.

In scientific work it is often necessary to consider special sequences of numbers in which there is a definite law for the determination of any particular term. Such sequences are typified by the following:

$$1, 8, 27, 64, 125, 216. \tag{1}$$

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{8}, \tfrac{1}{10}, \tfrac{1}{12}. \tag{2}$$

$$1, 4, 9, 16, 25, 36. \tag{3}$$

$$2, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{5}{4}, \tfrac{6}{5}, \tfrac{7}{6}, \tfrac{8}{7}. \tag{4}$$

The law for the determination of the terms of each of the above sequences is made more apparent when they are written as follows:

$$1^3, 2^3, 3^3, 4^3, 5^3, 6^3. \tag{1}$$

$$\frac{1}{2}, \frac{1}{(2)(2)}, \frac{1}{(2)(3)}, \frac{1}{(2)(4)}, \frac{1}{(2)(5)}, \frac{1}{(2)(6)}. \tag{2}$$

$$1^2, 2^2, 3^2, 4^2, 5^2, 6^2. \tag{3}$$

$$1 + \tfrac{1}{1}, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{3}, 1 + \tfrac{1}{4}, 1 + \tfrac{1}{5}, 1 + \tfrac{1}{6}, 1 + \tfrac{1}{7}. \tag{4}$$

Thus, any particular term of one of the sequences (1), (2), (3), and (4) is determined by the use of the appropriate one of the following rules:

(1) $a_n = n^3;$ (2) $a_n = \dfrac{1}{2n};$

(3) $a_n = n^2;$ (4) $a_n = 1 + \dfrac{1}{n};$

where in each case n designates the number of the desired term.

In the consideration of such special sequences it is often desirable to evaluate a_n when n is given, or to find n if a_n is given. Also, it is desirable for many applications to find a formula which will give the sum of any number of consecutive terms of a given sequence.

A great variety of laws are employed in the construction of various kinds of sequences. The simplest types of such laws result in sequences that are often met in practice and will now be studied.

99. ARITHMETICAL PROGRESSION

Definition: An arithmetical progression is a sequence of numbers in which the difference between any term and the preceding term is a constant, which is called the *common difference.* The first term of the progression must be specified independently.

Thus, 5, 7, 9, 11 is an arithmetical progression whose common difference is 2, and whose first term is 5.

Similarly, 5, 4, 3, 2 is an arithmetical progression whose common difference is -1, and whose first term is 5.

EXERCISES 65

Determine which of the following sequences are arithmetical progressions:

1. 2, 4, 6, 8
2. 2, 0, -2, -4
3. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$
4. 2, 4, 8, 16
5. 1, 4, 9, 16
6. $a, a + d, a + 2d, a + 3d$
7. $x, 2x - 2y, 3x - 4y, 4x - 6y$
8. $1, \sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}$
9. $1 - \sqrt{3}, 1, 1 + \sqrt{3}, 1 + 2\sqrt{3}$
10. $a - d, a, a + d$
11. $x, \dfrac{x + y}{2}, y$

100. FORMULAS FOR THE ARITHMETICAL PROGRESSION

If we consider the sequence $a_1, a_2, a_3, \cdots, a_n$ with the understanding that it is an arithmetical progression whose common difference is d, we note that

$$\begin{aligned} a_1 &= a_1; \\ a_2 &= a_1 + d; \\ a_3 &= a_2 + d = a_1 + 2d; \\ a_4 &= a_3 + d = a_1 + 3d; \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n &= a_1 + (n - 1)d. \end{aligned} \tag{1}$$

The factor $(n - 1)$ in Formula (1) is obtained from observation of the fact that the coefficient of d is always 1 less than the number of the term. Equation (1) expresses a relationship between the four quantities a_n, n, a_1, and d. If we are given any three of these four quantities, the fourth may readily be found. In fact, Equation (1) may be solved for any one of the

quantities involved in terms of the others. Thus, three other useful formulas may be obtained.

Since n must be a positive integer, we should note that if we assign numerical values to a_n, a_1, and d, in order to find n, it is necessary that the assigned values shall actually be elements of an arithmetical progression; otherwise, n will not result in a positive integer.

If we let S_n denote the sum of the first n terms, that is, $a_1 + a_2 + a_3 + \cdots + a_n$, we may derive a formula for S_n as follows:

$$(A) \quad S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + [a_1 + (n - 1)d].$$

If we consider the same progression, but with its terms written in the reverse order, we have

$$(B) \quad S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d].$$

After adding the corresponding members of Equations (A) and (B), we have

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n),$$

there being n terms upon the right. Consequently,

$$2S_n = n(a_1 + a_n),$$

or

$$S_n = \frac{n}{2}(a_1 + a_n). \tag{2}$$

From this Formula (2), we may obtain three additional formulas by solving for each quantity in terms of the others.

We thus have shown that it is possible to have eight formulas for use in the consideration of arithmetical progressions, although we may solve all problems met in practice by using only Formulas (1) and (2).

The quantities a_1, d, a_n, n, and S_n are referred to as the elements of an arithmetical progression; if any three elements of an arithmetical progression are given, the remaining two elements may be found.

Illustration 1. The arithmetical progression 20, 18, 16, $\cdots$ is given. Find the 20th term and the sum of the first 20 terms.

Here, $\quad a_1 = 20, d = -2, n = 20.$

From Equation (1) $\quad a_{20} = 20 + (19)(-2) = -18.$

From Equation (2) $\quad S_{20} = \frac{20}{2}(20 - 18) = 20.$

Illustration 2. Given $a_1 = 15$, $d = 3$, $a_n = 30$, find n and S_n.

We have, from Equation (1),

$$\begin{aligned} 30 &= 15 + (n - 1)3 \\ &= 12 + 3n. \\ \therefore 3n &= 18, \\ n &= 6. \end{aligned}$$

From Equation (2),

$$\begin{aligned} S_n &= \tfrac{6}{2}(15 + 30) \\ &= 135. \end{aligned}$$

Illustration 3. Given $a_1 = 7$, $S_n = 7$, and $d = -2$ Find a_n and n.

From Equation (1), $a_n = 7 + (n - 1)(-2)$

or $a_n = 9 - 2n.$

From Equation (2), $7 = \frac{n}{2}(7 + a_n).$

After substituting in this latter equation the value of a_n previously found, we have

$$7 = \frac{n}{2}(16 - 2n),$$

or

$$7 = 8n - n^2.$$

This quadratic equation may be simplified to

$$n^2 - 8n + 7 = 0,$$

or

$$(n - 1)(n - 7) = 0.$$

Thus, $n = 1$ and 7.

EXERCISES 66

1. Solve Formula (1) (Section 100) for each element in terms of the others.

2. Solve Formula (2) (Section 100) for each element in terms of the others.

3. Given the arithmetical progression $-1, 3, 7, \cdots$, find the tenth term and the sum of 20 terms.

4. Given the arithmetical progression $-1, -3, -5, \cdots$, find the twentieth term and the sum of 20 terms.

5. Given the arithmetical progression $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, find the sum of 15 terms.

6. Given the arithmetical progression $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$, find the sixteenth term and the sum of 16 terms.

7. Given $a_1 = \sqrt{2}$, $d = 2$, $n = 12$. Find a_n and S_n.

8. Given $d = \sqrt{2}$, $n = 20$, and $a_n = 5\sqrt{2}$. Find a_1 and S_n.

9. Given $a_1 = 39$, $a_n = 67$, $d = \frac{7}{2}$. Find n and S_n.

10. Given $a_1 = 2\frac{1}{2}$, $d = 2\frac{1}{2}$, $S_n = 165$. Find n and a_n.

11. If the first and third terms of an arithmetical progression are, respectively, 25 and 4, what is the second term?

12. The sum of three terms in an arithmetical progression is 9, and the sum of their squares is 135. Find the numbers. HINT: Let the three terms be $a - d$, a, and $a + d$.

13. A drilling company charges the following rates for drilling wells: 40 cents for the first foot and an increase of 2 cents for each additional foot. What would be the charge for drilling a well 100 ft deep?

14. Suppose that two positions are available, one at an annual salary of $1300 with a yearly increase of $100, the other at a fixed annual salary of $2000. In how many years would the total income from the two positions be the same?

15. Twenty potatoes are placed in a straight line on the ground at intervals of 5 ft. A basket is placed on this same line and 10 ft from the first potato. A runner starts from the basket, picks up the first potato, and carries it to the basket; he then continues to the second potato and carries it to the basket; and so on.

How far must he run before all the potatoes are deposited in the basket?

16. A student in need of money decided to raise it by raffling off his watch. He decided to sell tickets numbered consecutively and charge for each ticket as many cents as the number on the ticket. How many tickets must he sell to raise at least $40?

17. A body falling freely in a vacuum falls $\frac{1}{2}g$ ft the first second, and each second after the first the distance fallen increases g ft. Find a formula for the distance S fallen in t sec.

18. A term b of the proper magnitude is introduced between a and c so that the three terms form an arithmetical progression. Show that b is the ordinary average of a and c.

19. Prove that the sum of the first n odd numbers is equal to n^2.

20. A flywheel 5 ft in diameter is revolving at a speed of 50 rps just as the power is shut off. If the speed then decreases 2 rps, how far will a point on the rim travel before the wheel stops?

21. A harmonic progression, by definition, is a sequence of numbers whose reciprocals form an arithmetical progression. The first term of a harmonic progression is $\frac{1}{2}$ and the fourth term is $\frac{3}{16}$; find the second term and the seventh term.

101. GEOMETRICAL PROGRESSION

Definition: A geometrical progression is a sequence of numbers in which the ratio of any term divided by the preceding term is a constant, called the *common ratio*. The first term is given independently.

Thus, the sequence 3, 6, 12, 24 is a geometrical progression whose common ratio is 2. The first term is 3.

Similarly, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$ is a geometrical progression whose common ratio is $\frac{1}{2}$ and whose first term is 4.

EXERCISES 67

Determine which of the following sequences are geometrical progressions:

1. 2, 4, 8, 16

2. 2, 3, 4, 5

3. 2, 3, $\frac{9}{2}$, $\frac{27}{4}$

4. 10, -5, $+\frac{5}{2}$, $-\frac{5}{4}$

5. $1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}$

6. $\sqrt{2}, 2, 2\sqrt{2}, 4$

7. $1, 0, -1, -2$

8. $ax, ax^{3/2}, ax^2, ax^{5/2}$

9. $1, a + b, (a + b)^2, (a + b)^3$

10. $1 + \sqrt{2}, -1, \sqrt{2} - 1$

102. FORMULAS FOR THE GEOMETRICAL PROGRESSION

If we consider the sequence

$$a_1, a_2, a_3, \cdots, a_n$$

with the understanding that it is a geometrical progression whose common ratio is r, we note that

$$\begin{aligned}
a_1 &= a_1. \\
a_2 &= a_1 r. \\
a_3 &= a_2 r = a_1 r^2. \\
a_4 &= a_3 r = a_1 r^3. \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Hence,

$$a_n = a_1 r^{n-1}. \tag{1}$$

The nth term is obtained from observation of the fact that the exponent of each term is 1 less than the number of the term.

Equation (1) expresses a relationship between the four quantities a_n, n, a_1, and r. If we are given any three of these four quantities, the fourth may be found.

We should note that if we assign numerical values to a_n, a_1, and r in order to find n, it is necessary that the assigned values shall be actual elements of a geometrical progression; otherwise, n will not be a positive whole number.

Let S_n denote the sum of the first n terms, namely,

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}.$$

After multiplying each member of the preceding expression for S_n by r, we have

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^{n-1} + a_1 r^n.$$

After subtracting the members of the equation for S_n from the corresponding members of the equation just obtained, there results

$$rS_n - S_n = a_1 r^n - a_1,$$

$$S_n(r - 1) = a_1(r^n - 1),$$

or

$$S_n = \frac{a_1(r^n - 1)}{r - 1}. \tag{2}$$

This formula fails if $r = 1$. In that case,

$$S_n = a_1 + a_1 + \cdots + a_1 = na_1.$$

Illustration 1: Given the geometrical progression 2, 3, $\frac{9}{2}$, $\frac{27}{4}$, $\cdots$, find the tenth term and the sum of 10 terms. Here,

$$a_1 = 2, \quad r = \tfrac{3}{2}, \quad n = 10.$$

From Equation (1), $$a_{10} = 2\left(\frac{3}{2}\right)^9 = \frac{3^9}{2^8}.$$

From Equation (2), $$S_{10} = \frac{2\left[\left(\frac{3}{2}\right)^{10} - 1\right]}{\frac{3}{2} - 1}$$

$$= 4\left[\left(\frac{3}{2}\right)^{10} - 1\right] = 4\left[\frac{59{,}049}{1{,}024} - 1\right]$$

$$= \frac{59{,}049}{256} - 4 = 226.66, \text{ approximately.}$$

Illustration 2: Given $a_1 = 50$, $r = \frac{1}{2}$, $a_n = \frac{25}{64}$. Find n and S_n.

From Equation (1), $$\tfrac{25}{64} = 50(\tfrac{1}{2})^{n-1}.$$

Therefore, $$\tfrac{1}{128} = (\tfrac{1}{2})^{n-1},$$

or $$128 = 2^{n-1}.$$

Since $128 = 2^7$, it follows that

$$n - 1 = 7,$$

or $$n = 8.$$

From Equation (2), $$S_8 = 50\,\frac{[(\frac{1}{2})^8 - 1]}{\frac{1}{2} - 1} = -100\left[\frac{1}{256} - 1\right]$$

$$= \frac{25{,}500}{256} = \frac{6375}{64}.$$

EXERCISES 68

1. Solve Formula (1) (Section 102) for each of the elements a_1, r, and n in terms of the other elements.

2. Solve Formula (2) (Section 102) for each of the elements a_1 and n in terms of the other elements.

3. Given the geometrical progression -1, $\frac{1}{2}$, $-\frac{1}{4}$, $\cdots$, find the ninth term and the sum of nine terms.

4. Given the geometrical progression $5\sqrt{2}$, 10, $10\sqrt{2}$, $\cdots$, find the eighth term and the sum of eight terms.

5. Given $a_1 = 21$, $r = \frac{1}{3}$, and $n = 8$. Find a_n and S_n.

6. Given $a_1 = \dfrac{\sqrt{3}}{12}$, $a_n = 864$, and $n = 8$. Find r and S_n.

7. A man invested \$1000 on January 1, 1930, at $4\frac{1}{2}$ per cent compounded annually. If no withdrawals are made, what will be the value of his investment at the end of 20 years?

8. If the enrollment of a school is 1500 and has been increasing at the rate of 10 per cent per year, what was the enrollment 10 years ago? What will it be 10 years from now?

9. If a student invests $100 on each anniversary of the date of his graduation from college, and these investments earn 5 per cent compounded annually, how much will he have to his credit on the twentieth reunion of his class after making his regular investment upon that date?

10. A painter agreed to paint a flag pole at the following rate: $5 for the first 20 ft, $10 for the second 20 ft, $20 for the third ft, and so on. What would be his total bill if the flag pole is 120 ft high?

11. An air pump used for removing the air from a tank removes with each stroke one tenth of the weight of the air remaining in the tank.

(a) What fractional part of the original air, by weight, will remain in the tank after 10 strokes? Assume the original weight of air in the tank to be w lb.

(b) How many strokes would be necessary to remove 98 per cent of the air from the tank?

12. A number b is inserted between the two numbers a and c so that the three form a geometrical progression. Show that b must be the mean proportional between a and c.

13. According to legend, an Indian prince once agreed to pay a wiseman 1 grain of wheat for the first square of a chess board, 2 for the next, 4 for the next, 8 for the next, and so on. Recalling that a chess board contains 64 squares, compute the approximate number of grains of wheat involved in the transaction.

103. INFINITE GEOMETRICAL PROGRESSIONS

A sequence with an unlimited number of terms is said to be an infinite sequence. Such a sequence may be displayed symbolically as follows:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots,$$

wherein the three dots at the end indicate that no last term may be specified.

As an illustration, the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ possesses the property that every term is followed by another term, that is, there is no last term; it is an infinite sequence. Moreover, this sequence possesses the characteristic property of a geometrical progression, for each term after the first is one half its predecessor; so it is called an *infinite geometrical progression.* The fraction $\frac{1}{3}$, expressed as a decimal, results in another infinite geometrical progression, namely,

$$0.33333\cdots = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots,$$

in which the r is equal to 0.1.

A fundamental problem in the study of an infinite geometrical progression pertains to the behavior of the sum of the first n terms as n becomes large. In general, let the infinite geometrical progression be denoted by

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots.$$

The sum of the first n terms, of course, is

$$S_n = \frac{a_1(r^n - 1)}{r - 1}.$$

This formula may be rewritten in the form

$$S_n = \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r}.$$

If r is numerically less than 1, and if n increases, then r^n decreases, becoming, in fact, arbitrarily small as n becomes sufficiently large. This may be indicated symbolically by $r^n \to 0$, as $n \to \infty$, which is read "r^n approaches zero (that is, becomes and remains numerically smaller than any quantity specified in advance) as n increases without limit."

Since as $n \to \infty$, $r^n \to 0$, then $\frac{a_1 r^n}{1 - r}$ also tends to 0; consequently, S_n tends to $\frac{a_1}{1 - r}$. We indicate these facts by writing

$$\lim_{n \to \infty} S_n = \frac{a_1}{1 - r},$$

where the symbol $\lim\limits_{n \to \infty} S_n$ is read "limit of S_n as n increases without limit." We define the "sum" of an infinite geometrical progression whose common ratio is numerically less than 1 by $\lim\limits_{n \to \infty} S_n$.

If r is numerically greater than 1, r^n increases numerically without limit as n increases without limit; thus, the infinite geometrical progression does not have a definite finite sum.

If $r = +1$, the progression is

$$a_1 + a_1 + a_1 + a_1 + \cdots,$$

and since a_1 is a finite number, the infinite geometrical progression does not have a finite sum.

If $r = -1$, the progression is

$$a_1 - a_1 + a_1 - a_1 + \cdots;$$

evidentally, the sum oscillates between a_1 and 0; so the progression does not have a definite sum.

Hence, an infinite geometrical progression has a *sum* only when r is numerically less than 1. In practice, therefore, the study of the infinite geometrical progression is restricted to this case.

It is important to note the significance of $\lim\limits_{n \to \infty} S_n = \frac{a_1}{1 - r}$. This means that as n increases, the difference between S_n and $\frac{a_1}{1 - r}$, in absolute

value, becomes smaller and smaller and may be made arbitrarily small by selecting n large enough.

Illustration: A rubber ball falls from a height of 50 ft and rebounds two thirds of that distance. As the process continues, each rebound is two thirds of the distance of fall. Let us find the total distance traversed by the ball.

The distance traversed may be obtained by adding the two sums that follow:

$$\text{Total drops} = 50 + \tfrac{2}{3}(50) + \tfrac{4}{9}(50) + \tfrac{8}{27}(50) + \cdots;$$

$$\text{Total rises} = \tfrac{2}{3}(50) + \tfrac{4}{9}(50) + \tfrac{8}{27}(50) + \cdots.$$

Each sequence is an infinite geometrical progression in which $r = \frac{2}{3}$; hence they may be summed by the formula just derived. The results obtained are as follows:

$$\text{Total drops} = \frac{50}{1 - \frac{2}{3}} = 150;$$

$$\text{Total rises} = \frac{\frac{2}{3}(50)}{1 - \frac{2}{3}} = 100.$$

Hence, the limiting value of the distance covered by the ball is $100 + 150 = 250$ ft.

EXERCISES 69

Find the value of the sum of each of the following infinite geometrical progressions:

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

2. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$

3. $2 - \frac{2}{5} + \frac{2}{25} - \cdots$

4. $3 + 1 + \frac{1}{3} + \cdots$

5. $(0.9) + (0.9)^2 + (0.9)^3 + \cdots$

Find the rational fraction that represents the limiting value of each of the following repeating decimals:

6. $0.222\cdots$

7. $0.121212\cdots$

8. $0.243243\cdots$

9. $0.48545454\cdots$

10. $32.7363636\cdots$

11. When an electric circuit containing a galvanometer is closed, the needle of the galvanometer vibrates back and forth across the point where it finally comes to rest. If on the first swing to the right it turns through an angle of 30 degrees from the point of rest, and on the swing to the left it turns through an angle only one half as great as the previous swing to the right, what is the limiting value of the total number of degrees through which the needle turns? Assume that the successive swings right, left, right, and so on, are in geometrical progression.

12. If a weight is suspended from the end of a coiled spring and allowed to drop suddenly, the spring will be elongated and then contracted so that the weight vibrates up and down above and below a certain point. If, when the weight is

dropped, the spring is elongated 2 in. longer than its final length, what is the limiting value of the total distance the weight will travel? Assume that the successive distances below, above, below, and so on, are in a geometrical progression of common ratio $\frac{2}{3}$.

13. Determine the limit of the sum of the geometrical progression

$$\frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3} + \cdots, \quad \text{where } x > 0.$$

14. In adding successively the terms of the following geometrical progression

$$1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots,$$

what is the least number the sum will never exceed?

15 Mathematical Induction

104. MATHEMATICAL INDUCTION

Mathematical induction is a method frequently employed to investigate the validity of certain assumed formulas or laws in one variable, whose range is restricted to an infinite collection of consecutive integers. The general method will be analyzed by considering its application to particular examples.

Illustration 1: We note that

$$1 = 1^2,$$

$$1 + 3 = 2^2,$$

$$1 + 3 + 5 = 3^2.$$

It appears that the sum of the n consecutive, positive odd integers beginning with 1 is n^2. Let us examine the validity of this conjecture in general; that is, let us investigate whether

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2,$$

where n is any positive integer.

The method of mathematical induction is as follows:

First, we test the assumed law for at least one permissible value of n. This test has been applied to the formula under consideration for $n = 1, n = 2, n = 3$. If the formula is valid for the particular choice of n, we move to the next step. Of course, if the formula is invalid for that value of n, there is no need to go further.

Second, we assume that the formula is valid for an arbitrary value of n, such as $n = k$. Thus, for the formula at hand, we assume that

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2. \tag{1}$$

Using this assumption as a basis, we obtain a formula for the case where $n = k + 1$. Here, we add $2k + 1$, the next term of the progression, to both sides of (1), thereby obtaining

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1. \tag{2}$$

But the right member of Equation (2) is evidently $(k + 1)^2$, the original formula with n replaced by $k + 1$. Hence, the method shows that if

the assumed law is true for the arbitrary positive integer k, it is true for the next higher integer $k + 1$.

We have, however, verified that the law is true for $n = 1$, 2, and 3; hence, the induction just completed shows that the assumed law is true for the next value, $n = 4$; but, if the formula is valid for $n = 4$, the induction shows that the assumed law is true for $n = 5$, and thus, by a continuation of the process, we have established the law,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2,$$

as a general law for every positive integer n.

Illustration 2: We can readily verify by division that $a^n - b^n$ is divisible by $a - b$, when $n = 1, 2, 3$, and we now wish to investigate whether $a^n - b^n$ is divisible by $a - b$ when n is any positive integer.

To apply mathematical induction, we assume that the law is applicable when n is some arbitrary positive integer k; thus, we are assuming that $a^k - b^k$ is divisible by $a - b$. Basing our analysis upon this assumption, we ask the question whether $a^{k+1} - b^{k+1}$ is divisible by $a - b$. We write $a^{k+1} - b^{k+1}$ in the form

$$a^{k+1} - ab^k + ab^k - b^{k+1}.$$

But this expression may be written as

$$a(a^k - b^k) + b^k(a - b). \tag{3}$$

Thus, it is apparent that if $a^k - b^k$ is divisible by $a - b$ for the positive integer k, then $a^{k+1} - b^{k+1}$ is divisible by $a - b$. Since we know that $a^3 - b^3$ is divisible by $a - b$, this latter step demonstrates that $a^4 - b^4$ must be divisible by $a - b$ and so on. We have therefore established that $a^n - b^n$ is divisible by $a - b$ when n is any positive integer.

In summary, we note that mathematical induction involves

(A) Testing an assumed law, expressed as a function of n, for a numerical integral value of n, say $n = 1, 2, 3$;

(B) Investigating the assumed law for an arbitrary value of n, such as $n = k$, to determine whether the assumption of the law for $n = k$ results in the law holding when $n = k + 1$.

If (A) and (B) hold, then the reasoning shows that the law is true for all integral values of n higher than the lowest value used under (A).

The student must note that both parts (A) and (B) are essential in the establishment of an assumed law by mathematical induction. In fact, we shall now illustrate that the application of either (A) or (B), but not both (A) and (B), is not sufficient in mathematical induction.

Illustration 3: Let us investigate the validity of the formula

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1} + (n - 1)(n - 2)(n - 3),$$

when n is any positive integer.

We first apply step (A) and readily verify that the assumed law is true when $n = 1, 2, 3$.

If, however, we proceed to step (B) and assume for an arbitrary $n = k$ that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1} + (k-1)(k-2)(k-3), \quad (1)$$

and then add $\dfrac{1}{(k+1)(k+2)}$ to both members, we have

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + (k-1)(k-2)(k-3) + \frac{1}{(k+1)(k+2)}. \quad (2)$$

It is not difficult to see that the right member of (2) is not identical with the assumed formula when $n = k + 1$, that is, with

$$\frac{k+1}{k+2} + k(k-1)(k-2).$$

Hence, we see that although part (A) applies for the values of n tried, part (B) does not apply, and the law is not established. In fact, the proposed formula ceases to be valid when $n = 4$.

Illustration 4: Let us investigate whether the following formula is valid when n is any positive integer:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} + 5.$$

Without applying step (A), that is, without verifying the proposed formula for any value of n, let us apply (B). Thus, we assume that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} + 5. \quad (1)$$

After adding $(k + 1)^3$ to both members, we have

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + 5 + (k+1)^3. \quad (2)$$

The student can readily show that the right member of (2) is identical with the proposed formula when $n = k+1$, that is, with $\dfrac{(k+1)^2(k+2)^2}{4} + 5$.

Hence, the assumed law holds for $n = k + 1$ if it holds for $n = k$. If, however, we now apply step (A), that is, if we try the assumed law for $n = 1, 2, 3$, we find that it is not true for any of these cases. Therefore, we see that although step (B) applies, step (A) does not apply, and the law is not established.

105. PROOF OF THE BINOMIAL THEOREM FOR A POSITIVE INTEGER

In Section 26 we stated the binomial theorem for positive integral exponents. It is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\lfloor 3} a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$$

This formula was accepted at that time without proof, although it was confirmed for such values as $n = 2, 3, 4$.

If, in this binomial expansion, we designate $\frac{n(n-1)}{\lfloor 2}$ by ${}_nC_2$,

$$\frac{n(n-1)(n-2)}{\lfloor 3} \text{ by } {}_nC_3, \text{ and } \frac{n(n-1)\cdots(n-r+1)}{\lfloor r} \text{ by } {}_nC_r,$$

where r is any positive integer equal to or less than n, we note that the expansion may be written as

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + {}_nC_nb^n,$$

where the $(r+1)$th term is ${}_nC_ra^{(n-r)}b^r$.

This formula is readily proved when n is a positive integer by the use of mathematical induction.

As already stated, we know the formula is true for $n = 1, 2, 3$.

Hence, we assume the validity of the expansion for $n = k$. Then we multiply the left and right members of

$$(a+b)^k = a^k + {}_kC_1a^{k-1}b + {}_kC_2a^{k-2}b^2 + \cdots + {}_kC_ra^{k-r}b^r + \cdots + b^k$$

by $a + b$, and obtain

$$\begin{aligned}(a+b)^{k+1} = [a^{k+1} &+ {}_kC_1a^kb + {}_kC_2a^{k-1}b^2 + \cdots + {}_kC_ra^{k-r+1}b^r \\ &+ \cdots + ab^k] + [a^kb + {}_kC_1a^{k-1}b^2 + \cdots \\ &+ {}_kC_{r-1}a^{k-r+1}b^r + \cdots + b^{k+1}].\end{aligned}$$

After collecting terms, this equation becomes

$$\begin{aligned}(a+b)^{k+1} = a^{k+1} &+ ({}_kC_1 + 1)a^kb + ({}_kC_2 + {}_kC_1)a^{k-1}b^2 + \cdots \\ &+ ({}_kC_r + {}_kC_{r-1})a^{k-r+1}b^r + \cdots + b^{k+1}.\end{aligned}$$

We now show that

$${}_kC_r + {}_kC_{r-1} = {}_{k+1}C_r.$$

$${}_kC_r = \frac{k(k-1)(k-2)\cdots[k-(r-1)]}{\lfloor r},$$

and

$${}_kC_{r-1} = \frac{k(k-1)(k-2)\cdots[k-(r-2)]}{\lfloor r-1}.$$

Hence, after multiplying numerator and denominator of the expression

for ${}_kC_{r-1}$ by r, we obtain

$$
\begin{aligned}
{}_kC_r + {}_kC_{r-1} &= \frac{k(k-1)(k-2)\cdots[k-(r-2)][k-(r-1)+r]}{\lfloor r} \\
&= \frac{k(k-1)(k-2)\cdots[k-(r-2)](k+1)}{\lfloor r} \\
&= \frac{(k+1)(k)(k-1)\cdots[(k+1)-(r-1)]}{\lfloor r} \\
&= {}_{k+1}C_r.
\end{aligned}
$$

Hence,
$$
(a+b)^{k+1} = a^{k+1} + {}_{k+1}C_1a^kb + {}_{k+1}C_2a^{k-1}b^2 + {}_{k+1}C_3a^{k-2}b^3 + \cdots + b^{k+1}.
$$

From this equation we see that if the proposed expansion is true for $n = k$, it is true for $n = k + 1$. Hence, we have established by mathematical induction that the binomial theorem is true for all positive integral values of n.

EXERCISES 70

Prove by mathematical induction that

1. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

2. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$

3. $4 + 4^2 + 4^3 + \cdots + 4^n = \frac{4}{3}(4^n - 1)$

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

5. $\dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{(n+1)(n+2)} = \dfrac{n}{2(n+2)}$

6. $\dfrac{1}{5\cdot 6} + \dfrac{1}{6\cdot 7} + \cdots + \dfrac{1}{(n+4)(n+5)} = \dfrac{n}{5(n+5)}$

7. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

8. $a^n - 1$ is divisible by $a - 1$ when n is a positive integer.

9. $a^{2n} - 1$ is divisible by $a + 1$ when n is a positive integer.

10. Prove the formula for the nth term of an arithmetical progression by mathematical induction.

11. Prove the formula for the nth term of a geometrical progression by mathematical induction.

12. Show that step (A) of the process of mathematical induction applies to

$$2 + 4 + 6 + \cdots + 2n = n(n+1) + (n-1)(n-2)(n-3),$$

but that step (B) fails.

13. Show that step (B) of mathematical induction applies to

$$2 + 4 + 6 + \cdots + 2n = n(n+1) + 3,$$

but that step (A) fails.

14. Evaluate ${}_6C_2$; ${}_8C_3$; ${}_{11}C_4$; ${}_5C_5$.

15. Show by numerical analysis that

$${}_7C_4 + {}_7C_3 = {}_8C_4.$$

16 Permutations, Combinations, and Probability

106. PERMUTATIONS

Definition: Each of the arrangements which can be made by taking some or all of a number of things is called a *permutation*. Thus, if we are considering the three letters a, b, c, the different arrangements of these three letters, taking them two at a time, are *ab, ac, ba, bc, ca, cb*; in general, there are six arrangements or permutations of three things taken two at a time. We symbolize this as ${}_3P_2 = 6$. The permutations of the three letters taking them three at a time are *abc, acb, bac, bca, cab, cba*. Thus, ${}_3P_3 = 6$.

107. FUNDAMENTAL THEOREM

If a first act can be performed in m ways and a second act can be performed in n ways, and if it is assumed that the doing of the first act in m ways does not exclude the doing of the second in n ways, then the two can be done, in that order, in mn ways.

Thus, if we can cross a river in 10 different boats and return in 5 other different boats, the journey can be performed in $(5)(10) = 50$ different round trips, assuming that the round trips are different when at least one different boat is used in each round trip.

It is immediately apparent that the fundamental theorem as it pertains to two acts can be generalized to the case of n acts. Thus, if one act can be performed in a_1 ways, and, if after it has been done in some one of these a_1 ways, a second act can be performed in a_2 ways, and, if after this has been done, a third act can be performed in a_3 ways, and so on for n acts, then the n acts can be performed together, in that order, in $(a_1)(a_2)(a_3)\cdots(a_n)$ ways.

Illustration 1: In how many ways can individual portraits of five people be arranged in groups of three?

Figure 41 displays a group of three portraits. In this problem, in contrast with many problems, any one of the five portraits can be hung in any one of the three positions. Thus, starting with the first position to the left, there are five possible choices. After one portrait has been hung, there are four choices for the second position. After a selection has been made for the second position, there are only three choices for the

third position. Thus, the portraits may be arranged in $(5)(4)(3) = 60$ different ways.

FIG. 41

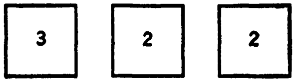


FIG. 42

Illustration 2: In how many ways may a collection of family portraits be arranged by threes if each collection shows a parent in the middle and one child on each side, there being three children in the family?

Figure 42 depicts the situation this time. In the middle there are two definite choices. On the left there are three possible choices, but, after a selection has been made, there are two choices on the right. Thus, the portraits may be arranged in $(3)(2)(2) = 12$ different ways.

EXERCISES 71

1. If there are five paths up one side of a mountain and three down the opposite side, in how many different ways may a person go over the mountain?

2. If there are two different railroads, three different bus lines, and three different air routes joining A and B, in how many ways may a traveler make the round trip from A to B and back to A if he decides to go by rail or by bus and return by air?

3. In how many ways may three positions be filled if there are five applicants for the first position, three for the second, and ten for the third? It is to be understood that no applicant is eligible for a position other than the one for which he has applied.

4. There are five pitchers and three catchers on a certain baseball squad. In how many ways may the coach choose a battery for a game?

5. If there are 23 men on the squad of Exercise 4, and each of the other 15 can play any one of the remaining seven positions equally well, in how many ways may the coach choose a team?

6. In how many different ways may a man dress if he has five suits, two hats, and three pairs of shoes?

7. In how many ways may a program consisting of four musical numbers and three speeches be arranged if the first number must be music and the other numbers alternate?

8. (*a*) How many even three-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if no repetition of digits is allowed?
   (*b*) Answer part (*a*) when the number must be between 300 and 400.
   (*c*) Answer part (*a*) when repetition of digits is allowed.

9. How many three-letter words may be formed with the four letters a, b, c, d, if it is understood that any arrangement of three of the letters with a vowel in the middle is a word. Repetition of letters is permitted.

10. In how many ways may the seven speakers at a banquet be seated at the seven places along one side of the head table?

108. THE FORMULA FOR ${}_nP_r$, $r \leqq n$

We shall now derive a formula for ${}_nP_r$, where $r \leqq n$; that is, we shall construct a formula for the total number of permutations of n things, taking them r at a time.

Consider n different things, $a_1, a_2, a_3, \cdots, a_n$. Evidently a first a can be chosen in n ways. Once a first a is selected, a second may be chosen in $n - 1$ ways [one out of the $(n - 1)a$'s left]. Once the second a has been selected, a third may be chosen in $(n - 2)$ ways, and so on. Hence, by the fundamental theorem, the r a's can be selected in $n(n - 1) \times (n - 2) \cdots [n - (r - 1)]$ ways.

Hence, $${}_nP_r = n(n - 1) \cdots [n - (r - 1)].$$

When $r = n$, we have

$${}_nP_n = \underline{|n}\,.$$

109. PERMUTATIONS OF n THINGS, q OF WHICH ARE ALIKE

If of the n things to be permuted n at a time q are alike, we would not have so many distinguishably different permutations as when all things are different, for the permutation of the q alike things among themselves would not give distinguishably different arrangements. But the q things may be permuted among themselves ${}_qP_q = \underline{|q}$ ways. Hence, if x is the number of distinguishably different permutations possible, we have a total of $x\underline{|q}$ permutations. But, the total number of permutations of n things, if they are all different, is

$${}_nP_n = \underline{|n}\,.$$

Hence, $$x\underline{|q} = \underline{|n}\,,$$

or $$x = \frac{\underline{|n}}{\underline{|q}},$$

this result representing the number of permutations of n things, n at a time, q of which are alike.

This result may be generalized immediately for the case where we desire the total number of permutations of n things, of which there are q alike of one kind, r alike of a second kind, s alike of a third kind, and so on, it being understood that

$$n = q + r + s + t + \cdots.$$

If x is the required number, we now have $x\,\underline{|q}\;\underline{|r}\;\underline{|s}\;\underline{|t} \cdots = \underline{|n}$; hence,

$$x = \frac{n!}{\underline{|q}\;\underline{|r}\;\underline{|s}\;\underline{|t} \cdots}.$$

EXERCISES 72

1. How many odd numbers of four figures each could be formed from the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 without repeating a digit in the same number?

2. In how many ways can a rowing crew of eight men be arranged?

3. In how many different orders can six debutantes be introduced at a "coming-out party"?

4. How many different signals may be formed from five different colored flags arranged horizontally, using any number at a time?

5. In how many ways can a party of four seat themselves in a seven-passenger car?

6. How many different permutations can be formed from the letters of the word *different,* using all the letters each time?

7. In how many ways can a football team of 11 men be arranged if the quarterback, fullback, and center must always play the same positions, the ends may be interchanged but can play in no other position, and the halfbacks may be interchanged but play in no other position?

8. How many different signals using a vertical array of flags can be formed from five red flags, three white flags, and three blue flags, using all eleven flags in each signal?

9. Five red squares and four green squares of the same size are to be put together to form one large square. How many different designs are possible?

10. In how many ways may six men and five women be arranged in a chorus if men and women must alternate?

110. COMBINATIONS

Definition: Each of the group selections, ignoring order, which can be made by taking some or all of a number of things is called a *combination.* That is, the word *combination* refers to the variety of groups and not to the arrangement within each group; the arrangement of the objects within the group does not alter the combination. If we have the five different things, a_1, a_2, a_3, a_4, a_5, and we desire the number of combinations of them in groups of three, we write this symbolically as ${}_5C_3$. These combinations are $a_1a_2a_3$, $a_1a_2a_4$, $a_1a_2a_5$, $a_1a_3a_4$, $a_1a_3a_5$, $a_1a_4a_5$, $a_2a_3a_4$, $a_2a_3a_5$, $a_2a_4a_5$, $a_3a_4a_5$; hence, ${}_5C_3 = 10$.

To analyze this particular illustration further, we note that if we were to permute the five things three at a time, we would have ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$ permutations, but since a permutation involves a rearrangement of the three things within each possible group, we note that the number of permutations is ${}_3P_3$ times the number of combinations. That is, such a combination as $a_1a_2a_3$ is actually counted in the number of permutations six different times, for $a_1a_2a_3$, $a_1a_3a_2$, $a_2a_3a_1$, $a_2a_1a_3$, $a_3a_1a_2$, $a_3a_2a_1$ are different permutations, even if they designate the same combination. Consequently,

$$\underline{|3}\,({}_5C_3) = {}_5P_3 \qquad \text{or} \qquad {}_5C_3 = \frac{{}_5P_3}{\underline{|3}}.$$

Similarly, if we require ${}_nC_r$, we first find

$${}_nP_r = n(n-1)(n-2)\cdots[n-(r-1)],$$

and then note that this result includes all the possible rearrangements within each group of r things, which may be done in ${}_rP_r = \underline{|r}$ ways; hence,

$$\underline{|r}\,({}_nC_r) = {}_nP_r \qquad \text{or} \qquad {}_nC_r = \frac{{}_nP_r}{\underline{|r}}.$$

Illustration: How many different committees of three men may be formed from six men?

Since order within each committee is not significant, we simply require

$${}_6C_3 = \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20.$$

EXERCISES 73

1. How many committees of three each could be formed from a group of ten people?

2. How many different baseball nines can be formed from a squad of 30 players, assuming that each one can play any position?

3. If only 3 of the 30 players in Exercise 2 can pitch and these three can play in no other position, how many nines can be formed?

4. If of the 30 players of Exercise 2, only 3 can pitch and only 3 others can catch, and if these men cannot play in any other position, how many nines could be formed? Suppose these 6 men can also play the outfield, how many nines could be formed?

5. In how many ways can a person make up a dinner party consisting of from 1 to 5 invited guests from a list of 10 friends?

6. How many committees consisting of 3 men and 2 women can be formed from 20 men and 15 women?

7. How many straight lines can be drawn through pairs of points selected from eight points if no three of the points are in the same straight line?

8. Prove that ${}_nC_r = {}_nC_{n-r}$.

9. By use of the formula of Exercise 8, find the value of ${}_{100}C_{95}$.

10. If we draw 5 balls at random from a bag containing 10 red and 15 white balls, in how many ways may we get 3 red and 2 white balls?

11. How many different bridge hands (13 cards) can be made from a complete pack of cards (52 cards)?

12. How many different collections of five cards are possible from the cards in a complete pack if it is specified that exactly three of the cards are to be aces?

EXERCISES 74

Miscellaneous Problems Involving Permutations and Combinations

1. A basket of fruit contains 1 doz oranges, 10 apples, and 5 pears. In how many ways may a selection of 3 be made that shall contain 1 orange, 1 apple, and 1 pear?

2. From the basket in Exercise 1, in how many ways may a selection of three be made that shall contain at least one orange?

3. From the basket in Exercise 1, in how many ways may a selection of three be made that shall contain no oranges?

4. Fifteen examination papers are to be distributed among 15 students, 1 paper to each student. In how many ways may it be done?

5. Three students are to be chosen from a group of 15 students for a special assignment. How many different groups of 3 may be selected?

6. A true-false test of 10 questions is such that each question may be answered by the words "true" or "false." In how many different ways may the set of questions be answered by students who guess?

7. If a cable contains 50 wires, in how many ways may they be connected in pairs?

8. There are seven subjects that a student desires to study, but he is allowed to register for only five. In how many ways may he select the five subjects?

9. Three points determine a plane. How many different planes are determined by 10 points, no 4 of which are in the same plane?

10. If there are 30 divisions on the dial of a "combination" lock and 3 settings must be made to operate the lock, how many settings are possible? Ignore the direction and number of turns between settings.

11. There are 20 people at a party and each person must shake hands with each of the others. How many handshakes are there?

12. Ten hockey players decide to choose two teams of five each for a game. In how many ways could this be done?

13. A roominghouse has 10 rooms, and there are 6 applicants for the rooms. In how many ways may the rooms be assigned if no 2 people are assigned to the same room?

14. In how many ways may the eight men in a squad be arranged in military formation if the same man must always act as corporal?

15. Twelve men and twelve women attend a contract-bridge party. In how many ways may teams consisting of a man and a woman be formed?

16. A mixed-doubles team (man and woman) from Club A is to compete against a mixed-doubles team from Club B. In how many ways may the tennis match be staged if Club A is composed of 10 men and 10 women and Club B has 12 men and 8 women?

17. In how many ways can 12 books be arranged on a shelf if one set of 3 volumes is kept together?

111. PROBABILITY

Let us assume that some event, if given a "trial," must happen or fail to happen in one of a limited number of ways, each of which is equally likely. By trial we mean any operation which gives an event an opportunity to happen. Thus, if we have a bag containing m white and n black tickets, a single ticket of a designated color withdrawn from the bag is such an event. Also, the turning of a particular face uppermost when a die is thrown is such an event. In our illustrations the actual drawing of a ticket and the tossing of a die are called *trials*. It is an important part of our consideration that any one of the $m + n$ tickets is equally likely to be drawn or that any one of the faces of the die is equally likely to fall uppermost. Under such assumptions, we define the probability

that an event occurs under trial as the ratio of the number of favorable cases to the entire number of possible cases, favorable and unfavorable.

Thus, if the problem is to determine the probability of drawing a white ticket under the conditions described in the previous paragraph, we note that we have a total of $m + n$ tickets (all the possible cases), of which m are white (the favorable cases); and so, according to the definition, the probability p of drawing one of the white tickets is given by the ratio

$$p = \frac{m}{m+n}.$$

By similar reasoning, the probability q of drawing one of the black tickets is

$$q = \frac{n}{m+n}.$$

It may also be said that $m/(m + n)$ is the probability of drawing a white ticket and $n/(m + n)$ is the probability of failing to draw a white ticket; or that $n/(m + n)$ is the probability of drawing a black ticket and $m/(m + n)$ is the probability of failing to draw a black ticket.

112. EXCLUSIVE EVENTS

If two or more events are so related that but one of them can occur, they are said to be *mutually exclusive.*

Theorem. The probability that some one of a set of mutually exclusive events will occur is the sum of the probabilities of the single events.

This theorem follows immediately from the definition of probability and the fact that the events are mutually exclusive.

Illustration 1: In throwing a die, the probability of throwing an ace is $\frac{1}{6}$, since there is only one ace out of six faces. Likewise, the probability of throwing a deuce is $\frac{1}{6}$. Hence, the probability of throwing an ace or a deuce is evidently

$$\tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}.$$

Illustration 2: We may note as a consequence of the above theorem that the probability of drawing *either* a black *or* a white ticket in the example of Section 111 is

$$\frac{m}{m+n} + \frac{n}{m+n} = 1.$$

In general, a probability of 1 indicates a certainty. By contrast, if an event is certain not to happen, the probability of its occurrence is zero.

Illustration 3: If four coins are tossed simultaneously, find the probability that there will be two heads and two tails.

The total number of ways in which these coins can fall is evidently $2^4 = 16$. The total number of ways of obtaining two heads is ${}_4C_2 = 6$.

Therefore, the probability of obtaining two heads (and, of course, two tails) is $\frac{6}{16} = \frac{3}{8}$.

Illustration 4: What is the probability of drawing two kings from a pack of cards if only one draw of two cards is made?

Two kings may be drawn from the four kings in the pack in ${}_4C_2$ ways, or six ways. The total number of pairs of every variety that may be drawn from the pack is ${}_{52}C_2 = \dfrac{(52)(51)}{(1)(2)} = 1326$. Hence, the required probability is $\dfrac{6}{1326} = \dfrac{1}{221}$.

EXERCISES 75

1. A single cubical die with its faces marked from 1 to 6 is thrown once. What is the probability that the face marked 6 will come up?

2. Three balls are drawn simultaneously from a bag containing six red and nine white balls. (*a*) What is the probability that all will be white? (*b*) That two will be white and one red?

3. The American Experience Mortality Table shows that of 92,637 people living at age twenty, 723 will die within a year. What is the probability that a person aged twenty will die before his next birthday? What is the probability that he will live?

This problem is typical of those problems in the field of probability that can be studied only after the gathering of data. Such considerations are frequently treated under the heading *empirical probability.*

4. Of the 92,637 people alive at age twenty (Exercise 3), 69,804 will be alive at fifty, according to the mortality table. What is the probability that a person aged twenty will live to be fifty? Will die before he is fifty?

5. A man has a flock of 20 hens, 7 of which are layers. If he selects 1 of the hens at random, what is the probability that the one selected will be a laying hen?

6. In drawing four cards from a pack, what is the probability that all will be aces?

7. In naming a date at random, what is the probability that it will fall on Sunday?

8. If 5 balls are drawn at random from a bag containing 10 red and 15 white balls, what is the probability that 3 will be red and 2 white?

9. Find the probability of throwing exactly 7 in a single throw of two dice.

10. Tickets numbered from 1 to 100 are placed in a box. If a ticket is drawn, what is the chance that it will be a predesignated number? What is the probability that it will be an even number? What is the probability that it will be less than 10?

113. INDEPENDENT EVENTS

Definition: Events are said to be independent or dependent according as the occurrence of any one of them does not or does affect the occurrence of others in the set.

Theorem. *The probability that all of a set of independent events will occur is the product of the probabilities that each of the single events will occur.*

Thus, if an event can happen in a_1 ways and fail in b_1 ways, and if another event independent of the first can happen in a_2 ways and fail in b_2 ways, the probability that both events will happen is

$$\frac{a_1 \cdot a_2}{(a_1 + b_1)(a_2 + b_2)}.$$

The proof for this is a direct result of the fundamental theorem (Section 107).

Illustration: What is the probability of throwing a 3 on the first throw of a single die and then throwing a 5 on the second throw?

These events are obviously independent, and the probability in each case is $\frac{1}{6}$. So the desired probability is $(\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}$.

114. DEPENDENT EVENTS

If the probability of a first event is p_1, and if after this event has happened the probability of a second event is p_2, the probability that both events will occur in the order stated is $p_1 \cdot p_2$, and in general for a series of events the probability for the order stated is $p_1p_2p_3 \cdots$.

Illustration: A box contains three white tickets and four black tickets. What is the probability that successive draws of single tickets from the box will yield white tickets if the ticket drawn first is not returned to the box?

The probability of obtaining a white ticket upon the first draw is $\frac{3}{7}$. After a white ticket is drawn, the box contains two white tickets and four black tickets, so the probability of drawing a white ticket the second time is $\frac{2}{6}$, or $\frac{1}{3}$. Consequently, the probability that these dependent events will occur as stated is $(\frac{3}{7})(\frac{1}{3}) = \frac{1}{7}$.

EXERCISES 76

1. Four cards are drawn from a pack one at a time. (*a*) What is the probability of drawing four aces, if each card drawn is returned before the next is drawn? (*b*) If the card drawn is not returned?

2. If two dates are named at random, what is the probability (*a*) that both will fall on Sundays? (*b*) that the first will fall on Sunday and the second on Saturday?

3. In a certain locality 80 per cent of the days in June are clear. If four successive days are named in advance, what is the probability (*a*) that all will be clear? (*b*) that the first two will be clear and the next two not?

4. A traveler has three railroad connections to make. If the probability that he will make any one of them, taken alone, is 0.6, what is the probability that he will make all his connections?

5. If a man and woman are married when each is twenty years of age, what is the probability that both will be living at fifty years of age? (Use the data of Exercise 4 in the previous list.) What is the probability that the man will be living but the woman will not?

6. A man has a flock of 20 hens, 7 of which are layers. If he selects two hens at random, what is the probability that both are layers? Check your

result by also finding the probability that neither is a layer and that one is a layer and the other not, and adding all three probabilities.

7. If from the flock in Exercise 6 the man selects three hens at random, what is the probability that all are layers? Check your result by finding the probability that all are nonlayers, that one is a layer and two are not, and that two are layers and one is not.

8. Tickets numbered from 1 to 100 are placed in a box. If two tickets are drawn in succession, what is the probability that both are even? That both are less than 10? That the two numbers drawn will be 75 and 62 in that order?

9. A man holds five tickets in a lottery in which there is a single prize. If there are 100 tickets, what is his probability of winning?

10. If there are two prizes in the lottery of Exercise 9, what is the probability that he will win both?

11. (*a*) If a single card is drawn from a pack of cards, what is the probability that the ace of spades will be drawn at least once in four tries? (*b*) What is the probability that at least one ace will be drawn in four tries?

12. If a bag contains five white balls, seven black balls, and three red balls, and one ball is drawn at random, what is the probability (*a*) that it is either red or white? (*b*) that it is neither red nor white?

13. If the probability that A will live 10 years is $\frac{5}{6}$, and that B will live 10 years is $\frac{9}{10}$, what is the probability that one or the other but not both will be alive in 10 years? What is the probability that both will be dead?

115. VALUE OF AN EXPECTATION

If p denotes the probability that a person will win a sum of money S, the product pS is called the *value of his expectation.*

Illustration: What is the value of the expectation of a person who is to have any two coins that he may draw at random from a purse that contains five \$1 pieces and seven 50-cent pieces?

The probability of drawing two \$1 pieces is

$$\frac{{}_5C_2}{{}_{12}C_2} = \frac{5 \cdot 4}{12 \cdot 11} = \frac{5}{33}.$$

So, the value of the expectation of drawing two \$1 pieces, that is, winning \$2, is

$$\$2 \cdot \frac{5}{33} = \$0.30.$$

Of course, there is also the possibility of drawing two 50-cent pieces. The value of the expectation of drawing two 50-cent pieces is

$$\$1 \left(\frac{{}_7C_2}{{}_{12}C_2}\right) = \$0.32.$$

Likewise, a draw of a \$1 piece and a 50-cent piece is a possibility. The value of the expectation of drawing one \$1 piece and one 50-cent piece is

$$(\$1.50)\left(\frac{5 \cdot 7}{{}_{12}C_2}\right) = \$0.80.$$

The total value of the expectation will be the sum of the individual expectations, or \$1.42.

EXERCISES 77

1. A gambler holds 4 numbers in a game of chance in which there are 60 numbers around the rim of a wheel. Since the winner is to receive a prize of $1 if the wheel stops at one of his numbers, what is the value of his expectation?

2. A man buys a ticket in a lottery for which there are 200 tickets. If there are one $500 prize, three $100 prizes, ten $50 prizes, and twenty $5 prizes, what is the value of his expectation?

3. A box contains 5 packages valued at 25 cents each, 10 packages valued at 50 cents each, and 20 packages valued at $1 each. If a person is allowed to draw one package at random, what is the value of his expectation? If he draws two packages, what is the value of his expectation? If he draws five packages?

4. What is the value of a ticket in a lottery of 100 tickets if there are one $10 prize, two $5 prizes, five $2 prizes, and ten $1 prizes?

5. A man aged twenty is to receive $5000 if he is living at age twenty-five. According to the American Experience Mortality Table, of 92,637 persons alive at age twenty, 89,032 of them will be alive at age twenty-five. What is the value of the man's expectation?

MISCELLANEOUS EXERCISES 78

1. A bag contains five red, four black, and six white balls. If three balls are drawn at random, what is the probability (*a*) that all are red? (*b*) that all are black? (*c*) that either all are black or all are red? (*d*) that one is red, one is black, one is white? (*e*) that at least one is white?

2. What is the probability of throwing either an ace or a deuce in two throws of a die?

3. A drawer contains 12 black socks and 8 brown socks. If a student reaches in the drawer and pulls out a pair at random, what is the probability that the socks match?

4. If three pieces of fruit are drawn at random from a basket of four oranges, five apples, and six pears, what is the probability (*a*) that all will be oranges? (*b*) that one apple, one orange, and one pear will be drawn? (*c*) that two pears and one orange will be drawn?

5. If the probabilities that A and B will survive 20 years are 0.7 and 0.8, respectively, what is the probability (*a*) that one or the other will live 20 years? (*b*) that one will be dead in 20 years? (*c*) that both will be dead in 20 years? (*d*) that A will be alive and B will be dead?

6. If the probability is $\frac{1}{2}$ that an ear of corn selected at random from a field is between 8 and 9 in. long, and the probability is $\frac{1}{3}$ that it is between 7 and 8 in. long, what is the probability that an ear selected is between 7 and 9 in. long?

7. Two cards are drawn at random from a pack. What is the probability (*a*) that they are both of one suit? (*b*) that they are the ace and king of spades? (*c*) that they are aces?

8. Seven couples attended a dance, and the men drew lots for their partners. Each man drew his own wife. What is the probability that this will occur?

9. A woman is to win a $10 prize if each of two consecutive draws from a pack of cards produces an ace. The first card is to be replaced in the pack after it is drawn. What is the value of her expectation?

10. Two women have tickets for the same row at a concert. If the row has 26 seats, what is the probability that they will be seated side by side?

17 Partial Fractions

116. PARTIAL FRACTIONS

The process of changing a fraction of the form $\frac{f(x)}{\phi(x)}$, where $f(x)$ and $\phi(x)$ are rational integral functions, into an equivalent algebraic sum of simpler fractions is called *reducing to partial fractions.*

Thus, the student can readily verify that $\frac{1}{x(x+1)}$ may be written in the form $\frac{1}{x} - \frac{1}{x+1}$; and that $\frac{x}{(x+1)(x+2)}$ may be written in the form $\frac{2}{x+2} - \frac{1}{x+1}$.

In scientific work, and especially in the calculus, it is frequently necessary to reduce a given fraction $\frac{f(x)}{\phi(x)}$ to its partial fractions. We shall assume that the degree of $f(x)$ is lower than that of $\phi(x)$; otherwise, we must first divide and write

$$\frac{f(x)}{\phi(x)} = Q(x) + \frac{f_1(x)}{\phi(x)},$$

where $Q(x)$ is the rational integral function obtained as the quotient after division and where $f_1(x)$ is a rational integral function of a lower degree than $\phi(x)$. We may then apply the method of partial fractions to $\frac{f_1(x)}{\phi(x)}$.

It is desirable to distinguish between several cases, depending on the nature of the zeros of $\phi(x)$. These various cases will be studied through the use of specific examples.

CASE 1. The zeros of $\phi(x)$ are all real and distinct. Thus, as an illustration, let us consider

$$\frac{f(x)}{\phi(x)} = \frac{x-1}{(x+1)(x+2)(x+3)}.$$

Assume that

$$\frac{x-1}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3},$$

where A, B, and C are constants to be determined.

Clearing of fractions, we have

$$x - 1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2). \quad (1)$$

The two members of Equation (1) are to be equal for all values of x, except possibly for $x = -1$, $x = -2$, $x = -3$, since the two members of the assumed equation are not defined for these special values of x. However, it is demonstrable that if two polynomials of degree n are equal for more than n distinct values of the variable, they are equal for all values. Hence, the two members of Equation (1) are equal for all values of x.

For the purposes of this discussion, we look upon (1) as an equation from which we may determine A, B, C so that the right member of (1) will reduce to $x - 1$. We may determine A, B, C in two distinct ways.

As a first process, we may substitute any three numerical values for x, and obtain three equations from which A, B, and C may be determined. However, the work of determining A, B, C is easier if we select the specific values $x = -1$, -2, and -3, and obtain the respective equations

$$\begin{aligned} -2 &= 2A \quad &\text{or}\quad A &= -1,\\ -3 &= -B \quad &\text{or}\quad B &= 3,\\ -4 &= 2C \quad &\text{or}\quad C &= -2. \end{aligned}$$

Hence, the fraction

$$\frac{x-1}{(x+1)(x+2)(x+3)}$$

may be reduced to the partial fractions

$$-\frac{1}{(x+1)} + \frac{3}{(x+2)} - \frac{2}{(x+3)}.$$

This result may easily be checked by observing that the three fractions may be combined to obtain the given fraction.

As a second method, we may equate the coefficients of the same power of x in the left and right members of Equation (1); this operation leads to three equations from which A, B, and C may be determined.

Thus, Equation (1), namely,

$$x - 1 = (A + B + C)x^2 + (5A + 4B + 3C)x + (6A + 3B + 2C),$$

may be written in the form

$$0 \cdot x^2 + x - 1 = (A + B + C)x^2 + (5A + 4B + 3C)x + (6A + 3B + 2C).$$

Hence, after equating the coefficients of like powers of x, we have

$$A + B + C = 0,$$
$$5A + 4B + 3C = 1,$$
$$6A + 3B + 2C = -1.$$

After solving this system, we have $A = -1$, $B = 3$, $C = -2$. Thus, the previous result is again obtained.

CASE 2. $\phi(x)$ contains real, multiple zeros, and others that are real and distinct.

For the consideration of this case let us take as an illustration

$$\frac{f(x)}{\phi(x)} = \frac{x^2}{(x+1)^2(x+2)(x+3)}.$$

Assume this time that

$$\frac{x^2}{(x+1)^2(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{x+3}.$$

It should be observed in this partial-fraction development that the repeated factor $(x + 1)$ is employed as a denominator to both the first power and the second power. In general, if $(x - k)^n$ appears as a factor of $\phi(x)$, separate partial fractions employing the denominators $(x - k)^n$, $(x - k)^{n-1}$, $(x - k)^{n-2}$, $\cdots$, $(x - k)$ should be set up.

If we use the first method employed in treating Case 1, the values for B, C, and D are found at once upon putting $x = -1$, $x = -2$, $x = -3$. The use of any other value for x and the values found for B, C, and D will determine A. The values are $A = -\frac{7}{4}$, $B = \frac{1}{2}$, $C = 4$, $D = -\frac{9}{4}$.

The second method employed in treating Case 1 results in four linear equations in A, B, C, and D from which the values of A, B, C, and D may be found. It is left as an exercise for the student to determine the values of A, B, C, and D by the second method.

CASE 3. $\phi(x)$ contains imaginary zeros, and thus $\phi(x)$ contains an irreducible quadratic factor.

NOTE: A quadratic expression

$$ax^2 + bx + c, \qquad \text{where } b^2 - 4ac < 0,$$

is called an *irreducible quadratic expression*. This simply means that $ax^2 + bx + c$ has for its zeros imaginary numbers.

Thus, as an illustration, let

$$\frac{f(x)}{\phi(x)} = \frac{x}{(x^2+1)(x+2)^2(x+3)},$$

wherein $(x^2 + 1)$ is irreducible. Assume that

$$\frac{x}{(x^2+1)(x+2)^2(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{x+3}.$$

In the development upon the right, it is observed that the general linear expression $Ax + B$ is employed as the numerator that corresponds to the quadratic denominator. Hence,

$$x = (Ax + B)(x + 2)^2(x + 3) + C(x^2 + 1)(x + 2)(x + 3) + D(x^2 + 1)(x + 3) + E(x^2 + 1)(x + 2)^2.$$

If we let $x = -2$ and $x = -3$, we obtain

$$-2 = 5D$$

and

$$-3 = 10E.$$

If we now choose any other three values for x, we obtain equations from which we may determine A, B, C. Thus, if we choose $x = 0, x = 1, x = 2$, we obtain, respectively,

$$0 = 12B + 6C + 3D + 4E,$$

$$1 = 36A + 36B + 24C + 8D + 18E,$$

and

$$2 = 160A + 80B + 100C + 25D + 80E.$$

From the five equations we may determine the values of the five constants. The solutions of these equations gives

$$A = \tfrac{1}{50},\quad B = \tfrac{3}{50},\quad C = \tfrac{7}{25},\quad D = -\tfrac{2}{5},\quad E = -\tfrac{3}{10}.$$

The determination of these constants by the second method discussed in connection with Case 1 is left as an exercise for the student.

Summary. It can be proved that in an integral rational function the imaginary zeros occur in pairs and that for every such pair we have a factor of the form $x^2 + px + q$; hence, the denominator of the fraction $\dfrac{f(x)}{\phi(x)}$ can always be written as the product of quadratic factors and linear factors in the form

$$(x^2 + p_1x + q_1)^r(x^2 + p_2x + q_2)^s \cdots (x - a)^t(x - b) \cdots.$$

For every irreducible factor $x^2 + px + q$ repeated k times, we assume k partial fractions of the form

$$\frac{A_1x + B_1}{x^2 + px + q},\quad \frac{A_2x + B_2}{(x^2 + px + q)^2}, \ldots, \frac{A_kx + B_k}{(x^2 + px + q)^n},$$

and for every factor of the form $(x - l)$ repeated n times, we assume n partial fractions of the form

$$\frac{L_1}{x - l},\quad \frac{L_2}{(x - l)^2}, \ldots, \frac{L_n}{(x - l)^n}.$$

Then we proceed to determine the constants as illustrated above.

EXERCISES 79

Reduce each of the following to partial fractions:

1. $\dfrac{5x-1}{(x-1)(x^2-5x+6)}$

2. $\dfrac{7x+2}{(x+1)(x+3)x}$

3. $\dfrac{5}{(x-1)^2(x+2)}$

4. $\dfrac{3x}{(x-1)^2(x+2)^2}$

5. $\dfrac{6x^2+5}{(x^2+1)(x+2)}$

6. $\dfrac{3x^2+7}{(x^2+1)(x+1)^2}$

7. $\dfrac{8x^3+5x^2+7}{(x-1)(x-2)}$

8. $\dfrac{1}{(x^2+1)(x^2+2)^2}$

9. $\dfrac{1-2x}{2-x-3x^2}$

10. $\dfrac{3x+4}{x^3-x}$

11. $\dfrac{3x^2+8}{x^2-5x+6}$

12. $\dfrac{x^3+6x^2-2x-41}{x^3+4x^2+x-6}$

13. $\dfrac{x^3+3}{x^4+4x^2}$

14. $\dfrac{6}{x^3-3x+2}$

15. $\dfrac{7x^2}{(x-1)^2(x^2+x+2)}$

16. $\dfrac{5x^2-3}{x^3-x}$

17. $\dfrac{w^2}{(w-1)^3}$

18. $\dfrac{3-x}{x^3+4x^2+3x}$

19. $\dfrac{x^3+3x}{(x^2+1)^2}$

20. $\dfrac{8}{x^4-1}$

18 Inequalities

117. GENERAL PRINCIPLES

It is frequently necessary to consider the truth of an assertion that one number is greater than another, or the conditions under which one variable is greater than another variable. Such studies are classified under the heading *inequalities.* If the symbol $>$ designates "is greater than," the symbol $<$ denotes "is less than" (this symbol always points toward the smaller quantity), and $\neq$ means "is not equal to," the following statements are typical inequalities:

$$3 > -2;$$
$$7 < 13;$$
$$a \neq b.$$

The type of inequality considered in this chapter applies only to real numbers, that is, those numbers which possess the property of order upon the usual number scale. The student should bear this restriction in mind.

Two inequalities in which the inequality signs point in the same direction are said to *have the same sense.* If the signs point in opposite directions, the inequalities are said to be *opposite in sense.* Thus, the statements $4 > 3$ and $-1 > -5$ have the same sense; whereas $1 > 0$ and $3 < 7$ are opposite in sense. In some studies based upon inequalities it is desirable to use the combination symbol $\geqq$ to mean "is greater than or equal to" and $\leqq$ to denote "is less than or equal to."

As in the study of equalities, there are two kinds of inequalities involving variables, namely, *absolute* inequalities and *conditional* inequalities. An absolute inequality is an inequality that is valid for all permissible values of any variables which may be involved. The statement $x^2 + 1 > 0$ is a typical absolute inequality; for it is true for any real value of x. A conditional inequality in one variable is an inequality that is valid for only a specific set of the permissible values of the variable. Thus, the inequality $2x + 1 > 3$ is only valid when $x > 1$.

118. OPERATIONS UPON INEQUALITIES

Many of the permissible operations upon inequalities resemble the corresponding principles employed in dealing with equalities; yet there are some striking differences. The rules employed in operating upon

inequalities are stated below without proof; they will probably seem quite reasonable, however.

(1) *The addition of the same real number to, or the subtraction of the same real number from, the two members of an inequality leaves the sense of the inequality unchanged.*

(2) *The multiplication or division of the two members of an inequality* ***by the same positive number,*** *leaves the sense of the inequality unchanged.*

(3) *The multiplication or division of the two members of an inequality* ***by the same negative number*** *changes the sense of the inequality.*

(4) *If both members of two inequalities of the same sense are* ***positive,*** *and if the corresponding members of the inequalities are multiplied, an inequality of the same sense is obtained.*

An interesting consequence of this rule is the proposition that if both members of an inequality are positive, any positive power of both members yields an inequality having the same sense.

119. ABSOLUTE INEQUALITIES

The discussion of this section will be introduced by means of an example.

Confirm the fact that for any positive number x, $x + \frac{1}{x} \geqq 2$. This, of course, is an interesting theorem and is typical of the absolute inequalities frequently met in practice.

Let us start the demonstration of the validity of this inequality for all $x > 0$ by considering the known inequality

$$(x - 1)^2 \geqq 0.$$

Since the square of any real number is greater than or equal to zero, this statement is true without the restriction that $x > 0$. After expanding the left member, we have

$$x^2 - 2x + 1 \geqq 0.$$

The addition of $2x$ to each member yields

$$x^2 + 1 \geqq 2x \qquad \text{(by Rule 1).}$$

We may now divide each member by x. Now, if $x > 0$, we may write

$$x + \frac{1}{x} \geqq 2 \qquad \text{(by Rule 2).}$$

This completes the demonstration.

The student undoubtedly wonders how we knew to start our demonstration with the known inequality $(x - 1)^2 \geqq 0$. To know what to start with, it is common to assume the validity of the inequality that was proposed, whereupon any of the four laws are applied in an attempt to obtain an inequality that is known to be valid. The process is then reversed, making certain that each step is justified.

As a second illustration let us show that

$$a + b > \frac{4ab}{a + b}, \qquad \text{if } a > 0,\ b > 0, \text{ and } a \neq b.$$

Since we are in doubt how to proceed, let us attempt to go backward, using the four basic laws, in an attempt to obtain a valid inequality.

Let us multiply each member of the given inequality by the positive quantity $a + b$; of course, the sense will remain the same, so we have

$$a^2 + 2ab + b^2 > 4ab.$$

Then, after subtracting $4ab$ from each member, the inequality becomes

$$a^2 - 2ab + b^2 > 0,$$

or

$$(a - b)^2 > 0.$$

This inequality is known to be true since $a \neq b$.

The desired confirmation of the proposed inequality is readily accomplished by starting with the latter inequality and reversing the steps as follows:

$(a - b)^2 > 0.$	Known since $a \neq b$.
$a^2 - 2ab + b^2 > 0.$	Expanding the square.
$a^2 + 2ab + b^2 > 4ab.$	Adding $4ab$ to each member.
$(a + b)^2 > 4ab.$	
$a + b > \dfrac{4ab}{a + b}.$	Dividing each member by the positive quantity, $a + b$.

EXERCISES 80

1. If $a > 1$, prove that $a^2 > a$.
2. If $a > b$, prove that $a^2 > b^2$.
3. Show that $\dfrac{a}{2b} + \dfrac{2b}{a} > 2$, if $a > 0$, $b > 0$, and $a \neq 2b$.
4. Show that the arithmetic average of two unequal positive numbers a and b $\left(\text{that is, } \dfrac{a + b}{2}\right)$ is greater than their geometric average (that is, $\sqrt{ab}$).
5. Show that $x^2 + y^2 + z^2 > xy + xz + yz$, if $x \neq y \neq z$.
6. Show that $\dfrac{a + b}{2a} > \dfrac{b}{a + b}$, if $a > 0$ and $b > 0$.
7. Show that $\dfrac{a + b}{4a} > \dfrac{b}{a + b}$, if $a > 0$, $b > 0$, and $a \neq b$.
8. If x and y are positive and $x > y$, prove that
$$x^3 + x^2 + x + 1 > y^3 + y^2 + y.$$
9. If a and b are positive and $a > b$, show that
$$a^2 + b^2 > ab.$$
10. Show that $\dfrac{1}{x^3} + \dfrac{1}{y^3} > \dfrac{1}{x^2y} + \dfrac{1}{xy^2}$, if $x > 0$, $y > 0$, and $x \neq y$.

120. CONDITIONAL INEQUALITIES

The discussion of this section is confined to conditional inequalities involving only one variable. Unlike a conditional equation, the solution of such an inequality usually comprises a range of values of the variable rather than a finite number of specific values. For example, in considering the inequality

$$3x - 2 > x + 7,$$

we may add $2 - x$ to each member to obtain

$$2x > 9,$$

which, after dividing each member by 2, becomes

$$x > \tfrac{9}{2}.$$

Thus, the given inequality is satisfied by any value of x in the range $x > \frac{9}{2}$.

The consideration of inequalities with members that are polynomials of degree higher than the first presents a somewhat more interesting situation. For instance, let us solve the inequality

$$3x^2 - 4x + 7 > 2x^2 + x + 1.$$

It is desirable to obtain an equivalent inequality in which the right member is zero. This is accomplished by subtracting the right member from each member, thereby giving

$$x^2 - 5x + 6 > 0.$$

In the consideration of an inequality $f(x) > 0$, where $f(x)$ is a polynomial, we usually find first the roots of $f(x) = 0$. Since $f(x)$ cannot change sign except at the points where $f(x) = 0$, the solution of the inequality $f(x) > 0$ can readily be obtained by an examination of the sign of $f(x)$ on each side of a root of $f(x) = 0$. The roots of the equation

$$x^2 - 5x + 6 = 0$$

are 2 and 3; so we must examine the sign of $f(x) = x^2 - 5x + 6$ on each side of these roots. After selecting a value such as $x = 1$ to the left of 2, we observe that the function is positive. For the value $x = 2\frac{1}{2}$, to the right of 2 but to the left of 3, the function is negative. To the right of 3 the function is positive. This may all be seen very clearly by an examination of the graph of $y = f(x) = x^2 - 5x + 6$; it appears as Figure 43. Consequently, the solution of the given inequality comprises the ranges $x < 2$ and $x > 3$.

Although a more elaborate discussion of conditional inequalities in one variable might be presented, the procedure just discussed is adequate for the solution of most inequalities met in practice.

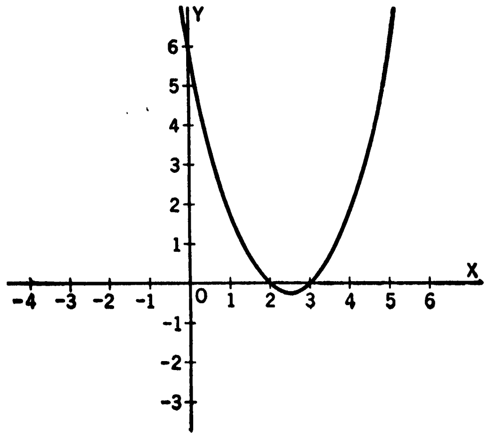


FIG. 43

EXERCISES 81

Solve each of the following inequalities:

1. $3x + 8 > x + 10$
2. $7x - 3 > 8x + 5$
3. $5v + 6 > 9v$
4. $(x + 3)(x - 1) < (x + 2)(x - 3)$
5. $x^2 - 4x + 3 > 0$
6. $2x^2 - 6x + 10 > x^2 + x$
7. $x^2 + x < 3 - x^2$
8. $x^2 + x + 1 > 0$
9. $(x + 3)(x - 2)(x - 7) > 0$
10. $(x - 3)(x - 1)(x)(x + 4) < 0$
11. $x^2 + 3x \geqq 0$
12. $2x^2 + 7x + 2 > 0$

For what range of values of x will each of the following expressions be imaginary?

13. $\sqrt{2x - 3}$
14. $\sqrt{x^2 + 11x + 10}$
15. $\sqrt{-x^2 + 3x + 1}$

Use your ingenuity in solving each of the following inequalities:

16. $\dfrac{1}{x^2} > 1$

17. $\dfrac{x}{x - 1} > 0$

NOTE: $x = 1$ is not a permissible value.

18. $\dfrac{x(x + 3)}{x - 1} > 0$

HINT: $x = 1$ is not a permissible value of x, so $(x - 1)^2$ is positive for all permissible values. Consequently, the sense of the inequality is unaltered for all permissible values of x if each member is multiplied by $(x - 1)^2$.

19. $\dfrac{(x + 2)^2}{x} < 0$

20. $\dfrac{(x - 1)(x + 2)}{x} < 0$

19 Review of Algebra

EXERCISES 82

1. Given $r_1 = \dfrac{Nr}{1 + (N - 1)r}\cdot$ If $r = \frac{1}{2}$ and $N = 90$, find the value of r_1 to three significant figures.

2. If $P = 2c - 0.01c^2$, solve for c in terms of P.

3. If $C = \dfrac{nE}{nr + r}$, solve for n, E, and r each in terms of the other letters in the formula.

4. Factor each of the following expressions:

$$a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$$

$$m^6 - n^6$$

$$9x^4 + 6x^2y^2 + 49y^4$$

$$2x^3 - 11x^2 - 21x$$

$$84 + 5a - a^2$$

$$24ab - 18ay - 20bx + 15xy$$

5. Find the first four terms and the ninth term of the expansion of $\left(x^{1/2} - \dfrac{y}{3}\right)^{13}$.

6. Simplify

$$\frac{\dfrac{2x}{x^2 - 1} - \dfrac{2x(x^2 + 1)}{(x^2 - 1)^2}}{\dfrac{x^2 + 1}{x^2 - 1}\sqrt{\dfrac{(x^2 + 1)^2}{(x^2 - 1)^2} - 1}}.$$

7. Simplify

$$\left(\frac{2a^8x^{7/2}}{3b^4}\cdot\frac{5a^4b^{5/2}}{6c^3x^3}\right) \div \left(\frac{bc^{3/4}}{a^{5/2}x}\cdot\frac{25a^{9/2}x}{18abc^{3/2}}\right).$$

8. Two men walk in opposite directions at the rates, respectively, of $3\frac{1}{2}$ and $4\frac{1}{2}$ mph, starting at the same time from the same place. In how many hours will they be 20 miles apart?

9. The denominator of a certain fraction exceeds the numerator by 13. If 5 is subtracted from each term of the fraction, the resulting fraction has the value $\frac{1}{2}$. What is the fraction?

10. A milk distributor buys raw milk containing $4\frac{1}{4}$ per cent butter fat. He wishes to standardize this milk by adding skim milk so that it will contain only $3\frac{3}{4}$ per cent butter fat. How many pounds of skim milk must he add to each 100 lb of raw milk?

11. Solve for x:

$$\frac{7x}{x+3} - \frac{5x}{1-x} = \frac{12(x^2-1)}{x^2+2x-3}.$$

12. Solve for x:

$$\frac{a}{x+b} - \frac{b}{x+a} = \frac{a-b}{x+a+b}.$$

13. Solve the following system of equations:

$$\begin{aligned} 2x - y + 2z &= -16 \\ x + 3y + z &= 41 \\ 2x + y + 4z &= 22 \end{aligned}$$

14. Find the first four terms and the ninth term of the expansion of $\left(\frac{x^{1/2}}{2} - \sqrt{3y}\right)^{13}$.

15. An hour after starting, a train meets with an accident, after which it proceeds at three fifths of its former speed and arrives 2 hr and 40 min late. If the accident had happened 50 miles farther on the line, the train would have been only $1\frac{1}{2}$ hr late. Find the length of the journey.

16. From the general equation $px^2 + 2qx + r = 0$, derive a formula for solving any quadratic equation.

17. Find the values of x which give $x + \frac{1}{x}$ twice the value it has for $x = 3$.

18. The diameter d of the rivet holes for a certain type of riveted joint is determined from the equation $p = 0.56\frac{d^2}{t} + d$. If $p = 1.50$ and $t = 0.25$, find d correct to two decimal places.

19. (*a*) Find the values of K for which the equation $3x^2 - 2Kx + 1 = 0$ will have equal roots.

(*b*) What values of K will give this equation roots that are not equal?

20. Show graphically that the function $-2 + x - x^2$ cannot be equal to any positive value for a real value of x.

21. Solve

$$\sqrt{x+1} + \sqrt{3x+1} = 2$$

22. Solve

$$x^2 + 3x - 1 - \sqrt{2x^2 + 6x + 1} = 0$$

23. Divide $18xy^{-2} - 23 + x^{-1/2}y + 6x^{-1}y^2$ by $3x^{3/4}y^{-1} + x^{1/4} - 2x^{-1/4}y$.

24. Simplify and express with positive exponents

$$\frac{\left(\frac{y}{27} \div \frac{y^{-2}}{8}\right)^{-2/3} - x^3}{\frac{3y^{-1} + 2x^{-1}}{2}}$$

25. Solve

$$2x^2 - 21 > 11x$$

26. Find the numerical value of $16^{-3/2} + 16x^0 - 16^0 - \left(\frac{8}{31}\right)^{-2}$ to the nearest thousandth.

27. Simplify $7\sqrt{\frac{A}{3}} - \frac{5}{3}\sqrt{27A} + 7\sqrt{\frac{225A}{3}}$.

28. Simplify the expression $\frac{2-\sqrt{3}}{2+\sqrt{3}}(1 + 2\sqrt{3})$, expressing your result without radicals in the denominator.

29. Solve

$$\frac{\sqrt{5x-4} + \sqrt{5-x}}{\sqrt{5x-4} - \sqrt{5-x}} = \frac{2\sqrt{x}+1}{2\sqrt{x}-1}.$$

30. Solve the system

$$2x^2 - xy = 6y$$
$$x + 2y = 7$$

31. Solve the system

$$x^2 - xy = 3$$
$$y^2 + xy = 10$$

32. Two travelers, A and B, set out at the same time on a trip; A is to go from the first town to the second, and B is to go from the second to the first. Both travel at uniform rates. When they meet, A has traveled 25 miles farther than B. A finishes his journey in 4 days and B in $6\frac{1}{4}$ days after they meet. Find the distance between the towns and the number of miles each travels per day.

33. Is 1 a root of $x^{10} - 1 = 0$? Is -1 a root of $x^{10} - 1 = 0$?

34. Show by two methods that 3 is a root of $x^3 - x^2 - 7x + 3 = 0$.

35. Find all the roots of $x^4 + 5x^3 - 3x^2 - 31x - 12 = 0$.

36. Find all the roots of $6x^4 - 13x^3 - 6x^2 + 5x + 2 = 0$.

37. Form the equation whose roots are $0, 7, -2, \frac{1}{2}$.

38. Write the equation whose roots will be 3 less respectively than the roots of $2x^3 - 5x - 3 = 0$.

39. Draw the graph of $y = x^3 - 6x^2 - x + 27$, and find the value of each real root of $x^3 - 6x^2 - x + 27 = 0$ to the nearest thousandth.

40. Find the value of $\log_3 27 + \log_{10}\sqrt[5]{0.01}$.

41. Find the value of $\log_7 49 - \frac{1}{3}\log_2 64 + \log_6 216 + \log_{81} 3 - \log_8 2^{3/4}$.

42. Find the value of $\frac{(0.07536)^2\sqrt[3]{1.0573}}{(0.89304)^{2/3}}$.

43. If $2.3713 = (1.045)^n$, find the value of n.

44. If $S = R\left[\frac{(1+i)^n - 1}{i}\right]$, find n when $S = 1000$, $R = 200$, and $i = 0.045$.

45. Draw the graph of $y = \dfrac{e^{2x} - e^{-2x}}{2}$.

46. A body moves 20 ft the first minute, three times 20 ft in the second minute, five times 20 ft in the third minute, and so on. How far does it move in a half-hour?

47. Find to the nearest thousandth the positive geometric mean between 25 and 41.

48. The geometric mean of several values $a_1, a_2, a_3, a_4, \cdots, a_n$ is defined by the following formula:

$$G = \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots a_n}.$$

Find the geometric mean of 25, 31, 28, 34, 36, and 27.

49. A young man just graduating from college arranged to invest \$100 at the end of each year for 12 yr, the money to earn compound interest at 5%. Find the amount to his credit at the end of the 12 yr, provided that he made all his payments as agreed.

50. What is the least number that the sum of the terms of the infinite geometric progression $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$, will never exceed?

51. In how many ways may a committee of 5, consisting of a college president, a dean, and 3 professors, be chosen from a faculty consisting of a president, five deans, and 70 professors?

52. In forming the committee in Exercise 51, what is the probability that a particular dean will be chosen? A particular professor? Assume that the committee is chosen by chance.

Book II · TRIGONOMETRY

1

1. TRIGONOMETRY

Trigonometry is a word of Greek origin which means the *measurement of triangles.*

While the measurement or solution of triangles still forms an essential part of the study of trigonometry, the subject in its modern sense includes the study of the properties of certain "functions of the angles" and their applications in pure and applied mathematics.

In our use of the word *triangle,* or rectilinear figures having three angles, we assume that the student is already somewhat familiar with the idea of an angle and, possibly, its measurement. For our purposes, however, it is desirable to define carefully what is meant by an angle and its measurement.

2. DIRECTED LINE SEGMENTS

A portion of a line between two of its points A, B is called a *line segment.* We distinguish between two possible directions of the segment. The line segment AB means the line segment from A to B; while the line segment BA means the line segment from B to A. Hence, the line segment AB is opposite in direction to the line segment BA; this fact is denoted by the symbolic statement $AB = -BA$. A directed line segment, therefore, is a line segment measured in a definite direction by designating one of the end points as the initial point and the other end point as the terminal point.

3. DEFINITION OF AN ANGLE

An angle is a geometric figure formed when two line segments have an end point in common. The common end point is called the *vertex of the angle.* To define what is meant by the magnitude of an angle, it is desirable to think of an angle as having been "generated" by the rotation of one line segment about the vertex into coincidence with the other segment. Thus, line-segment AB rotating in the same plane about A as a pivot from the position AB to the position AB' (Figure 1), or from AB to AB'' (Figure 2), is said to generate the angles α and β, respectively. The original position, that is, AB, is referred to as the *initial line* and the other position as the *terminal line.*

It is evident from the figures that the line segment AB rotating about the vertex A may generate an angle either in a counterclockwise manner, as in Figure 1, or clockwise, as in Figure 2. It is customary to designate

the rotation in Figure 1 as giving a positive measure, and the rotation in Figure 2 as providing a negative measure. As a consequence of this statement, it is apparent that any angle may be measured positively or negatively.

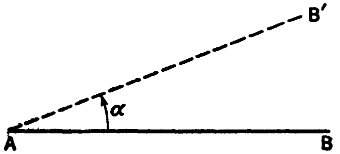


Fig. 1

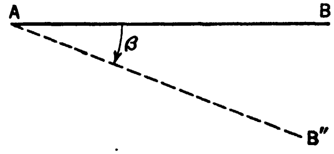


Fig. 2

4. MAGNITUDE OF AN ANGLE

The magnitude of an angle is the amount of rotation about the vertex required to bring the line segment occupying the initial position to the terminal position. The magnitude, or measure, of an angle, therefore, depends upon the position of the initial side, the position of the terminal side, and the extent of the rotation.

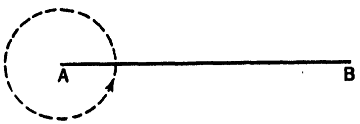


Fig. 3

Obviously, it is necessary to have a unit of measurement in terms of which we may measure magnitudes and compare angles.

If AB (Figure 3) rotates about A in the same plane and in the direction indicated by the circular arrow, from its initial position completely around to that position, it is said to generate an angle of one positive revolution.

For many purposes of measurement the revolution is not a suitable unit, so other systems of units have been invented. One of the oldest is the sexagesimal system characterized as follows:

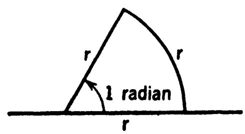


Fig. 4

1 positive revolution = 360 degrees, written 360°.

1 degree = 60 minutes, written 60′.

1 minute = 60 seconds, written 60″.

Hence, 1 degree = $\frac{1}{360}$ of a revolution.

Another unit frequently used in measuring angles is called a *radian.* A radian is defined as the measure of a central angle subtended by an arc equal in length to the radius of the arc (Figure 4).

From plane geometry we have the formula $C = 2\pi r$, where C is the circumference and r the radius of the same circle. This formula means that the ratio of the circumference to the radius of any circle is 2π, where $\pi = 3.14159$, approximately.

From the above definition of a radian we note that

$$2\pi \text{ radians} = 360°,$$
$$\pi \text{ radians} = 180°,$$

Hereafter, whenever we express a magnitude of an angle by a number symbol and with no unit indicated, it is to be understood that the radian is the unit of measurement. Thus, we write

$$2\pi = 360°,$$

$$\pi = 180°.$$

Hence, $$1 = \frac{180°}{\pi} = 57°17'45'', \quad \text{approximately,}$$

or $$1° = \frac{\pi}{180} = 0.01745, \quad \text{approximately.}$$

From the definition of a radian we also note that in a circle of radius r, a central angle θ, subtended by an arc of length a, contains a/r radians. This follows from the fact that arcs of the same circle are proportional to their subtended angles. Therefore,

$$a = r\theta,$$

where θ is expressed in radians, and a and r are expressed in the same units of length.

Hence, if we are given any two elements of the equation $a = r\theta$, we may determine the third.

Illustration 1: Given $a = 16$ ft and $r = 10$ ft, find θ. From the relation $a = r\theta$,

$$\theta = \frac{a}{r}.$$

After substituting the given values for a and r, we obtain $\theta = \frac{16}{10} = 1.6$ radians.

Since 1 radian $= 57°17'45''$, θ may now be calculated in degrees, minutes, and seconds.

Illustration 2: Given $a = 112$ ft and $\theta = 32°$, find r. Since θ in the formula $a = r\theta$ is expressed in radians, it is necessary to change 32° to its equivalent value in radians. Since $1° = \pi/180$ radians, it follows that

$$32° = \frac{32\pi}{180} \text{ radians.}$$

Therefore, $$r = \frac{a}{\theta} = \frac{112}{\frac{32\pi}{180}} = 112 \times \frac{180}{32\pi} \text{ ft} = 200.5 \text{ ft.}$$

EXERCISES 1

1. How many degrees are equivalent to $\pi/2$ radians? $\pi/3$ radians? $\pi/12$ radians? $\frac{2}{3}$ radians? 1.5 radians?

2. Express 30°, 45°, 120°, 300° in radian measure. In each case the result may be written as a multiple of π.

3. Express each of the following in radians: $38°23'$; $72°16'$; $126°32'18''$; $86.7°$; $142°17.3'$; $47°22'46''$.

4. Express each of the following in degrees and minutes: 3.2 radians; 1.62 radians; 2.74 radians; 2.86 radians.

5. Given $r = 10$ ft and $\theta = 72°$, find a.

6. If the spoke of a wheel is 3 ft long, find the central angle subtended by a portion of the rim 2 ft long.

7. A pendulum 18 in. long swings through an angle of $16°13'$. Through how great an arc does the bob at the end swing?

8. A railroad curve is an arc of a circle of radius 927 ft. What is the length of the arc if it subtends an angle of $20°18'$ at the center?

9. The minute hand of a clock is 4.3 in. long. Through how great a distance does its end move in 22 min?

10. A point on a rotating wheel of radius 23 ft moves through a distance of 9.2 ft in 1 sec. What is the angular velocity of the wheel in radians per sec? in degrees per sec?

5. IMPORTANT FACTS AND DEFINITIONS FROM GEOMETRY

(1) A positive right angle $= \frac{1}{4}$ revolution $= 90°$.

(2) A positive straight angle $= \frac{1}{2}$ revolution $= 180°$.

(3) If α and β are two angles such that $\alpha + \beta$ equals $90°$, one is said to be the complement of the other; for example, $\alpha = 30°$ and $\beta = 60°$ are complementary angles.

(4) If α and β are two angles such that $\alpha + \beta = 180°$, then one is said to be the supplement of the other; for example, $\alpha = 50°$ and $\beta = 130°$ are supplementary angles.

(5) The sum of the interior angles of any triangle equals $180°$.

(6) Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

(7) In a right triangle one acute angle is the complement of the other.

(8) In two similar triangles the corresponding sides are proportional.

(9) In a right triangle the sum of the squares of the two legs equals the square of the hypotenuse.

(10) In a right triangle, if one acute angle is $30°$, the leg opposite it is equal to one half the hypotenuse.

EXERCISES 2

1. Find y and r from the data in Figure 5.

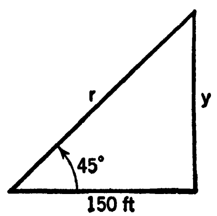


FIG. 5

2. Find x and r from the data in Figure 6.

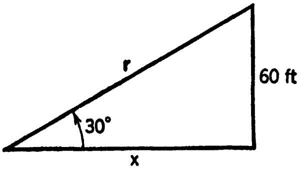


FIG. 6

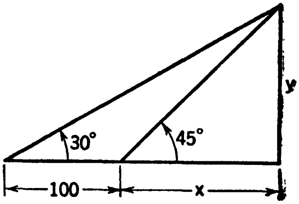


FIG. 7

3. In a certain right triangle one of the acute angles is 30° and the hypotenuse is a ft; find the legs in terms of a.

4. Find x and y from the data of Figure 7.

5. A pole 6 ft high casts a shadow 11.2 ft long at the same time that a taller pole casts a shadow 23.7 ft long. Find the height of the taller pole.

6. TRIGONOMETRIC FUNCTIONS

If we consider the angle A in any of the figures of this article, where OX represents the initial position and OP the terminal position, and where P, whose coordinates are (x, y), is any point on the line OP except O, and if we let $OM = x$, $MP = y$, and $OP = r = \sqrt{x^2 + y^2}$, it is possible to construct exactly six ratios of the lengths x, y, and r; namely, y/r, x/r, y/x, x/y, r/x, and r/y. These ratios are defined as the sine of $\angle A$, cosine of $\angle A$, tangent of $\angle A$, cotangent of $\angle A$, secant of $\angle A$, and cosecant of $\angle A$, respectively. In these definitions r is always considered positive when measured from O in the direction OP, and x and y possess signs following the conventions usually associated with the coordinates of a point.

The important definitions just given should be memorized by the student; the names of the various ratios, which are called *trigonometric functions*, may be abbreviated as follows:

$$\sin A = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc A = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}},$$

$$\cos A = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec A = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}},$$

$$\tan A = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot A = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}.$$

We note that the coordinate axes divide the entire plane into four parts called *quadrants*. For convenience of reference these are numbered I, II, III, IV, as in Figure 14, the order being that given by the positive direction of rotation about the origin.

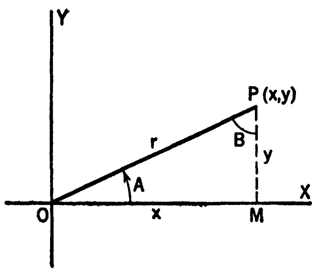


FIG. 8

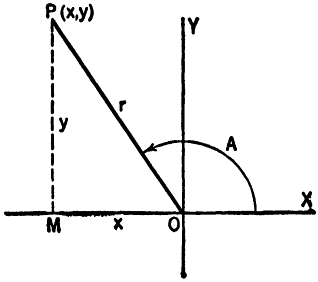


FIG. 9

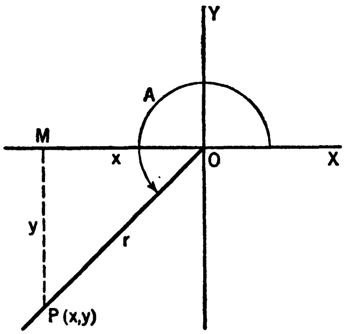


FIG. 10

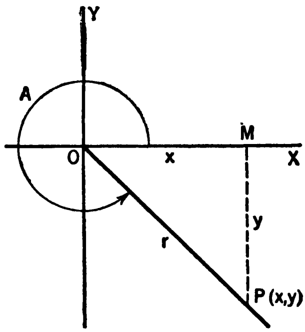


FIG. 11

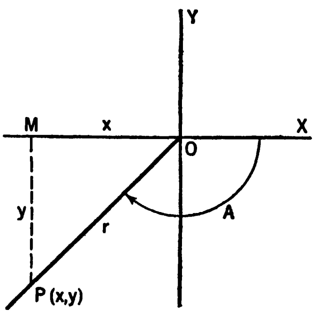


FIG. 12

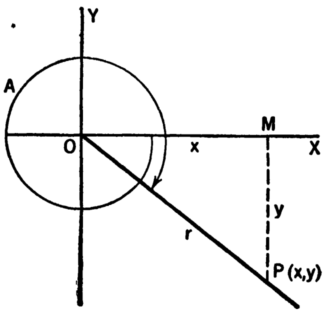


FIG. 13

When an angle is drawn with reference to a set of rectangular axes, as described at the start of Section 6, the signs of the ordinate and abscissa of point P may be positive or negative, depending upon the quadrant in which the $\angle A$ terminates. It is evident, then, that the signs of the trigonometric functions may be positive or negative also, depending upon the quadrant in which the $\angle A$ terminates. Thus, it is possible to construct a table such as the following:

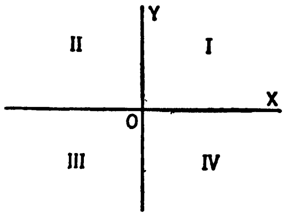


FIG. 14

	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
sin A		+		
cos A		−		
tan A		−		
cot A		−		
sec A		−		
csc A		+		

It is left as an exercise for the student to complete this table from a study of Figures 8 to 13. The table should not be memorized, however.

It is to be observed that the ratios y/x and r/x are not defined when $x = 0$, and the ratios x/y and r/y are not defined when $y = 0$. These special cases are considered later.

In Figure 8, where $\angle A$ is an acute angle, we note that y is opposite $\angle A$, x is described as adjacent to $\angle A$, and r is the hypotenuse of the right triangle containing $\angle A$. Hence, in the particular case of a right triangle it is often convenient to use the following special definitions for the trigonometric functions:

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}}, \qquad \csc A = \frac{\text{hypotenuse}}{\text{side opposite}},$$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}}, \qquad \sec A = \frac{\text{hypotenuse}}{\text{side adjacent}},$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}}, \qquad \cot A = \frac{\text{side adjacent}}{\text{side opposite}}.$$

The student must learn these definitions thoroughly, as the study of the trigonometry of the right triangle is based upon them.

In $\triangle OMP$, in Figure 8, we denoted the positive acute angle complementary to $\angle A$ as $\angle B$, then the side opposite $\angle A$ is adjacent to $\angle B$, and the side adjacent to $\angle A$ is opposite $\angle B$. Hence, if we use the special

definitions of the functions just given, we obtain the following relations:

$$\sin A = \cos B = \cos (90° - A),$$

$$\cos A = \sin B = \sin (90° - A),$$

$$\tan A = \cot B = \cot (90° - A),$$

$$\cot A = \tan B = \tan (90° - A),$$

$$\sec A = \csc B = \csc (90° - A),$$

$$\csc A = \sec B = \sec (90° - A).$$

Hence, any function of a positive acute angle is the cofunction of its complementary angle.

7. ADDITIONAL DISCUSSION OF TRIGONOMETRIC FUNCTIONS

The student must note that when the angle is given it determines, in general, the six numerical values which we have defined as the trigonometric functions of that angle. It is obvious that if the angle is constructed as in Section 6, and if from any point except the origin on the terminal side a perpendicular is dropped to the initial line, we may measure the ordinate (y), abscissa (x), and distance (r), and calculate the six required ratios, except for y/x, r/x, when $x = 0$, and x/y, r/y, when $y = 0$.

To describe the situation still more completely, we note that the trigonometric functions depend solely upon the position of the terminal line. They are independent of the direction of rotation and of the point on the terminal line from which the perpendicular is dropped.

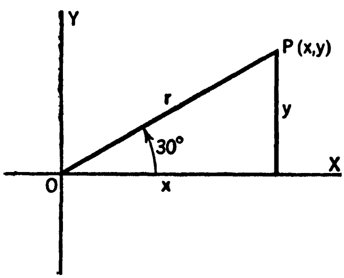


FIG. 15

8. TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

If the numerical measure of an angle is given, there are methods by means of which the values of the trigonometric ratios may be obtained. In general, this requires more mathematics than the student has at his command at this time. However, for certain special angles, such as integral multiples of 45° and 30°, we can readily calculate the trigonometric functions.

Illustration 1: Let us find the functions of 30°. Refer to Figure 15.

If we take any point P on the terminal line a distance r from the origin and drop a perpendicular to the initial line, we know from

elementary geometry that $y = r/2$. Consequently, by the Pythagorean theorem,

$$x = \sqrt{r^2 - \frac{r^2}{4}} = \frac{r}{2}\sqrt{3}.$$

Thus,

$$\sin 30° = \frac{\frac{r}{2}}{r} = \frac{1}{2} = 0.50000;$$

$$\cos 30° = \frac{\frac{r}{2}\sqrt{3}}{r} = \frac{\sqrt{3}}{2} = 0.86603, \quad \text{approximately};$$

$$\tan 30° = \frac{\frac{r}{2}}{\frac{r}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.57735, \quad \text{approximately};$$

$$\csc 30° = \frac{r}{\frac{r}{2}} = 2;$$

$$\sec 30° = \frac{r}{\frac{r}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.15470, \quad \text{approximately};$$

$$\cot 30° = \frac{\frac{r}{2}\sqrt{3}}{\frac{r}{2}} = \sqrt{3} = 1.73205, \quad \text{approximately}.$$

Of course, the last three values are the reciprocals, respectively, of the first three.

It is apparent that we might assign to any one of the three variables r, x, or y some convenient numerical value and calculate the numerical values of the other two corresponding variables and thus obtain the same values of the trigonometric functions as obtained above. Thus, when the angle is 30°, if we let $r = 2$, then $y = 1$ and $x = \sqrt{3}$; when the angle is 45°, if we let $y = 1$, then $x = 1$ and $r = \sqrt{2}$.

EXERCISES 3

1. Find the six trigonometric functions of the angles described in Figures 16, 17, and 18. Recall that x is negative in Figure 17.

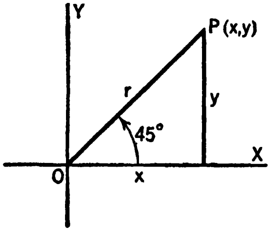


FIG. 16

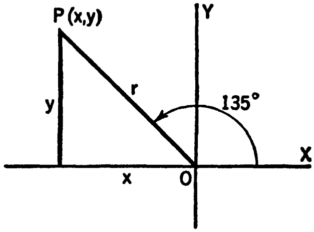


FIG. 17

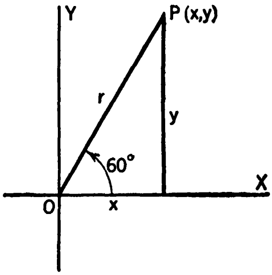


FIG. 18

2. Draw appropriate figures and find the six trigonometric functions of the angles listed in the following table:

Angle	sin	cos	tan	csc	sec	cot	
120°							
−60°							
−240°							
150°							
225°							
315°							
−135°							

3. Find the trigonometric functions of 30° by taking $r = 2$; of 45° by taking $y = 1$; of 60° by taking $x = 1$.

4. Find the six trigonometric functions of $\frac{4}{3}\pi$, $\frac{3}{4}\pi$, $\frac{7}{6}\pi$.

9. THE TRIGONOMETRIC FUNCTIONS OF 0°, 90°, 180°, AND 270°

The evaluation of the trigonometric functions of 0°, 90°, 180°, and 270° requires special consideration. The angle corresponding to 0° ter-

minates on the initial line. Hence, in our attempt to satisfy the previous definitions of the trigonometric functions for such an angle, we may take any point P on the x axis (see Figure 19) as r units from the origin and define our reference triangle to be such that $x = r$ and $y = 0$.

Thus, the six functions of 0° may be written down as follows:

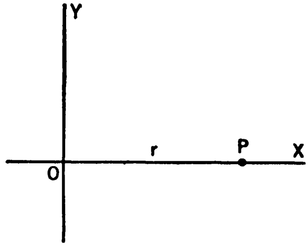


FIG. 19

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0;$$

$$\cos 0° = \frac{x}{r} = \frac{r}{r} = 1;$$

$$\tan 0° = \frac{y}{x} = \frac{0}{x} = 0;$$

$$\csc 0° = \frac{r}{y} = \frac{r}{0} \qquad \text{(does not exist)};$$

$$\sec 0° = \frac{r}{x} = \frac{r}{r} = 1;$$

$$\cot 0° = \frac{x}{y} = \frac{x}{0} \qquad \text{(does not exist)}.$$

It is observed that cot 0° and csc 0° involve division by zero, which is not permitted in mathematics; consequently, they are said not to exist.

However, let us consider the ratios x/y and r/y, where r is held fixed in length, if y is made to decrease and approach the value zero.

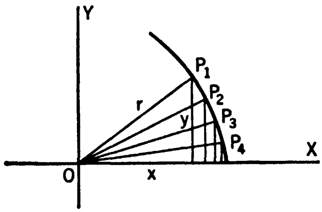


FIG. 20

Thus, in Figure 20, let

$$r = 1 \qquad \text{and} \qquad y = \tfrac{1}{100};$$

then,

$$x = \sqrt{1 - (\tfrac{1}{100})^2} = \text{approximately } 1.$$

Therefore, if the angle under consideration is designated by A, $\cot A = x/y$ is approximately equal to 100.

Similarly, let $y = \frac{1}{1000}$, then $\cot A = 1000$, approximately.

In fact, as y remains positive, but becomes smaller and smaller, the numerical value of $\cot A$ increases without limit. Of course, $\angle A$ is approaching 0°. This entire situation is frequently described symbolically as follows:

$$\lim_{A \to 0} \cot A = \infty.$$

The symbol ∞ is the sign for infinity; it does not represent a number.

The previous symbolic statement is read, "cot A approaches infinity as a limit when A approaches 0." This is a convenient way to express the fact that cot 0° does not exist, but that as $\angle A$ tends to decrease to zero, cot A tends to increase beyond any given positive value.

In the previous discussion it is assumed that the $\angle A$ approaches zero through positive values. Similar considerations will show that csc A increases without limit as $\angle A$ approaches zero, and one may write $\lim_{A \to 0} \csc A = \infty$ with a similar significance.

If the angle A approaches zero through negative values, then y approaches zero through negative values, and cot A and csc A are both negative, although numerically they become and remain larger than any given quantity. The situation may be described symbolically

$$\lim_{A \to 0} \cot A = -\infty \qquad \text{and} \qquad \lim_{A \to 0} \csc A = -\infty.$$

EXERCISES 4

1. In a manner similar to that of the previous illustrations, assign values to the functions of 90°, 180°, and 270°.

2. Fill in the appropriate number under the radical in each numerator of the following table:

x	0°	30°	45°	60°	90°
sin x	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$
cos x	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$	$\frac{\sqrt{}}{2}$

3. Find the numerical value of $\sin 60° + 2 \cos 45°$.

4. Find the numerical value of $\cos 0° \sin 45° + \sin 90° \sec^2 30°$.

NOTE: $\sec^2 30°$, by definition, means $(\sec 30°)^2$.

5. Find the value of x if $x \cot^3 45° \sec^2 60° = 11 \sin^2 90°$.

6. Find the value of x if $x(\cos 30° + 2 \sin 90° + 3 \cos 45°) = 2 \sec 180° - 5 \sin 90°$.

7. Draw an angle of 163°, and find the values of the functions by measurement.

NOTE: The approximate values of the trigonometric functions of any angle may be found graphically. We construct the angle by use of a protractor, select a point on the terminal side of the angle, and draw a perpendicular to the initial side. We then measure the lengths of the sides of the triangle formed and write the values of the trigonometric functions by use of the definitions.

8. Draw an angle of 320°, and find the values of the functions. (See the note in Exercise 7.)

9. Which trigonometric functions of 90°, 180°, and 270° do not exist?

10. THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Tables of trigonometric functions give the functions of angles from 0° to 90°. Frequently in solving practical problems it is necessary to

know the functions of an angle larger than 90° and sometimes of a negative angle. The functions of such angles may be found through the use of the limited tables, however, by a comparatively simple process.

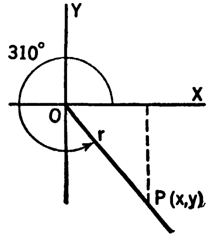


FIG. 21

To illustrate the process that we shall employ, let us consider the functions of 310°. This angle appears in Figure 21, wherein the vertical dotted line has been drawn to the horizontal axis from any point $P(x, y)$ on the terminal side of the angle. By definition, $\sin 310° = y/r$; since the angle terminates in the fourth quadrant, y is negative, thereby causing the ratio to be negative. Numerically, except for sign, the value of the ratio is obviously the same as $\sin 50°$, the angle 50° being the acute angle at O within the right triangle having the dotted line as one leg. Thus, it follows at once that

$$\sin 310° = -\sin 50°.$$

If $\sin 50°$ is obtained by reference to a table of trigonometric functions, $\sin 310°$ is completely determined.

Similarly,

$$\cos 310° = \frac{x}{r} = \cos 50°;$$

$$\tan 310° = \frac{y}{x} = -\tan 50°;$$

$$\csc 310° = \frac{r}{y} = -\csc 50°;$$

$$\sec 310° = \frac{r}{x} = \sec 50°;$$

$$\cot 310° = \frac{x}{y} = \cot 50°.$$

The method employed in the case of 310° may be generalized to obtain the functions of any angle not listed in a standard table. Specifically, any function of an angle θ is numerically equal to the same function of α, where α is the acute angle between the terminal side of θ and the horizontal axis, but the sign of the function must be determined from the quadrant in which the angle θ terminates. The idea is illustrated further for the trigonometric functions of $-167°$ and $737°$ (note Figures 22 and 23). The angle of measure $-167°$ terminates in the third quadrant, with the terminal side forming an acute angle of 13° with the horizontal axis. The angle of measure 737° terminates in the first quadrant, with the terminal

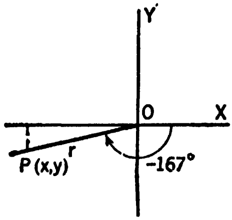


FIG. 22

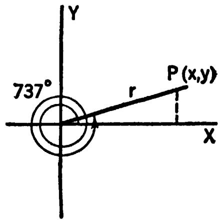


FIG. 23

side forming an acute angle of 17° with the horizontal axis. Consequently,

$$\sin(-167°) = \frac{y}{r} = -\sin 13°;$$

$$\cos(-167°) = \frac{x}{r} = -\cos 13°;$$

$$\tan(-167°) = \frac{y}{x} = \tan 13°;$$

$$\csc(-167°) = \frac{r}{y} = -\csc 13°;$$

$$\sec(-167°) = \frac{r}{x} = -\sec 13°;$$

$$\cot(-167°) = \frac{x}{y} = \cot 13°.$$

Also,

$$\sin 737° = \frac{y}{r} = \sin 17°;$$

$$\cos 737° = \frac{x}{r} = \cos 17°;$$

$$\tan 737° = \frac{y}{x} = \tan 17°;$$

$$\csc 737° = \frac{r}{y} = \csc 17°;$$

$$\sec 737° = \frac{r}{x} = \sec 17°;$$

$$\cot 737° = \frac{x}{y} = \cot 17°.$$

EXERCISES 5

1. Draw a figure and express the trigonometric functions of each of the following angles in terms of functions of a positive acute angle:

(*a*) $119°$	(*e*) $-37°$	(*i*) $372°$	(*m*) $-690°$
(*b*) $213°$	(*f*) $-165°$	(*j*) $544°$	(*n*) $800°$
(*c*) $296°$	(*g*) $-215°$	(*k*) $-544°$	(*o*) $-800°$
(*d*) $400°$	(*h*) $-340°$	(*l*) $690°$	(*p*) $540°$

2. Draw a diagram and find the functions of the angle $180° - A$ in terms of the functions of $\angle A$, where $\angle A$ is some positive acute angle.

3. Draw a diagram and find the functions of the angle $180° + A$ in terms of the functions of $\angle A$, where $\angle A$ is an acute angle.

4. Draw a diagram and find the functions of the angle $-A$ in terms of the functions of $\angle A$, where $\angle A$ is an acute angle.

5. Draw a diagram and find the functions of the angle $90° + A$ in terms of functions of $\angle A$, where $\angle A$ is some positive acute angle.

6. Consider the answer to Exercise 4 if $\angle A$ is any angle.

11. TO COMPUTE THE TRIGONOMETRIC FUNCTIONS IF THE VALUE OF ANY ONE FUNCTION IS GIVEN

Suppose we know that $\sin A = \frac{3}{5}$. Since $\sin A = y/r$, and since r is always positive, y must be positive. Moreover, as a practical expedient in drawing a figure to represent the situation, we may choose any two

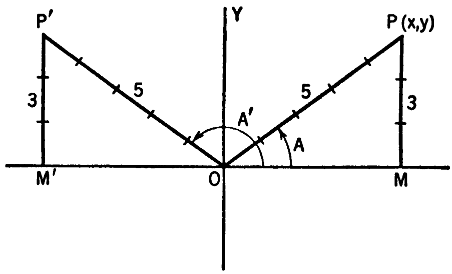


FIG. 24

positive numbers for y and r in the ratio $\frac{3}{5}$. If we choose $y = 3$ and $r = 5$, we may construct the two possible cases displayed in Figure 24, thereby determining where the possible angles must terminate.

It is evident that the given value for $\sin A$ determines an unlimited number of angles, but they must all terminate on OP or on OP'. We have then, by calculation, if we take $y = 3$ and $r = 5$, $OM = 4$ and $OM' = -4$. Hence, for the angles terminating on OP, we have $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$, $\cot A = \frac{4}{3}$, $\sec A = \frac{5}{4}$, $\csc A = \frac{5}{3}$; and for the angles terminating on OP', $\cos A' = -\frac{4}{5}$, $\tan A' = -\frac{3}{4}$, $\cot A' = -\frac{4}{3}$, $\sec A' = -\frac{5}{4}$, $\csc A' = \frac{5}{3}$.

We note that of the unlimited number of angles whose sine is $\frac{3}{5}$ there are two positive angles less than 360°. These two angles are called the

principal angles, determined by $\sin A = \frac{3}{5}$. In general, we are in a position to construct the principal angles, if the value of any one function is given, and we may then compute the remaining functions.

If either $\sin A$, $\cos A$, $\sec A$, or $\csc A$ is given, and we are required to construct the principal angles it is best to draw a circle with a chosen r as a radius, then determine y or x from the given function. It is then comparatively simple to determine where the required angles must terminate.

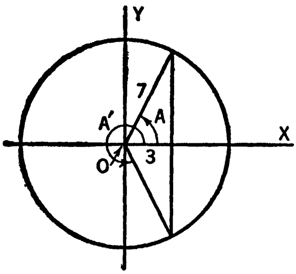


FIG. 25

If we are given either $\tan A$ or $\cot A$, to construct the principal angles, we measure the required values of y and x and thus determine the angles without the construction of a circle.

Illustration: If $\cos A = \frac{3}{7}$ and the problem is to construct the principal angles, we may choose $r = 7$; then $x = 3$. Of course, one may select any other two positive numbers for r and x in the ratio $\frac{3}{7}$. Next, construct a circle with $r = 7$ as a radius, and draw a line parallel to the y axis and three units to the right of it, as in Figure 25. We have thus determined the required angles A and A'. We may now readily compute the other functions of A and A' if desired.

EXERCISES 6

1. Given $\csc A = \frac{13}{5}$; construct the principal angles, and find the values of the remaining functions.

2. Given $\sec A = \frac{17}{15}$; construct the principal angles, and compute the values of the other functions.

3. Given $\sin A = -\frac{3}{5}$ and $\cos A$ is a positive number; construct the one principal angle determined by these two conditions, and find the value of the remaining functions.

4. Given $\cos A = -\frac{5}{8}$ and $\tan A$ is a negative number; construct the principal angle, and find the remaining functions.

5. Given $\tan A = \frac{5}{7}$; construct the principal angles, and find the remaining functions.

NOTE: Here we have

$$\frac{y}{x} = \frac{+5}{+7} \quad \text{or} \quad \frac{-5}{-7}.$$

6. Given $\cot A = \frac{9}{5}$ and $\sin A$ is negative; construct the principal angle, and find the remaining functions.

7. Given $\sin A = \frac{5}{13}$ and $\cos A$ is positive; find the value of

$$(a) \quad (\sec A)(\tan A) + (\cos A)(\cot^2 A);$$

$$(b) \quad \frac{\cos A}{\tan A} - \frac{\sec A}{\csc A}.$$

8. Given $\tan A = -\frac{8}{15}$ and $\sin A$ positive; find the value of

$$\frac{\cos A - 3\cot A}{\csc^2 A}.$$

9. Given $\cos A = \frac{12}{13}$ and $\tan A$ negative; find the value of

$$(a)\quad \frac{[(\sin A)^{-1} - (\cot A)^{-1}]^2}{(\sec A)^0};$$

$$(b)\quad \left(\frac{\sin^2 A}{\cos^2 A} + 1\right)\left(\frac{1}{\csc^2 A - \cot^2 A}\right).$$

10. Given $\sec A = -\frac{5}{4}$ and $\sin A$ negative; find the value of

$$\frac{(1 - \sin^2 A)^{1/2}}{\tan A}\left(\frac{1}{\cos A} - \cot A\right).$$

12. LINE VALUES OF THE TRIGONOMETRIC FUNCTIONS

It is evident from Figure 26 that if we construct any angle θ terminating in the first quadrant, and take $OP = r = 1$, then $\sin\theta = y/r = y/1 = y = MP$. Likewise, $\cos\theta = x/r = x/1 = OM$. If we draw ST and QV tangent to the circle at S and Q, respectively, then $\tan\theta = ST/OS = ST/1 = ST$, and $\sec\theta = OT/OS = OT/1 = OT$.

Since $\angle OVQ = \angle\theta$ (why?), it is also seen that

$$\csc\theta = \frac{OV}{OQ} = \frac{OV}{1} = OV, \quad \text{and} \quad \cot\theta = \frac{QV}{OQ} = \frac{QV}{1} = QV.$$

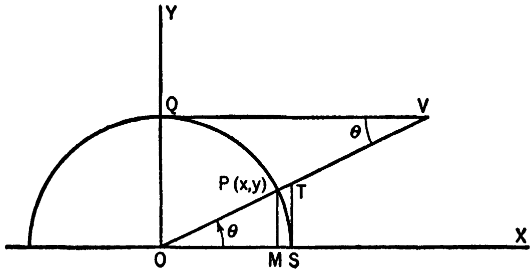


FIG. 26

Hence, we speak of y, x, ST, OT, OV, and QV as the line values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$, respectively, since the lengths of these lines in terms of the length of OP as a unit are the values of these functions of θ.

If $\angle\theta$ terminates in the second quadrant, as in Figure 27, the line values of the functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$ are still numerically equal to the lengths of the lines y, x, ST, OT, OV, and QV, respectively. Irrespective of the quadrant in which the angle terminates, S is taken as the point $(1, 0)$ and Q as $(0, 1)$. It is important to note this

time, however, that for an angle terminating in the second quadrant, x, ST, OT, and QV are to be regarded as negative magnitudes. The fact that the vertical or horizontal distances x, ST, and QV are to be taken as negative is readily called to our attention because of the conventional

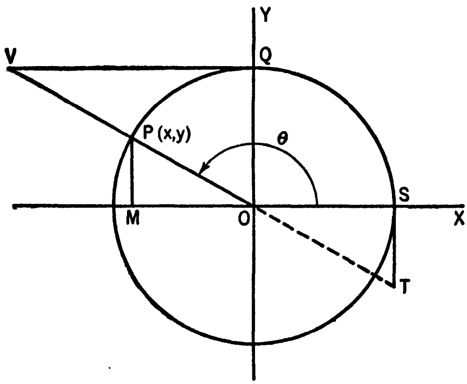


FIG. 27

scheme of signs associated with our axis system. A negative sign is associated with OT, since its direction is opposite to that of the terminal side OP.

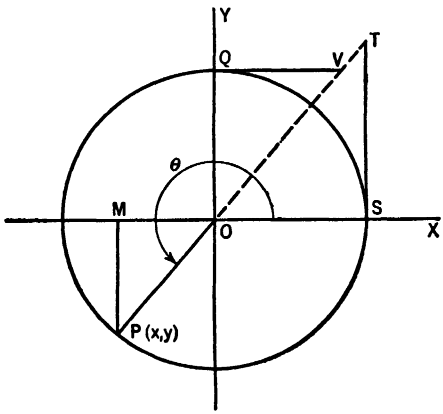


FIG. 28

If $\angle\theta$ terminates in the third quadrant or fourth quadrant, the line values of the functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$ are y, x, ST, OT, OV, and QV, respectively, of the corresponding Figures 28 and 29. But it is to be noted that in Figure 28, y, x, OT, and OV are negative magnitudes and that in Figure 29, y, ST, OV, and QV are negative magnitudes.

The student should be able to reproduce these figures for any angle terminating in any quadrant.

In each figure it is also possible to describe the line value of $\angle\theta$ in radians as the measure of the arc subtended by θ, in the direction of θ.

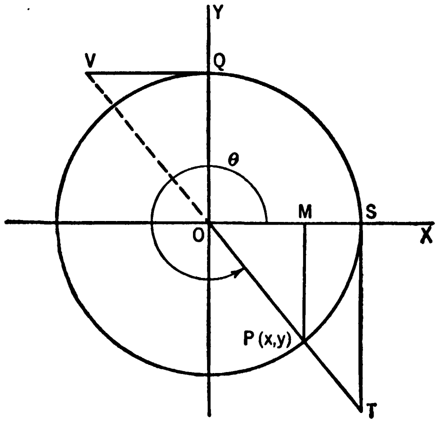


FIG. 29

From an examination of the line values of the functions it may be seen that $\sin\theta$ and $\cos\theta$ cannot equal any number greater than 1 or less than -1; that $\sec\theta$ and $\csc\theta$ may equal any finite number less than or equal to -1, or any finite number equal to or greater than 1; and that $\tan\theta$ and $\cot\theta$ may equal any finite number.

13. GRAPHS OF TRIGONOMETRIC FUNCTIONS

We now have a convenient method of graphing the trigonometric functions, namely, by laying off the line values of the angles as abscissas and the line values of the corresponding functions as ordinates.

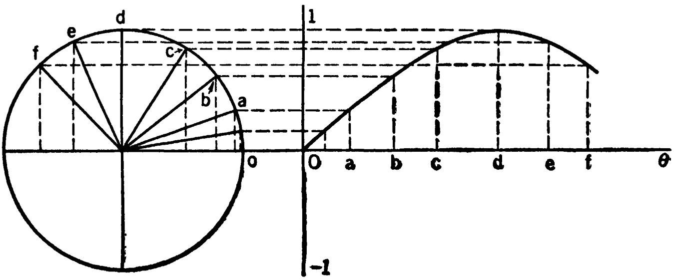


FIG. 30

Thus, to graph $\sin\theta$, draw a circle of unit radius and various arbitrary angles θ_1, θ_2, θ_3, and so on, whose terminal lines intersect the circle in such points as a, b, c etc., as shown in Figure 30.

Next choose some convenient point O as an origin, located on a straight line through the center of the circle, and lay off the line values of the various angles as abscissas and the corresponding sines of the angles as ordinates. A smooth curve drawn through these points, as in Figure 30, is a portion of the graph of $\sin\theta$. Figure 31 shows the graph and its mechanical

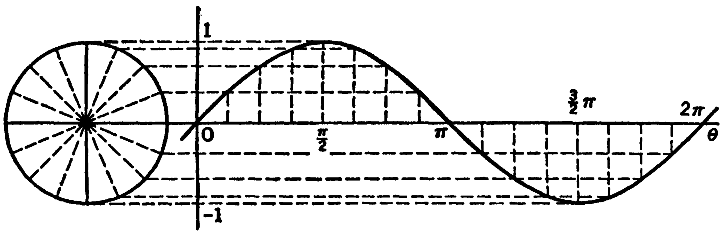


FIG. 31

construction when θ ranges from 0 to 2π. When constructing the curve, it should be noted that in terms of the radius of the circle as the unit, π is approximately $3\frac{1}{7}$ times the unit.

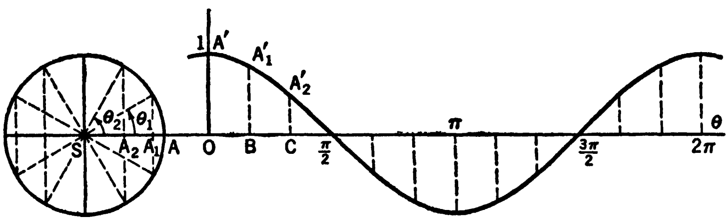


FIG. 32

The graph for $\cos\theta$, when θ ranges from 0 radians to 2π radians, appears as Figure 32.

$$SA = OA' = \cos 0 = 1;$$

$$SA_1 = BA_1' = \cos\theta_1;$$

$$SA_2 = CA_2' = \cos\theta_2;$$

$$\cdots\cdots\cdots$$

From the graph of $\sin\theta$ we see that as θ increases from 0 to $\pi/2$, $\sin\theta$ increases from 0 to 1; as θ increases from $\pi/2$ to π, $\sin\theta$ decreases from 1 to 0; as θ increases from π to $3\pi/2$, $\sin\theta$ decreases from 0 to -1; and as θ increases from $3\pi/2$ to 2π, $\sin\theta$ increases from -1 to 0. Moreover, this segment of the curve repeats itself from 2π to 4π, from 4π to 6π, and so on. In general, for any value of θ,

$$\sin\theta = \sin(\theta + 2n\pi),$$

when $n = 0, 1, 2, 3, \cdots$. We therefore describe $\sin\theta$ as a periodic function with a period 2π.

Similarly, for any value of θ,

$$\cos\theta = \cos(\theta + 2n\pi),$$

when $n = 0, 1, 2, 3, \cdots$; thus, $\cos\theta$ is also a periodic function with a period of 2π.

14. THE GRAPH OF SIN $b\theta$

If we consider the graph of the function $\sin 2\theta$, it is observed, for any value of θ, that

$$\sin 2\theta = \sin(2\theta + 2n\pi) = \sin 2(\theta + n\pi),$$

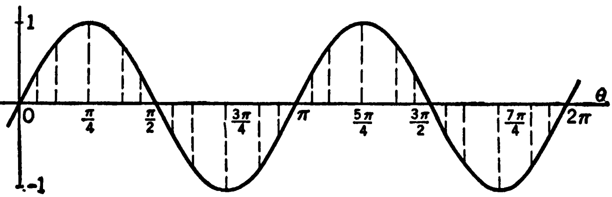


FIG. 33

when $n = 0, 1, 2, 3, \cdots$. Hence, $\sin 2\theta$ is a periodic function with a period π; that is, the curve representing $\sin 2\theta$ between 0 and π is repeated between π and 2π, between 2π and 3π, and so on (note Figure 33).

In the same way it can be shown that

$$\sin 3\theta = \sin 3\left(\theta + \frac{2n\pi}{3}\right)$$

when $n = 0, 1, 2, 3, \cdots$. Thus, the values of $\sin 3\theta$ corresponding to θ in the range from 0 to $2\pi/3$ will be repeated over and over again; so $\sin 3\theta$ is a periodic function with a period $2\pi/3$.

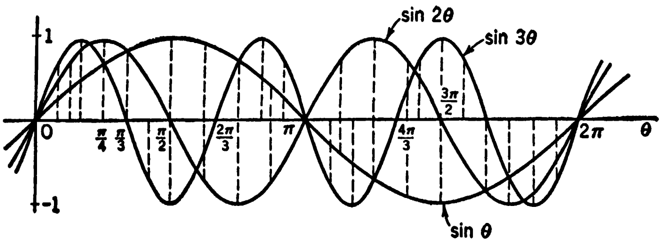


FIG. 34

If we graph the functions $\sin\theta$, $\sin 2\theta$, and $\sin 3\theta$ relative to the same axes and to the same scale, we have the situation depicted in Figure 34.

The figure shows $\sin \theta$ for one period, $\sin 2\theta$ for two periods, and $\sin 3\theta$ for three periods. This pictorial representation makes clear the difference that exists between the various curves and emphasizes the fact that $\sin \theta$ is a periodic function whose period is 2π, that $\sin 2\theta$ is a periodic function whose period is π, and that $\sin 3\theta$ is a periodic function whose period is $2\pi/3$.

In general, the function $\sin b\theta$ is a periodic function whose period is $2\pi/|b|$. As an illustration of this general conclusion, the graph of $\sin \theta/2$ is given in Figure 35, from which it is readily seen that $\sin \theta/2$ is a periodic function whose period is $2\pi/\frac{1}{2}$ or 4π.

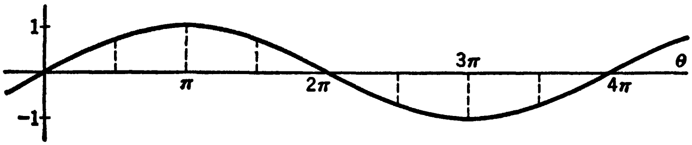


FIG. 35

EXERCISES 7

1. Construct the graph of $3 \sin \theta$ for $0 \leqq \theta \leqq 2\pi$.

NOTE: The ordinate of the graph of $3 \sin \theta$, for each value of θ, is three times the corresponding ordinate of the graph of $\sin \theta$. The period of $3 \sin \theta$ is the same as the period of $\sin \theta$.

2. Construct the graph of $\frac{1}{2} \sin \theta$ for $0 \leqq \theta \leqq 2\pi$. (See the note after Problem 1.)

3. Construct the graph of $2 \sin 3\theta$ for one period of the function.

4. Construct the graph of $3 \sin (2\theta/3)$ for one period of the function.

5. Construct the graph of $\cos 2\theta$ for $0 \leqq \theta \leqq 2\pi$.

6. Construct the graph of $\cos 3\theta$ for one period of the function.

7. Construct the graph of $\cos (\theta/2)$ for one period of the function.

8. Construct the graph of each of the following for one period:

(*a*) $2 \cos 3\theta$; (*b*) $\frac{1}{2} \cos 2\theta$; (*c*) $\frac{1}{2} \cos 3\theta$

9. (*a*) By use of a construction similar to that used in drawing the graph of $\sin \theta$, and by employing the line values for $\tan \theta$, draw a graph for $\tan \theta$ as θ increases from $\theta = 0°$ to $\theta = 360°$.

(*b*) From your graph discuss the variation in the value of $\tan \theta$ as θ increases from 0° to 180°.

10. Construct the graph of $\cot \theta$ from $\theta = 0°$ to $\theta = 360°$. From your graph discuss the variation in the value of $\cot \theta$ as θ increases from $\theta = 0°$ to $\theta = 180°$.

11. (*a*) By use of line values for $\sec \theta$, construct the graph of $\sec \theta$ from $\theta = 0°$ to $\theta = 360°$.

(*b*) From your graph discuss the variation of $\sec \theta$ as θ increases from 0° to 180°.

12. (*a*) Construct the graph of $\csc \theta$ from $\theta = 0°$ to $\theta = 360°$.

(*b*) Discuss the variation of $\csc \theta$ as θ increases from 0° to 180°.

15. THE GRAPH OF $a \sin b(\theta + c)$

If we graph $\sin \theta$, $\sin (\theta + \pi/3)$, and $\sin (\theta - \pi/3)$, either by use of line values or by use of a table of sines, we have Figure 36.

These curves show that the three given functions are periodic, each with a period of 2π. In fact, if every point of $\sin \theta$ is moved to the left parallel to the axis of θ a distance $\pi/3$ units, we obtain the graph of $\sin (\theta + \pi/3)$; if every point of $\sin \theta$ is moved to the right parallel to the

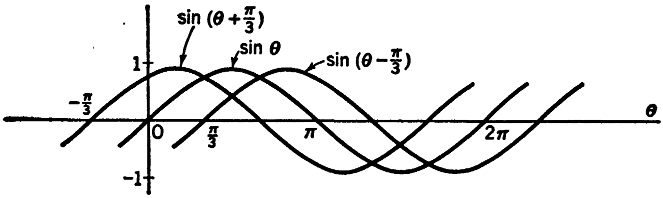


FIG. 36

axis of θ a distance of $\pi/3$ units, we obtain the graph of $\sin (\theta - \pi/3)$. The graph of $\sin (\theta + \pi/3)$ is said to have a lead of $\pi/3$ relative to the graph of $\sin \theta$; whereas the graph of $\sin (\theta - \pi/3)$ is said to have a lag of $\pi/3$ relative to the graph of $\sin \theta$. In general, the graph of $\sin (\theta + c)$ has a lead of c relative to $\sin \theta$ if c is positive, or it has a lag of c relative to $\sin \theta$ if c is negative.

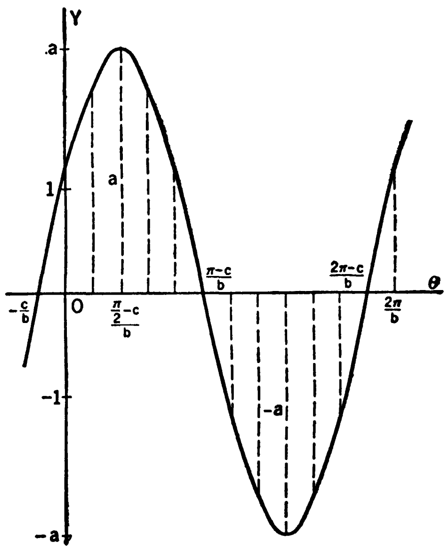


FIG. 37

Let us now consider the function $y = a \sin (b\theta + c)$. The graph of this function is constructed in Figure 37. It is evident from the figure that the greatest value of y is a, and the smallest value of y is $-a$. This value a is designated as the amplitude of the function. We have seen

that the function $y = \sin b\theta$ is zero when $\theta = 0$, but in the case of the function $y = a \sin (b\theta + c)$, $y = a \sin c$ when $\theta = 0$, and $y = 0$ when $\theta = -c/b$. Hence the graph $y = a \sin (b\theta + c)$ is said to be in the lead of $y = \sin b\theta$ by the angular value c/b. If c and b are of opposite sign, the graph of $y = a \sin (b\theta + c)$ is said to lag in reference to $y = \sin b\theta$ by the angular value $-c/b$. It is not expected that c will be numerically greater than π; if such be the case, however, the lag or lead will be $(c - 2n\pi)/b$, where n is an integer large enough to make $| c - 2n\pi | \leqq \pi$.

We note that since $\sin (b\theta + c) = \sin [b(\theta + 2n\pi/b) + c]$, the values of y corresponding to θ from 0 to $2\pi/b$ will be repeated over and over. Hence $2\pi/| b |$ is said to be the period of the function $y = a \sin (b\theta + c)$, and the function is said to be a periodic function of period $2\pi/| b |$.

EXERCISES 8

1. What is the amplitude, lag or lead, and period of $5 \sin (3x - 5)$?

2. What is the amplitude, lag or lead, and period of $\frac{1}{2} \cos (4x + \pi/2)$?

3. In the function $y = a \sin (b\theta + c)$, assign values to a, b, and c so that the amplitude is 10 and the period is $\pi/3$ radians.

4. Is the value of c in Exercise 3 fixed by the given conditions?

5. Fix the value of c in Exercise 3 so that the function will have a lag of $\pi/4$ radians.

6. Show that $y = \cos \theta$ has the same period as $y = \sin \theta$, but has a lead of $\pi/2$ relative to $y = \sin \theta$. Note that $y = \sin (\theta - \pi/2)$.

Sketch the graphs of each of the following functions and state the amplitude, lag or lead, and period for each function.

7. $y = 2 \sin (2x + 2)$

8. $y = \sin (3x - \pi/3)$

9. $y = 3 \sin (x/2 + \pi)$

10. $y = 10 \sin (10x - 5)$

11. $y = 2 \sin (2x - 7)$

12. $y = \cos (x + \pi/3)$

13. $y = \cos (x - \pi/3)$

14. $y = \cos (2x + \pi/2)$

15. $y = \cos (3x - \pi/6)$

16. $y = 5 \cos (x/2 + \pi)$

17. $y = \frac{1}{2} \cos (3x - 2)$

18. $y = 10 \cos (20x - 5)$

19. The graph of the function $y = F_1(x) + F_2(x) + F_3(x)$ may be sketched by graphing $y_1 = F_1(x)$, $y_2 = F_2(x)$, $y_3 = F_3(x)$ on the same axes and noting that for any value of x, $y = y_1 + y_2 + y_3$. Hence, any ordinate y on the desired graph may be obtained by adding graphically through the use of a ruler the corresponding ordinates of y_1, y_2, and y_3. By this device, sketch the graphs of each of the following:

(*a*) $y = \sin \theta + 2 \sin 2\theta + 3 \sin 3\theta$
(*b*) $y = 2 \sin \theta - \cos 2\theta$
(*c*) $y = x + \sin x$
(*d*) $y = 3 + \sin x$
(*e*) $y = 2x - 3 + 2 \sin x$
(*f*) $y = 3x - \sin 3x$

16. INVERSE TRIGONOMETRIC FUNCTIONS

In the previous sections we have graphed the trigonometric functions by designating the values of a given function as the ordinates and the

corresponding angles as the abscissas. It was assumed that the angle was the independent variable and the function was the dependent variable. The study was obviously facilitated by the fact that the trigonometric functions are single-valued; that is, by assigning a value to the angle, one and only one value of the function is determined.

In many applications of trigonometry we meet an inverse problem; that is, we are required to find the angle or angles when the value of the function is assigned. In this case the independent variable, chosen as the abscissa, denotes values of the function while the angle or angles corresponding to a particular function are designated as ordinates. To be more specific, if x denotes values of $\sin y$, then y is said to be the inverse sine of x; the fact is represented symbolically by $y = \sin^{-1} x$. This is read, as just implied, y is the angle (or angles) whose sine is x. We must note that the -1 is not used as an exponent in this case; rather $\sin^{-1} x$ is merely a new symbol for the angle or angles y. This particular example may be generalized to apply to the inverse cosine, the inverse tangent, and so on.

Illustration 1: Let us graph $y = \sin^{-1} x$.

Since the statement

$$y = \sin^{-1} x$$

means

$$x = \sin y,$$

we note that as y increases from 0 to $\pi/2$, x increases from 0 to 1; as y increases from $\pi/2$ to π, x decreases from 1 to 0; as y increases from π to $3\pi/2$, x decreases from 0 to -1; and as y increases from $3\pi/2$ to 2π, x increases from -1 to 0. Hence, the graph of $y = \sin^{-1} x$ is merely the sine curve as previously studied, but drawn relative to the y axis. Since the sine of an angle varies between -1 and $+1$, we can only assign values to x between -1 and $+1$.

This now brings us to an important feature of such a curve as $y = \sin^{-1} x$; namely, any value assigned to x will determine an unlimited number of values for y. Thus, if $x = 0$,

$$y = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \text{etc.};$$

if $x = \frac{1}{2}$,

$$y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{etc.}$$

For this reason, the inverse sine is said to be multivalued.

If in Figure 38 we *restrict* ourselves to the portion AB of the graph of $y = \sin^{-1} x$, where $-\pi/2 \leqq y \leqq \pi/2$, the function becomes single-valued, and the values of the function along AB are called the *principal values* of the $\sin^{-1} x$.

In Section 13 we graphed $y = \sin x$ by employing the line values of the angles as abscissas. But if $r = 1$, as in Section 13, the line value of the angle equals the arc that intercepts the angle. Hence, the angle is

sometimes indicated by arc sin x and the relation $y = \sin^{-1} x$ is written as $y = \text{arc} \sin x$.

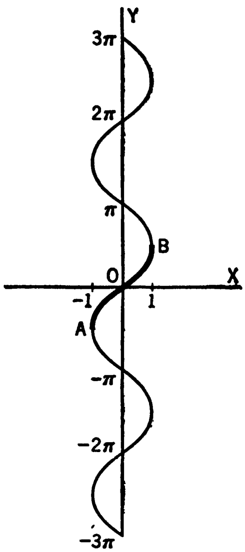


FIG. 38

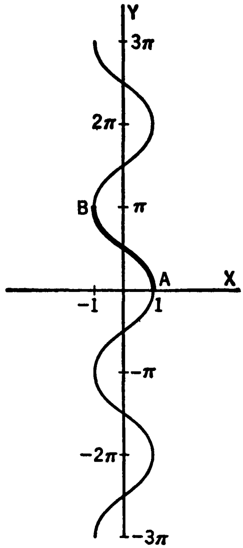


FIG. 39

Illustration 2: Let us graph $y = \cos^{-1} x$ (note Figure 39).

Since the cosine of an angle varies between -1 and $+1$, we can assign values to x only between -1 and $+1$. As in Illustration 1, however, the assignment of any value to x between -1 and $+1$ will determine an infinite number of values for y. Thus, if $x = 0$,

$$y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \text{etc.},$$

if $x = \frac{1}{2}$,

$$y = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \text{etc.}$$

Hence, we note that the function $y = \cos^{-1} x$ is also a multivalued function, but if we restrict ourselves in Figure 39 to the portion AB, where $0 \leqq y \leqq \pi$, the function $y = \cos^{-1} x$ becomes single-valued, and the values of the function along AB are called the *principal values* of $\cos^{-1} x$.

Similarly, $\csc^{-1} x$ and $\tan^{-1} x$ are frequently restricted to values between $-\pi/2$ and $+\pi/2$, and $\cot^{-1} x$ and $\sec^{-1} x$ are restricted to values between 0 and π in order to make the functions single-valued.

EXERCISES 9

Sketch the graph of each of the following functions:

1. $y = \text{arc}\sin x$

2. $y = \text{arc}\cos x$

3. $y = \tan^{-1} x$

4. $y = 2\sin^{-1} x$

5. $y = \sin^{-1} 2x$

6. $y = 2\sin^{-1} 2x$

7. $y = \cos^{-1} 3x$

8. $y = 2\cos^{-1} 3x$

9. $y = \sin^{-1} 2x - \pi/4$

10. $y = \cos^{-1} 3x - \pi/3$

2 Trigonometric Identities and Conditional Equations

17. FUNDAMENTAL TRIGONOMETRIC IDENTITIES

In Book I, we defined an algebraic identity as an equation that is valid for any permissible value of the unknown. Thus,

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$$

is an algebraic identity. Likewise, the equation

$$\frac{1}{x^2 - 1} = \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)}$$

is an identity, but in this case $x = 1$ and $x = -1$ are not permissible values of the unknown. When $x = 1$ or -1, $1/(x^2 - 1)$ does not exist. Similarly, the first fraction on the right does not exist when $x = 1$, and the second fraction on the right does not exist when $x = -1$. A definition of an identity, equivalent to the meaning just expressed, is

An identity is an equation that is true for all values of the unknowns for which both members are defined.

In this chapter, we shall study trigonometric identities. Just as in algebra, a few trigonometric identities are of sufficient importance to be developed and memorized.

The simplest identities follow immediately from the definitions of the trigonometric functions. These are

(1) $\sin A = \dfrac{1}{\csc A}$; (2) $\csc A = \dfrac{1}{\sin A}$;

(3) $\tan A = \dfrac{1}{\cot A}$; (4) $\cot A = \dfrac{1}{\tan A}$;

(5) $\cos A = \dfrac{1}{\sec A}$; (6) $\sec A = \dfrac{1}{\cos A}$.

Since $\sin A = \dfrac{y}{r}$ and $\cos A = \dfrac{x}{r}$, it follows that

$$\text{(7)} \qquad \frac{\sin A}{\cos A} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan A.$$

Similarly,

$$\text{(8)} \qquad \frac{\cos A}{\sin A} = \cot A.$$

By referring to the figures of Section 6, we see that $x^2 + y^2 = r^2$. After dividing each member of this equation by x^2, y^2, and r^2, respectively, we have

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2};$$

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2};$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

The various ratios appearing in these equations may be replaced by the appropriate trigonometric functions, thereby giving

$$\text{(9)} \qquad 1 + \tan^2 A = \sec^2 A;$$

$$\text{(10)} \qquad \cot^2 A + 1 = \csc^2 A;$$

$$\text{(11)} \qquad \cos^2 A + \sin^2 A = 1.$$

These three latter identities may appear in various forms; thus, from $1 + \tan^2 A = \sec^2 A$, we obtain

$$\tan A = \pm\sqrt{\sec^2 A - 1}, \qquad \text{and} \qquad \sec A = \pm\sqrt{1 + \tan^2 A}.$$

From $\cot^2 A + 1 = \csc^2 A$, we obtain

$$\cot A = \pm\sqrt{\csc^2 A - 1} \qquad \text{and} \qquad \csc A = \pm\sqrt{1 + \cot^2 A}.$$

Also, from $\cos^2 A + \sin^2 A = 1$, we obtain

$$\sin A = \pm\sqrt{1 - \cos^2 A} \qquad \text{and} \qquad \cos A = \pm\sqrt{1 - \sin^2 A}.$$

The identities involving radicals are ambiguous since they involve two signs before the radicals. The sign to be chosen in each case depends on the quadrant in which $\angle A$ terminates. Thus, if $\angle A$ terminates in the first or fourth quadrant,

$$\cos A = \sqrt{1 - \sin^2 A},$$

$$\sec A = \sqrt{1 + \tan^2 A};$$

but if $\angle A$ terminates in the second or third quadrant,

$$\cos A = -\sqrt{1 - \sin^2 A},$$

$$\sec A = -\sqrt{1 + \tan^2 A}.$$

These fundamental identities which have been proved may now be employed to establish an unlimited number of other trigonometric identities.

Illustration 1: Find various expressions that are identical to $\tan \phi + \cot \phi$.

From the Fundamental Identities (7) and (8), we obtain

$$\tan \phi + \cot \phi = \frac{\sin \phi}{\cos \phi} + \frac{\cos \phi}{\sin \phi}.$$

The right member may be written

$$\frac{\sin^2 \phi + \cos^2 \phi}{\sin \phi \cos \phi} \quad \text{or} \quad \frac{1}{\sin \phi \cos \phi},$$

since $\sin^2 \phi + \cos^2 \phi = 1$. The fraction

$$\frac{1}{\sin \phi \cos \phi} = \frac{1}{\sin \phi} \cdot \frac{1}{\cos \phi},$$

and

$$\frac{1}{\sin \phi} \cdot \frac{1}{\cos \phi} = \csc \phi \cdot \sec \phi,$$

by Fundamental Identities (2) and (6). Thus,

$$\tan \phi + \cot \phi = \frac{\sin \phi}{\cos \phi} + \frac{\cos \phi}{\sin \phi} = \frac{1}{\sin \phi \cos \phi} = \csc \phi \sec \phi.$$

As it appears from this illustration, we could obtain an unlimited number of expressions identical to a given expression.

Illustration 2: Express $\tan \phi + \cot \phi$ by an expression identical to it involving only $\tan \phi$.

From Identity (4),

$$\tan \phi + \cot \phi = \tan \phi + \frac{1}{\tan \phi}.$$

Similarly, if it had been requested, we could express the given sum as an expression involving only $\cot \phi$. Thus, by Identity (3),

$$\tan \phi + \cot \phi = \frac{1}{\cot \phi} + \cot \phi.$$

In later mathematical work it is often necessary to find various expressions that are identical to certain given expressions. It will also be necessary in many cases to state a given expression in terms of one trigonometric function.

Illustration 3: Determine whether or not

$$\tan \phi + \cot \phi$$

is identical to

$$\frac{\sec^2 \phi}{\tan \phi}.$$

This may be done in various ways. In general, an attempt is made to transform the left member into the right member by the use of fundamental identities; or an attempt is made to transform the right member into the left member; or an effort is made to transform both members into the same expression.

Since $\sec^2 \phi = 1 + \tan^2 \phi$, it follows that

$$\frac{\sec^2 \phi}{\tan \phi} = \frac{1 + \tan^2 \phi}{\tan \phi} = \frac{1}{\tan \phi} + \frac{\tan^2 \phi}{\tan \phi}.$$

But, knowing that $\cot \phi$ is the reciprocal of $\tan \phi$, we have

$$\frac{\sec^2 \phi}{\tan \phi} = \cot \phi + \tan \phi,$$

which establishes the identity by virtue of the fact that we have succeeded in transforming one member into the other.

Since the left member of the last equation is not defined for $\tan \phi = 0$, we shall understand that $\tan \phi \neq 0$.

Illustration 4: Prove that

$$\tan \phi + \cot \phi \equiv \frac{\sec^2 \phi}{\tan \phi},^*$$

by showing that both members are identical to some other expression. We may proceed as follows: In Illustration 1 we showed that $\tan \phi + \cot \phi$ is identical to $\csc \phi \sec \phi$.

Also, since

$$\sec^2 \phi = \frac{1}{\cos^2 \phi} \qquad \text{[Identity (6)]},$$

and

$$\tan \phi = \frac{\sin \phi}{\cos \phi} \qquad \text{[Identity (7)]},$$

then,

$$\frac{\sec^2 \phi}{\tan \phi} = \frac{\dfrac{1}{\cos^2 \phi}}{\dfrac{\sin \phi}{\cos \phi}} = \frac{1}{\cos \phi \sin \phi} = \sec \phi \csc \phi,$$

by Identities (5) and (6). So both members are identical to the same expression, namely, $\csc \phi \sec \phi$.

* The symbol $\equiv$ may be read "is identical to."

CAUTION: The student is cautioned to avoid a common type of error in working with identities. For example, it is true that $\sqrt{a^2 - 2ab + b^2} = \sqrt{b^2 - 2ab + a^2}$. But either radical may be simplified only to $a - b$ if $a - b > 0$, or to $b - a$ if $b - a > 0$. In other words, it must be kept in mind that the radical sign denotes the *positive square root.*

Thus, we cannot write

$$\sqrt{1 - 2\tan x + \tan^2 x} = 1 - \tan x,$$

unless it is known that $1 - \tan x > 0$. If $1 - \tan x$ is negative, the value of the radical is $\tan x - 1$.

EXERCISES 10

Transform the left member of each of the following identities into the form of the right member:

1. $\cos x \tan x + \sin x \cot x = \sin x + \cos x$
2. $\dfrac{\cos x}{\sin x \cot^2 x} = \tan x$
3. $(\tan x + \cot x) \sin x \cos x = 1$
4. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$
5. $\cot x + \dfrac{\sin x}{1 + \cos x} = \csc x$
6. $\dfrac{\tan x - \cot x}{\tan x + \cot x} = \dfrac{2}{\csc^2 x} - 1$
7. $\dfrac{\sec^3 x - (\tan x - 1)\sec x \tan x}{\sec^2 x} = \sin x + \cos x$
8. $\cos^2 x(1 + \tan^2 x) = 1$
9. $\cot^2 A - \cos^2 A = \cos^2 A \cot^2 A$
10. $\dfrac{\tan\theta}{1 + \sec\theta}\left[\dfrac{\tan^2\theta}{(1 + \sec\theta)^2} + 3\right] = \dfrac{2\sin\theta(\cos\theta + 2)}{(1 + \cos\theta)^2}$

Establish each of the following identities by any method.

11. $(\tan A + \cot A)^2 = \sec^2 A + \csc^2 A$
12. $\dfrac{\sec A}{\cos A} - \dfrac{\tan A}{\cot A} = 1$
13. $(\csc A - \cot A)^2 = \dfrac{1 - \cos A}{1 + \cos A}$
14. $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{1 - \cot A}{1 + \cot A}$
15. $\dfrac{1 + \cot^2 A}{1 + \tan^2 A} = \cot^2 A$
16. $\dfrac{\sqrt{1 + \tan^2\theta}\,\sec^2\theta}{\tan^4\theta} = \dfrac{\cos\theta}{\sin^4\theta}$
17. $\dfrac{\tan^2\theta}{\sec^3\theta} = \cos\theta\sin^2\theta$

18. $\sec x\,(\sec x + \tan x)^2 = \sec^3 x(1 + \sin x)^2$

19. $\dfrac{\tan x \cos x - \sin x \sec^2 x}{\tan^2 x} = -\sin x$

20. $\sec^2 x \tan x + \sec^3 x = \dfrac{\tan x + \sec x}{\cos^2 x}$

21. $\dfrac{\tan \theta}{\cos \theta}(1 + \cos^2 \theta) = \sin \theta(\sec^2 \theta + 1)$

22. $\cos^3 x - \sin^2 x \cos x - 4 \sin x \cos x = \cos x(1 - 4 \sin x - 2 \sin^2 x)$

23. $\tan^5 x = \sec^2 x(\tan^3 x - \tan x) + \tan x$

24. $\dfrac{(1 - \cos x)^2 \sin x \cos x - \sin^3 x(1 - \cos x)}{(1 - \cos x)^4} = \dfrac{-\sin x}{(1 - \cos x)^2}$

25. $\dfrac{\csc \theta[\tan \theta \sec \theta + 2 \sec^3 \theta \tan \theta] - \tan^2 \theta \sec \theta \csc \theta \cot \theta}{\csc^2 \theta}$
$= \tan^2 \theta(3 \tan^2 \theta + 1)$

26. $\dfrac{(\sec x - \tan x)(\sec x \tan x + \sec^2 x) - (\sec x + \tan x)(\sec x \tan x - \sec^2 x)}{(\sec x - \tan x)^2}$
$= 2 \sec^3 x(1 + \sin x)^2$

27. Express $\sin \theta \cos^2 \theta - \dfrac{\cot \theta}{\cos \theta \sin \theta} + \dfrac{\tan \theta}{\cos \theta}$ in terms of $\sin \theta$ only.

28. Express in terms of $\tan A$ the expression $\cot A + \sec A - \csc A$.

29. Express in terms of $\cos A$, $\dfrac{\sin^2 A}{\cos A} + \dfrac{\tan A}{\cot A}$.

18. TRIGONOMETRIC CONDITIONAL EQUATIONS

In trigonometry, as in algebra, an equation that is not true for all values of the unknown for which both members are defined is called a *conditional equation*. As in algebra, the determination of the values of the unknown for which the equation is true is called *solving the equation*.

Illustration 1: Solve the equation

$$(2 \sin \phi - 1)(\tan \phi + 1) = 0.$$

Since the right member of this equation is zero, and since the left member is already factored, the desired solution may be obtained by solving the two equations

$$2 \sin \phi - 1 = 0 \quad \text{and} \quad \tan \phi + 1 = 0.$$

From the first equation we have

$$\sin \phi = \tfrac{1}{2}.$$

We wish, therefore, to determine all the values of ϕ that satisfy this equation. The two positive angles less than 360° are $\phi = 30°$ and $\phi = 150°$. Hence, all the roots of $2 \sin \phi - 1 = 0$ are given by the formulas

$$\phi = 30° \pm n \cdot 360°, \; n = 0, 1, 2, 3, \cdots \tag{1}$$

and

$$\phi = 150° \pm n \cdot 360°, n = 0, 1, 2, 3, \cdots. \tag{2}$$

From the second equation, we have

$$\tan \phi = -1.$$

Hence, $$\phi = 135° \pm n \cdot 360°, n = 0, 1, 2, 3, \cdots \tag{3}$$

and $$\phi = 315° \pm n \cdot 360°, n = 0, 1, 2, 3, \cdots. \tag{4}$$

Thus, the roots of the original equation are given by (1), (2), (3), and (4).

The positive angles between 0° and 360° satisfying a conditional equation are the principal roots. The general or complete solution is obtained by adding $n(\pm 360°)$, $n = 0, 1, 2, 3, \cdots$, to the principal roots.

Illustration 2: Solve the equation $\sin^2 \phi + 3 \cos \phi - 3 = 0$.

First, we replace $\sin^2 \phi$ by $1 - \cos^2 \phi$, so that the equation involves only one function. Thus, we have

$$1 - \cos^2 \phi + 3 \cos \phi - 3 = 0,$$

or $$\cos^2 \phi - 3 \cos \phi + 2 = 0,$$

or $$(\cos \phi - 2)(\cos \phi - 1) = 0.$$

The roots of this equation are obviously obtained by solving the two equations,

$$\cos \phi = 2 \qquad \text{and} \qquad \cos \phi = 1.$$

The first equation has no roots. (Why?)

From $\cos \phi = 1$, the principal root is $\phi = 0$, and the general solution is $0° \pm n \cdot 360°$, $n = 0, 1, 2, 3, \cdots$.

Illustration 3: Solve the equation

$$\sin \phi + \cos \phi = \tfrac{1}{2}.$$

It is desirable to obtain an equivalent equation involving only one function. Since $\sin^2 \phi + \cos^2 \phi = 1$, we have

$$\cos \phi = \pm\sqrt{1 - \sin^2 \phi}.$$

After substituting this value for $\cos \phi$ into the given equation, we have

$$\sin \phi \pm \sqrt{1 - \sin^2 \phi} = \tfrac{1}{2},$$

or $$\pm\sqrt{1 - \sin^2 \phi} = \tfrac{1}{2} - \sin \phi.$$

After squaring each member, this equation becomes

$$1 - \sin^2 \phi = \tfrac{1}{4} - \sin \phi + \sin^2 \phi,$$

or $$2 \sin^2 \phi - \sin \phi - \tfrac{3}{4} = 0.$$

By means of the quadratic formula, it is determined that

$$\sin \phi = \frac{1 \pm \sqrt{7}}{4} = 0.91144 \text{ and } -0.41144.$$

The principal values of ϕ, when $\sin \phi = 0.91144$, are represented in Figure 40 by ϕ_1 and ϕ_2.

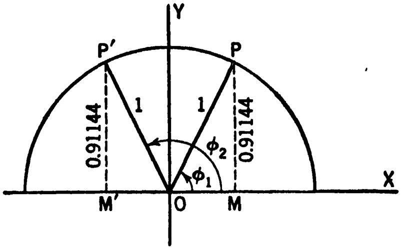


Fig. 40

Obviously, since $\sin \phi_1 = 0.91144$ and $\cos \phi_1$ is positive, we cannot have $\sin \phi_1 + \cos \phi_1 = \frac{1}{2}$. Hence, ϕ_1 is not a root of the original equation. It is not uncommon in trigonometry, as in algebra, to have an extraneous root when it has been necessary to square both members of an equation in order to solve it.

From Figure 40 we note that $\phi_2 = 180° - \phi_1$ and from Table 2 in the Appendix we find that $\phi_1 = 65°42'20''$. Hence, $\phi_2 = 114°17'40''$.

The principal angles ϕ_3 and ϕ_4, when $\sin \phi = -0.41144$, are represented in Figure 41. Since $\sin \phi + \cos \phi = \frac{1}{2}$, both sine and cosine cannot

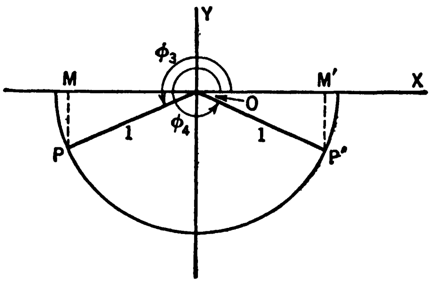


Fig. 41

be negative; hence, ϕ can terminate only in the fourth quadrant. It follows, then, that ϕ_3 is not a solution of the original equation.

From Figure 41 and by use of Table 2, which is explained in the next section, we have

$$\begin{aligned}\phi_4 &= 360° - 24°17'44'' \\ &= 335°42'16''.\end{aligned}$$

Hence, the two principal angles satisfying the equation are ϕ_2 and ϕ_4. The complete solution of the given equation is therefore

$$114°17'40'' \pm n(360°)$$

and $\quad 335°42'16'' \pm n(360°), \quad$ where $n = 0, 1, 2, 3, \cdots$.

We may summarize the method of solving a trigonometric equation as follows:

(1) Transform the given equation into one containing only a single function, or, when possible, express the equation in the form $\phi(x) = 0$, where $\phi(x)$ is factorable into factors, each of which contains only a single trigonometric function.

(2) Solve the equation, algebraically, for these functions as the unknown quantities.

(3) Find the value of the angles from the table or, in special cases, from a triangle.

(4) Test all solutions by substituting in the given equation.

19. THE USE OF TABLE 2

It is apparent from the previous discussion that Table 2 in the Appendix will be used frequently in the material that follows. Consequently, a brief explanation of it may be desirable. Two problems connected with its use will be considered.

Problem I. To find the value of a specified trigonometric function of an angle when the measure of the angle is given.

(1) Find the value of sin 32°20′. In the particular part of Table 2 headed 32°, look in the marginal column on the left for the number 20′. In the same row with this latter number and in the column headed *sin*, we find the value, .53484. This is the desired reading. Also in the same row we find tan 32°20′ = .63299; cot 32°20′ = 1.5798; and cos 32°20′ = .84495.

(2) Find the value of sin 62°20′. At the top of the various parts of Table 2 the headings only go as far as 44°. But it will be observed that the various parts are marked at the bottom also; these listings start at 45° and continue to 89°. When using the tabular headings that appear at the bottom, it is also necessary to use the column designations given at the bottom; this comment is most important, for it is noted that the designations at the bottom are different from those at the top. The corresponding marginal entries will now be found upon the right. This extension of the table to angles between 45° and 90° is made possible by the identities involving functions of complementary angles.

In the row corresponding to 62°20′, we find sin 62°20′ = .88566. Also, cos 62°20′ = .46433; cot 62°20′ = .52427; tan 62°20′ = 1.9074.

(3) Find the value of sin 28°13′15″. The desired reading obviously lies between the values of sin 28°13′ and sin 28°14′.

$$\sin 28°14' = .47306,$$

and

$$\sin 28°13' = .47281.$$

Thus, a difference of 1′, or 60″, makes a difference of .00025 in the tabulated values.

At this point we shall use simple interpolation, which is based on the assumption that a small change in the value of a function is proportional

to the change in the angle. Hence, the change x in the value of the sine corresponding to an increment of 15″ in the angle is given by

$$\frac{x}{.00025} = \frac{15''}{60''}.$$

Thus, $$x = .00006, \text{ approximately.}$$

So it follows that

$$\begin{aligned} \sin 28°13'15'' &= .47281 + .00006 \\ &= .47287. \end{aligned}$$

With a little experience, interpolation can be performed mentally.

(4) As a second problem involving interpolation, let us obtain cos 52°18′10″.

$$\cos 52°18' = .61153,$$

and $$\cos 52°19' = .61130.$$

Thus, a difference of 60″ represents a change of .00023 in the tabulated values, or 10″ corresponds to an increment of .00004 in the tabulated values.

Therefore, $$\begin{aligned} \cos 52°18'10'' &= .61153 - .00004 \\ &= .61149. \end{aligned}$$

Notice that the difference .00004 was subtracted this time. (Why?)

Problem II. To find the angle when the value of one of its trigonometric functions is given.

(1) Determine the acute angle θ if $\sin \theta = .60761$.

This time we investigate the columns headed *sin* until the proper angle is identified. The desired angle is 37°25′.

(2) Determine the acute angle θ if $\sin \theta = .34540$.

It is apparent from an examination of the tables that the desired angle is between 20°12′ and 20°13′. In fact,

$$\sin 20°13' = .34557,$$

and $$\sin 20°12' = .34530.$$

The increment x, to be added to 20°12′ to obtain an angle corresponding to the value .34540, may be obtained through interpolation by solving the equation

$$\frac{x}{60''} = \frac{.00010}{.00027},$$

wherein .00027 is the difference between the two readings displayed above, and .00010 is the difference between the value of sin 20°12′ and the given value .34540. The solution of this equation provides

$$x = 22''.$$

Thus, the desired angle is 20°12′22″.

20. THE ACCURACY OF TABLES

In general (exceptions are discussed below) for values of functions correct to two, three, four, and five significant figures, respectively, the corresponding angles may be determined from tables and through the use of interpolation correct to $1°$, $10'$, $1'$, and $0.1'$, respectively. Conversely, in general (exceptions are discussed below), for angles correct to $1°$, $10'$, $1'$, and $0.1'$, respectively, the corresponding function values can be determined from tables and through the use of interpolation correct to two, three, four, and five significant figures, respectively. We note that though an angle cannot be determined more accurately than to the nearest $0.1'$ when a function is given correct to five significant figures, it will be our practice to interpolate to the second in giving an angle, with the understanding that the angle may be in error as much as $3''$ in either direction.

The exceptions noted above in parentheses refer to the determination of angles from $0°$ to $4°$ by reference to a table of values of the cosine and the determination of angles from $86°$ to $90°$ by reference to a table of values of the sine. The exceptions also refer to the determination of the cotangent to five significant figures from a five-place table for angles from $0°$ to $4°$, and the determination of the tangent to five significant figures from a five-place table for angles from $86°$ to $90°$.

A reference to the graph of the cosine near $0°$ and to the graph of the sine near $90°$ shows that the cosine is changing very slowly near $0°$, and the sine is changing very slowly near $90°$. Thus, a five-place table of values for the cosine gives 1 as the cosine of all angles from $0°$ to $10'$. The table gives .99999 for all angles from $10'$ to $18'$. In fact, the table shows that the cosine varies only from 1.00000 to .99995 as the angle varies from $0°$ to $35'$. Hence, a small angle cannot be determined to the accuracy of $0.1'$ from its cosine. Similarly, an angle near $90°$ cannot be determined to an accuracy of $0.1'$ from its sine.

A reference to the graph of the cotangent near $0°$ and to the graph of the tangent near $90°$ shows that the cotangent is changing very rapidly near $0°$, and the tangent is changing very rapidly near $90°$. Thus, a five-place table of values for the cotangent gives 3437.7 for $2'$ and 1718.9 for $3'$.

The determination of the cotangent of $2'12''$, for example, from such a table gives 1603.9. If, however, we determine $\cot 2'12''$ from the formula $\log \cot x = -\log \tan x$ and the calculated value of the log tangent, by use of special tables for logs of functions for small angles, we have

$$\log \cot 2'12'' = -\log \tan 2'12'' = -(6.80615 - 10) = 3.19385.$$

Hence, $$\cot 2'12'' = 1562.6.$$

Thus, the value 1603.9 for $\cot 2'12''$ is correct to only two significant figures. This illustrates the fact that the cotangent cannot be determined correct to five significant figures from a five-place table for angles from $0°$

to 4°. Similarly, the tangent cannot be determined correct to five significant figures for angles from 86° to 90°.

In practical calculations the possible inaccuracies due to the exceptions discussed above may be avoided by a choice of formulas that will not involve these inaccuracies.

EXERCISES 11

Find the value of each of the following functions by using Table 2.

1. $\sin 27°18'$
2. $\cos 47°37'$
3. $\tan 36°8'$
4. $\cot 69°42'$
5. $\tan 29°32'18''$
6. $\sin 74°18'49''$
7. $\cos 82°7'13''$
8. $\sin 53°46'14''$
9. $\tan 47°19.2'$
10. $\cos 64°34.6'$

Find the acute angle θ corresponding to each of the following:

11. $\sin\theta = 0.41072$
12. $\cos\theta = 0.79300$
13. $\tan\theta = 1.2726$
14. $\cot\theta = 1.1731$
15. $\sin\theta = 0.63729$
16. $\cos\theta = 0.23726$
17. $\sin\theta = 0.63827$
18. $\tan\theta = 0.93824$
19. $\cos\theta = 0.83005$
20. $\sin\theta = 0.32116$

EXERCISES 12

Find the principal roots for each of the following equations, and construct the angle in each case. Look up the value of the angle in the table when necessary; also express the complete solution in each case.

1. (*a*) $\sin\theta = \frac{1}{2}$; (*b*) $\cos\theta = -\sqrt{\frac{3}{2}}$; (*c*) $\tan\theta = 6$
2. (*a*) $\sin\theta = -\frac{2}{3}$; (*b*) $\cos\theta = 0.26037$; (*c*) $\cot\theta = 0.55555$
3. $\tan\phi = 2 - \sin\phi\sec\phi$
4. $\tan^2\phi + 1 = 2\sec\phi$
5. $\tan\phi - \cos^2\phi = \sin^2\phi + 5$
6. $\tan\theta + \cot\theta = 2$
7. $\sec\theta + \tan\theta = 2$
8. $\sin\theta + \csc\theta = -\frac{5}{2}$
9. $\sec x\tan x = 2\sqrt{3}$
10. $\cos x\cot x = -\frac{5}{6}$
11. $\sec^2\phi - \tan\phi - 1 = 0$
12. $3\sin^2\phi + \cos^2\phi + \sin\phi = 0$
13. $10\cos^2\theta - 10\tan^2\theta - 3 = 0$
14. $\sin^2\theta - \tan\theta + \frac{3}{4} = 0$

HINT: Use Horner's method, if necessary.

15. $\cos^2 x - \sin^2 x + \cos x = 0$
16. $27\csc x\cot x = 8\sec x\tan x$
17. $2\csc^2\theta - \dfrac{\cos\theta}{\sin^2\theta} - 2 = 0$
18. $\sin\theta\tan\theta - 7\cos\theta + 5\sec\theta = 0$
19. $2\sin\theta - 3\cos\theta = 1$

Solve the following systems of trigonometric equations for r and θ, and properly pair your roots.

20. $r\sin\theta = 5$
    $r\cos\theta = 5$

HINT: Divide the members of the first equation by the corresponding members of the second to eliminate r. After determining θ, substitute to find r.

21. $r \cos \theta + r \sin \theta = 5$
$r \cos \theta = 3$

22. $r = 10 \cos \theta$
$r = 5(1 + \cos \theta)$

23. $r = \dfrac{8}{1 + \cos \theta}$
$r = 5(2 + \cos \theta)$

Solve the following systems of three trigonometric equations to determine the related sets of values of r, θ, ϕ.

24. $r \cos \theta \cos \phi = 3$
$r \cos \theta \sin \phi = 4$
$r \sin \theta = 5$

25. $r \cos \theta \cos \phi = -1$
$r \cos \theta \sin \phi = 2$
$r \sin \theta = \sqrt{5}$

21. SIN $(A + B)$ AND COS $(A + B)$

We shall next establish certain fundamental identities involving two different angles and, as special cases, relations involving the double angle and the half angle. We start with the sum of two angles and establish the important identities

$$\sin (A + B) = \sin A \cos B + \cos A \sin B,$$

and

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

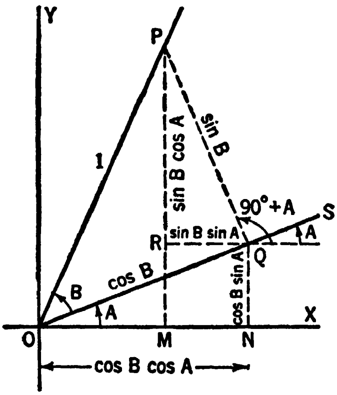


FIG. 42

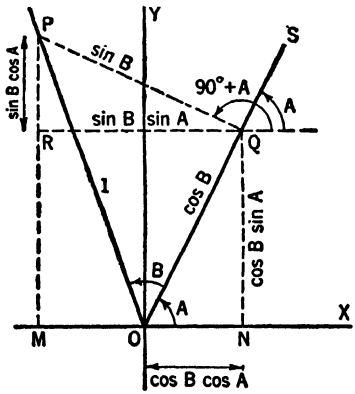


FIG. 43

These two identities will be proved when A and B are positive acute angles.

There are three cases: (a) when $(A + B) < 90°$ (Figure 42); (b) when $90° < (A + B) < 180°$ (Figure 43); and (c) when $A + B = 90°$.

We shall establish the identity for the first two cases. The third case is left as an exercise for the student.

In either Figure 42 or Figure 43, if we let $OP = 1$ and make the constructions $QP \perp OS$, $QN \perp OX$, $RQ \parallel OX$, $MP \perp OX$; then

$$QP = \sin B,$$

and
$$OQ = \cos B.$$

Also,
$$MR = NQ = OQ \sin A = \cos B \sin A,$$

and
$$ON = OQ \cos A = \cos B \cos A.$$

Since $\angle RPQ = \angle A$, it follows that

$$\begin{aligned} RP &= QP \cos A \\ &= \sin B \cos A, \end{aligned}$$

and

$$\begin{aligned} MN &= QP \sin A \\ &= \sin B \sin A. \end{aligned}$$

Hence,

$$\begin{aligned} \sin (A + B) = \frac{MP}{OP} &= MP = MR + RP \\ &= \cos B \sin A + \sin B \cos A \\ &= \sin A \cos B + \cos A \sin B. \qquad (1) \end{aligned}$$

Likewise,

$$\begin{aligned} \cos (A + B) = \frac{OM}{OP} &= OM = ON - MN \\ &= \cos B \cos A - \sin B \sin A. \qquad (2) \end{aligned}$$

These two identities hold when A and B are of any magnitude, positive or negative, although they have been established only under the conditions stated above.

22. SIN $(A - B)$ AND COS $(A - B)$

If we accept the universal nature of Identities (1) and (2), we may substitute $-B$ for B and write

$$\sin (A - B) = \sin [A + (-B)] = \sin A \cos (-B) + \cos A \sin (-B).$$

It has already been noted that $\cos (-B) = \cos B$ and $\sin (-B) = -\sin B$; hence,

$$\sin (A - B) = \sin A \cos B - \cos A \sin B. \qquad (3)$$

Similarly,

$$\begin{aligned} \cos (A - B) &= \cos [A + (-B)] \\ &= \cos A \cos (-B) - \sin A \sin (-B) \\ &= \cos A \cos B + \sin A \sin B. \qquad (4) \end{aligned}$$

These four identities are very important in the development of the formulas for the solution of oblique triangles to be used in the next chapter. They are also fundamental in the development of many important identities that are of value in simplifying formulas found in practice.

23. TAN $(A + B)$ AND TAN $(A - B)$

To obtain an identity for $\tan (A + B)$ in terms of the tangents of A and B, we may write

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

After dividing both the numerator and the denominator of the second member by $\cos A \cos B$, we have

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \tag{5}$$

Since
$$\tan (-B) = -\tan B,$$
we may substitute $-B$ for B in Identity (5) and obtain

$$\tan (A - B) = \frac{\tan A + \tan (-B)}{1 - \tan A \tan (-B)}$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{6}$$

Illustration: Given $\sin A = \frac{1}{3}$, where A terminates in the first quadrant, and $\sin B = \frac{3}{7}$, where B terminates in the second quadrant. Find the trigonometric functions of $(A + B)$ and $(A - B)$.

Since $\sin A = \frac{1}{3}$ and $\sin B = \frac{3}{7}$, it is readily determined that

$$\cos A = \frac{2\sqrt{2}}{3} \text{ and } \cos B = -\frac{2\sqrt{10}}{7}.$$

Therefore,

$$\sin (A + B) = \left(\frac{1}{3}\right)\left(\frac{-2\sqrt{10}}{7}\right) + \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{3}{7}\right) = \frac{-2\sqrt{10} + 6\sqrt{2}}{21}$$
$$= \frac{-2(3.1623) + 6(1.4142)}{21} = \frac{2.1606}{21} = 0.1029.$$

It is left as an exercise for the student to obtain the remaining functions of $A + B$ and $A - B$.

EXERCISES 13

1. Given $\tan A = \frac{1}{2}$, where A terminates in the third quadrant, and $\sin B = \frac{2}{3}$, where B terminates in the first quadrant. Find the functions of $(A + B)$ and $(A - B)$.

2. Evaluate $\sin 75°$.

HINT: $75° = 45° + 30°$. Hence,

$$\sin 75° = \sin (45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30° = ?$$

3. Evaluate $\cos 75°$.

4. Evaluate $\sin 15°$.

5. Evaluate $\sin (90° - B)$ in terms of functions of B.

6. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, find all possible values of $\tan (A + B)$.

7. If $\tan A = 1$ and $\cot B = -1$, find all possible values of $\tan (A - B)$.

8. Substitute $A = B$ in Formula (5) (Section 23) and thus obtain a formula for $\tan 2B$.

9. Show that $\sin (30° + A) - \sin (30° - A) = \sqrt{3} \sin A$.

10. Show that $\dfrac{\sin (A + B)}{\cos A \cos B} = \tan A + \tan B$.

11. Show that $\cos (A + 45°) + \sin (A - 45°) = 0$.

12. Show that $\tan (x + 45°) = \dfrac{1 + \tan x}{1 - \tan x}$.

13. Show that $\cos n\theta \cos \theta + \sin n\theta \sin \theta = \cos (n - 1)\theta$.

14. By the use of Formulas (1) and (2) (Section 21), find the sine and cosine of $90° + A$ in terms of functions of A. From your results find other functions of $90° + A$ in terms of functions of A.

15. By a method similar to that used in Section 23, establish the identity

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

24. SIN 2A AND COS 2A

If in Formula (1) (Section 21) we let $B = A$, we have

$$\sin (A + A) = \sin A \cos A + \cos A \sin A$$

or

$$\sin 2A = 2 \sin A \cos A. \tag{7}$$

Similarly, if we let $B = A$ in Formula (2) (Section 21), we have

$$\cos (A + A) = \cos A \cos A - \sin A \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A \tag{8}$$

$$= \cos^2 A - (1 - \cos^2 A)$$

$$= 2 \cos^2 A - 1 \tag{9}$$

$$= 1 - 2 \sin^2 A. \tag{10}$$

Hence, if we know the functions of any given angle, Formulas (7) to (10) enable us to obtain the functions of an angle twice as large as the given angle. These identities are very useful in simplifying complicated formulas and in solving trigonometric equations.

25. SIN $\frac{A}{2}$ AND COS $\frac{A}{2}$

If in (10) (Section 24) we replace A by $A/2$, we have

$$\cos A = 1 - 2 \sin^2 \frac{A}{2},$$

or

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}. \tag{11}$$

Similarly, when we employ (9) (Section 24), we have

$$\cos A = 2\cos^2\frac{A}{2} - 1,$$

or

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \qquad (12)$$

The sign before the radical in both (11) and (12) is determined by the quadrant in which $A/2$ terminates.

EXERCISES 14

1. If $\cos A = \frac{1}{2}$, where $0 < A < 90°$, find the functions of $2A$.
2. If $\sin A = -\frac{2}{3}$, where $180° < A < 270°$, find the functions of $2A$.
3. Derive a formula for $\tan 2A$ in terms of $\tan A$.
4. Express $\sin 3A$ in terms of $\sin A$.

HINT: Let $\sin 3A = \sin(2A + A)$. Then use Formulas (1) and (10).

5. Express $\cos 3A$ in terms of $\cos A$.
6. Express $\cos 6A$ in terms of functions of $3A$.
7. If $\cos A = -\frac{2}{3}$, where A is a positive angle less than 360° terminating in the second quadrant, find the functions of $A/2$.
8. Show that $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$.
9. Show that $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$.
10. Verify each of Formulas (7) and (8) (Section 24) and (11) and (12) (Section 25) for $A = 60°$ and for $A = 90°$.
11. Show that $\frac{\sin 2x}{1+\cos 2x} = \tan x = \frac{\tan 2x}{1+\sec 2x}$.
12. Show that $\frac{1-\sin 2x}{\cos 2x} = \frac{1-\tan x}{1+\tan x}$.
13. Show that $\csc x - \cot x = \tan\frac{1}{2}x$.
14. Show that $\sin 4A = \cos A(4\sin A - 8\sin^3 A)$.
15. Show that $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$.
16. Express $\sin 2\theta\cos 2\theta$ in terms of functions of θ.
17. Show that $\cos^4\theta = \frac{3+4\cos 2\theta+\cos 4\theta}{8}$.
18. Show that $\sin^4\theta = \frac{3-4\cos 2\theta-\cos 4\theta}{8}$.

26. SUM OF SINES OR COSINES

From Formulas (1) (Section 21) and (3) (Section 22), we have

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$

and

$$\sin(A-B) = \sin A\cos B - \cos A\sin B.$$

If we first add the corresponding members of these equations and then

subtract them, we obtain the two following relations:

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,$$

and $$\sin(A+B)-\sin(A-B)=2\cos A\sin B.$$

Now let $$A+B=x \quad\text{and}\quad A-B=y,$$

so that $$A=\frac{x+y}{2} \quad\text{and}\quad B=\frac{x-y}{2}.$$

After substituting these values in the two preceding relations, we obtain

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{13}$$

and $$\sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{14}$$

Similarly, from the formulas

$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$

and $$\cos(A-B)=\cos A\cos B+\sin A\sin B,$$

we may obtain

$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{15}$$

and $$\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{16}$$

MISCELLANEOUS EXERCISES 15

The following identities may be established by making use of the fundamental identities derived in the preceding sections.

Establish the following identities:

1. $2\sin\theta+\sin 2\theta=\dfrac{2\sin^3\theta}{1-\cos\theta}$

2. $\tan\theta\cot\theta=2\csc 2\theta$

3. $\dfrac{\sin 3x}{\sin x}=1+2\cos 2x$

4. $\tan\dfrac{x}{2}=\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x}$

5. $\dfrac{\sin 2x+\sin 2y}{\sin 2x-\sin 2y}=\dfrac{\tan(x+y)}{\tan(x-y)}$

6. $\dfrac{\sin x+\sin y}{\cos x+\cos y}=\tan\dfrac{x+y}{2}$

7. $\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}=\cos x$

8. $\dfrac{\cos^3 x+\sin^3 x}{\cos x+\sin x}=\dfrac{2-\sin 2x}{2}$

HINT: Use Problem 8 in Exercises 14.

9. $\dfrac{\sin 5x-\sin 2x}{\cos 2x-\cos 5x}=\cot\dfrac{7x}{2}$

10. $\cos^2\theta\sin^2\theta=\dfrac{1-\cos 4\theta}{8}$

11. $\sin x = \dfrac{2 \tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}$

12. $\sec \theta + \tan \theta = \tan\left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right)$

13. $\dfrac{\sin x}{\tan \frac{x}{2}} = \dfrac{2}{\sec^2 \frac{x}{2}}$

14. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

15. $\sin^2 \theta - \sin^2 \phi = \sin (\theta + \phi) \sin (\theta - \phi)$

16. $\cos^2 \theta - \sin^2 \phi = \cos (\theta + \phi) \cos (\theta - \phi)$

17. $\sin 30° + \sin 60° = \sqrt{2} \cos 15°$

18. $\cos 3x \sin x = \frac{1}{2} \sin 4x - \frac{1}{2} \sin 2x$

19. $\cos 5x \cos x = \frac{1}{2} \cos 6x + \frac{1}{2} \cos 4x$

20. $\cos 5x - \cos 3x = -8 \sin^2 x \cos x \cos 2x$

21. $\dfrac{\sin (x + y)}{\sin x \cos y} = \cot x \tan y + 1$

22. $\dfrac{\sin (x - y)}{\cos x \cos y} = \tan x - \tan y$

23. If $A + B + C = 180°$, $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$

24. If $A + B + C = 180°$,
$$\cos A + \cos B + \cos C = 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + 1$$

27. TRIGONOMETRIC EQUATIONS

We have previously outlined a method for solving certain trigonometric equations. The equations given below differ from those considered earlier, however, in that they involve functions of multiple angles. Of course, it is possible to express an equation involving multiple angles in terms of functions of the single angle.

Illustration 1. Solve

$$\cos 2x + \sin x = 1. \tag{1}$$

From Formula (8) (Section 24),

$$\cos 2x = 1 - 2 \sin^2 x. \tag{2}$$

After substituting this value for $\cos 2x$ in Equation (1), we have

$$1 - 2 \sin^2 x + \sin x = 1,$$

or

$$-2 \sin^2 x + \sin x = 0.$$

The left member may be factored to provide

$$\sin x \, (-2 \sin x + 1) = 0.$$

Therefore,

$$\sin x = 0 \text{ and } \tfrac{1}{2}.$$

Hence, the principal angles are $x = 0$, $x = \pi$, $x = \pi/6$, $x = 5\pi/6$.

Illustration 2. Solve

$$\sin 3x + \sin 2x + \sin x = 0. \tag{1}$$

After applying Formula (13) (Section 26) to the first and third terms

of this equation, we have

$$2 \sin 2x \cos x + \sin 2x = 0. \tag{2}$$

This equation may be rewritten in the form

$$\sin 2x(2 \cos x + 1) = 0. \tag{3}$$

Therefore, $\sin 2x = 0$ and $\cos x = -\frac{1}{2}$.

From $\sin 2x = 0$, we obtain

$$2x = 0 \pm 2n\pi, \quad \text{or } x = 0 \pm n\pi, \quad n = 0, 1, 2, 3, \cdots,$$

and

$$2x = \pi \pm 2n\pi, \quad \text{or } x = \frac{\pi}{2} \pm n\pi, \quad n = 0, 1, 2, 3, \cdots.$$

Hence, the principal angles are

$$x = 0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}.$$

From $\cos x = -\frac{1}{2}$, the principal angles are $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

Illustration 3. Solve

$$\sin mx = 0.$$

We have $mx = 0 \pm 2n\pi$ and $\pi \pm 2n\pi$, $n = 0, 1, 2, 3, \cdots$;

hence, $x = 0 \pm \dfrac{2n\pi}{m}$ and $\dfrac{\pi \pm 2n\pi}{m}$.

EXERCISES 16

Solve each of the following equations and check the results:

1. $\cos 2x = 2 \sin^2 x$

2. $2 \cos 2x = 1 - 4 \cos x$

3. $\tan^2 x - \cot^2 x = 4 \cot 2x$

4. $\tan 2x = 5 \cot x$

5. $\cos 2x = \sin 3x$

6. $\sin\left(2x + \frac{\pi}{3}\right) = \sin\left(x - \frac{\pi}{3}\right)$

7. $\cos\left(2x - \frac{\pi}{3}\right) = \sin\left(3x + \frac{\pi}{3}\right)$

8. $\sin\left(x - \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{3}\right) = 1$

9. $\cos 2x + \sin x + 2 = 0$

10. $2 \sin x = \sin 2x$

11. $\sin x + \sin 3x = \cos x - \cos 3x$

12. $\cos 2x - \tan 2x = 0$

13. $\cos 2x + 2 \cos^2 \frac{x}{2} = 1$

14. $\sin 3x + \sin x = \sin 2x$

15. $\tan x + \tan 2x = \tan 3x$

16. $\cot 2x = \tan x - 1$

17. $2 \sin x \sin 3x - \sin^2 2x = 0$

18. $3 \tan^2 3x + 8 \cos^2 3x = 7$

Solve the following systems of trigonometric equations for r and θ:

19. $r \sin 2\theta = 3$
$r \cos \theta = 4$

20. $r \sin 3\theta = 1$
$r(1 + 2 \cos 2\theta) = 2$

Eliminate θ from each of the following systems of equations:

21. $x = 2 \sin \theta$
$y = 3 \cos \theta$

HINT: $\sin^2 \theta + \cos^2 \theta = 1$.

22. $x = a \cos^3 \theta$
$y = b \sin^3 \theta$

23. $x = 2a \cos^2 \theta$
$y = \dfrac{2a \cos^3 \theta}{\sin \theta}$

24. $x = a\theta \cos \theta$
$y = a\theta \sin \theta$

25. $x = a \sin \theta$
$y = b \cos^3 \theta$

26. $x = a \tan \theta$
$y = a \cos^2 \theta$

27. $x = a \cos 2\theta$
$y = 2a \sin \theta$

28. $x = 3 \sin 2\theta$
$y = 2 \sin^2 \theta$

29. If the angle at the vertex of a cone is represented by θ, find θ for the cone which has a volume of 1500π cu in., and which has a base of radius 20 in.

30. It can be shown that in a reciprocating engine the crank angles for maximum velocity of the piston are represented by the solutions of the equation $\cos \theta + \frac{1}{8} \cos 2\theta = 0$. Find θ.

31. In studying the problem of balancing one sphere upon another, there arises the equation $m \cos^3 \theta = (M + m)(3 \cos \theta - 2)$, where M and m are the masses of the lower and upper spheres, respectively, and θ is the angle that the straight line joining the centers makes with the vertical. Find θ when $M = 50$ and $m = 30$.

28. EQUATIONS INVOLVING INVERSE FUNCTIONS

It is occasionally necessary to solve equations involving inverse functions. We shall confine ourselves to the principal values of the inverse functions (Section 16).

Illustration 1: Find x, if

$$\tan^{-1} x + \tan^{-1} (1 - x) = \tan^{-1} \tfrac{4}{3}. \tag{1}$$

Solution: Let

$$\alpha = \tan^{-1} x, \tag{2}$$

and

$$\beta = \tan^{-1} (1 - x). \tag{3}$$

Hence, we have from (1), (2), and (3),

$$\alpha + \beta = \tan^{-1} \tfrac{4}{3}, \tag{4}$$

or

$$\tan (\alpha + \beta) = \tfrac{4}{3}. \tag{5}$$

After expanding the left member, there results

$$\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{4}{3}. \tag{6}$$

But from (2) and (3), we have

$$\tan \alpha = x \qquad \text{and} \qquad \tan \beta = 1 - x. \tag{7}$$

Hence, after substituting the values from (7) in Equation (6), we obtain

$$\frac{x + 1 - x}{1 - x(1 - x)} = \frac{4}{3},$$

or

$$4x^2 - 4x + 1 = 0. \tag{8}$$

Therefore,

$$x = \tfrac{1}{2}. \tag{9}$$

Illustration 2. Find x if

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}. \tag{1}$$

Solution: Let

$$\alpha = \sin^{-1} x, \tag{2}$$

and

$$\beta = \sin^{-1} 2x. \tag{3}$$

Hence, from (1), (2), and (3), we have

$$\alpha + \beta = \frac{\pi}{3}. \tag{4}$$

So it follows that

$$\cos(\alpha + \beta) = \cos\frac{\pi}{3} = \tfrac{1}{2}, \tag{5}$$

or

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \tfrac{1}{2}. \tag{6}$$

From (2) and (3) we have

$$x = \sin\alpha \qquad \text{and} \qquad 2x = \sin\beta. \tag{7}$$

Since $x = \sin\alpha$, it follows that

$$\cos\alpha = \pm\sqrt{1 - x^2}, \tag{8}$$

and since $2x = \sin\beta$, we have

$$\cos\beta = \pm\sqrt{1 - 4x^2}. \tag{9}$$

Therefore, from (6), (7), (8), and (9), there results

$$(\pm\sqrt{1 - x^2})(\pm\sqrt{1 - 4x^2}) - 2x^2 = \tfrac{1}{2}, \tag{10}$$

or

$$\pm\sqrt{1 - x^2}\,\sqrt{1 - 4x^2} = 2x^2 + \tfrac{1}{2}.$$

After squaring each member, we obtain

$$(1 - x^2)(1 - 4x^2) = 4x^4 + 2x^2 + \tfrac{1}{4},$$

or

$$1 - 5x^2 + 4x^4 = 4x^4 + 2x^2 + \tfrac{1}{4}.$$

The solution of this last equation yields

$$x = \pm\frac{\sqrt{21}}{14}.$$

The positive value, $x = \frac{\sqrt{21}}{14}$, is the only value that satisfies equation (1). The remaining possible root $x = -\frac{\sqrt{21}}{14}$ is extraneous; it was introduced when the members of the equation were squared.

Illustration 3: Prove that

$$\sin^{-1}\tfrac{3}{5} + \sin^{-1}\tfrac{8}{17} = \sin^{-1}\tfrac{77}{85}. \tag{1}$$

Solution: Let

$$\alpha = \sin^{-1}\tfrac{3}{5} \qquad \text{and} \qquad \beta = \sin^{-1}\tfrac{8}{17}. \tag{2}$$

Hence, the given relation is equivalent to

$$\sin(\alpha + \beta) = \tfrac{77}{85}, \tag{3}$$

or

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \tfrac{77}{85}. \tag{4}$$

But $\sin\alpha = \frac{3}{5}$; so $\cos\alpha = \pm\frac{4}{5}$. (5)

Also, since $\sin\beta = \frac{8}{17}$, $\cos\beta = \pm\frac{15}{17}$. (6)

It is now necessary to establish that the left member of (4) is equal to the right member. The value of the left member is

$$\tfrac{3}{5}(\pm\tfrac{15}{17}) + (\pm\tfrac{4}{5})(\tfrac{8}{17}).$$

If we confine ourselves to the principal values of the functions, $\cos\alpha$ and $\cos\beta$ must be positive, and we have $\frac{45}{85} + \frac{32}{85} = \frac{77}{85}$. Statement (1), therefore, is correct.

EXERCISES 17

Solve the following equations, restricting the functions to principal values:

1. $\sin^{-1} x - \cos^{-1} x = \pi/6$
2. $\sin^{-1} 2x - \cos^{-1} x = \pi/6$
3. $\sin^{-1} x + \sin^{-1} 2x = \pi/3$
4. $\tan^{-1} x + 2\cot^{-1} x = 2\pi/3$
5. $\sin^{-1} x = 2\cot^{-1} x$
6. $\sin^{-1} 3x - \sin^{-1} x = \frac{1}{3}\pi$
7. $\sin^{-1} x = 2\tan^{-1} x$
8. $2\sin^{-1} x = \cos^{-1} x$
9. $\tan^{-1} 3x + \tan^{-1} 2x = \pi/4$

Justify each of the following:

10. $2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5}$
11. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$
12. $\tan^{-1} 2 + \cos^{-1}\frac{2}{5}\sqrt{5} = \pi/2$
13. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \pi/4$
14. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \pi/2$
15. $\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{8}{15} + \sin^{-1}\frac{13}{85} = \pi/2$

3 Solution of Triangles

29. SOLUTION OF TRIANGLES

A triangle has six elements, namely, three sides and three angles. If three elements, including at least one side, are given, it is possible, in general, to find the other three elements and the area of the triangle. The process of computing the unknown elements from those which are known is called *solving the triangle.*

A triangle may be solved by graphical methods or by formulas involving the trigonometric functions. A graphical method may be used when considerable approximation is permitted; whereas the second method may be employed to obtain a solution that is restricted in accuracy only by the accuracy of the given data and the accuracy of the tables.

The graphical method is used frequently as a check upon the solution obtained by the second method; moreover, it is also desirable that the solution be checked analytically by formulas that are independent of the formulas used in obtaining the solution.

30. GRAPHICAL METHOD

To solve a triangle graphically, merely construct the triangle to some convenient scale by means of a ruler and protractor so that it contains the given elements; then measure the unknown sides and angles.

The student should review the construction of a triangle as given in plane geometry. Also, it is recalled that a triangle is uniquely determined when two sides and an included angle, two angles and an included side, or three sides are given. On the other hand, when two sides and an angle opposite one of the sides are given, there may be one triangle, two triangles, or no triangle satisfying the given data. This latter case will be studied in greater detail later in this chapter.

In the following data the small letters indicate the sides of a triangle, and the capital letters indicate the angles opposite the sides designated by the corresponding small letter.

EXERCISES 18

Solve the following triangles graphically:

1. Given $A = 35°$, $B = 67°$, $a = 18$ ft
2. Given $A = 35°$, $c = 72$ ft, $b = 55$ ft

3. Given $C = 90°$, $c = 15$ ft, $b = 10$ ft
4. Given $B = 27°$, $a = 25$ ft, $b = 20$ ft (Two solutions)
5. Given $a = 16$ ft, $b = 20$ ft, $c = 25$ ft
6. Given $a = 9.3$ ft, $b = 12.4$ ft, $c = 15.5$ ft
7. Given $A = 42°$, $B = 37°$, $c = 11$ ft
8. Given $A = 62°$, $C = 62°$, $b = 17$ ft
9. What happens if $a = 16$ ft, $b = 20$ ft, and $c = 38$ ft?
10. What happens if $A = 30°$, $c = 20$ ft, $a = 9\frac{1}{2}$ ft?

31. LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

The numerical solution of triangles through the use of trigonometric formulas frequently involves considerable labor, which may be minimized by employing logarithms. Attention has already been called to the availability of Table 1, which gives the common logarithms of numbers. Now we desire to direct attention to Table 3 where the common logarithms of the trigonometric functions are listed.

The general organization of Table 3 is the same as that of Table 2. Certain special features of Table 3, however, should be noted. First, all sines and cosines of angles between 0° and 90° are less than 1, so their logarithms have negative characteristics; hence, part of the characteristic, namely, -10 has been omitted from the listed values of these two functions. For instance, $\log \sin 22°10' = 9.57669 - 10$. In the tangent column the quantity -10 is to be understood after all listed values until 45° is reached, after which the entire characteristic is written in the table, since $\tan \theta > 1$ when $45° < \theta < 90°$. To facilitate the process of interpolation when it is necessary to obtain the logarithm of a function of an angle involving a fractional part of a minute, columns have been introduced headed by d and cd that provide the differences between consecutive logarithms listed in the major columns.

As an illustration of interpolation as applied to the logarithms of the trigonometric functions, let us obtain $\log \cos 67°25'20''$.

$$\left.\begin{aligned} \log\cos 67°25' &= 9.58436 - 10 \\ \log\cos 67°25'20'' &= ? \\ \log\cos 67°26' &= 9.58406 - 10 \end{aligned}\right\} 30$$

The difference 30 (ignoring decimal points) between the two readings is conveniently written down in the d column. Since $67°25'20''$ is one third of the way from $67°25'$ to $67°26'$, it is an assumption employed in interpolation that $\log\cos 67°25'20''$ is one third of the way from $\log\cos 67°25'$ to $\log\cos 67°26'$. Thus,

$$\begin{aligned} \log\cos 67°25'20'' &= 9.58436 - 10 - \tfrac{1}{3}(0.00030) \\ &= 9.58426 - 10. \end{aligned}$$

EXERCISES 19

From Table 3 find the value of the following:

1. log sin 19°32′
2. log cos 49°8′
3. log tan 72°28′
4. log sin 32°17′20″
5. log cot 22°18′24″
6. log sin 72°25′42″
7. log tan 37°14.6′
8. log cos 11°7′36″
9. log tan 64°37′19″
10. log sin 40°6.7′

Determine the acute angle x that satisfies each of the following:

11. $\log \sin x = 9.58253 - 10$
12. $\log \tan x = 9.78618 - 10$
13. $\log \cos x = 9.76712 - 10$
14. $\log \cot x = 0.01946$
15. $\log \sin x = 9.76546 - 10$
16. $\log \cos x = 9.72283 - 10$
17. $\log \tan x = 9.93342 - 10$
18. $\log \cos x = 9.94447 - 10$
19. $\log \cot x = 0.37726$
20. $\log \sin x = 9.36367 - 10$

32. SOLUTION OF RIGHT TRIANGLES

We shall now consider the solution of right triangles. The discussion will involve the treatment of two possible cases, namely,

CASE 1. Given two sides.

CASE 2. Given one side and one acute angle.

Illustration: CASE 1. Given $C = 90°$, $c = 15$ ft, $b = 10$ ft. Determine a, B, A.

First construct the triangle approximately to scale, as in Figure 44. From the figure we have $\cos A = \frac{10}{15} = \frac{2}{3} = 0.66667$. Consequently, by reference to Table 2,

$$A = 48°11'22''.$$

Therefore, $\quad B = 90° - A = 41°48'38''.$

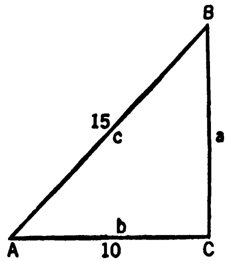


FIG. 44

Side a can be determined by use of the Pythagorean theorem as follows:

$$a = \sqrt{\overline{15}^2 - \overline{10}^2} = \sqrt{125} = 5\sqrt{5} = 11.180.$$

As a check upon a and B, observe that

$$\begin{aligned} a &= 10 \cot B \\ &= 10 \cot 41°48'38'' \\ &= 10(1.1180) \\ &= 11.180. \end{aligned}$$

These values of the unknown data have been determined much more accurately than the given measurements really justify. Side a, for instance, would probably be listed as 11.2.

Illustration: CASE 2. Given $C = 90°$, $A = 27°$, $c = 169$. Determine B, a, b.

Construct an appropriate triangle as in Figure 45.

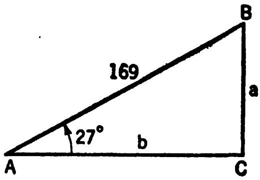


FIG. 45

$$B = 90° - 27° = 63°,$$

$$b = 169 \cos 27° = 169(0.89101) = 150.58,$$

$$a = 169 \sin 27° = 169(0.45399) = 76.724.$$

Check:

$$\cot B = \frac{a}{b} = \frac{76.724}{150.58} = 0.50953.$$

So $B = 63°$, thereby checking the value previously obtained. Also, $c^2 = 28561$, $a^2 = 22674$, and $b^2 = 5886.6$, thereby satisfying the Pythagorean relation, namely,

$$c^2 = a^2 + b^2.$$

Let us now consider a solution employing logarithms.

Illustration: Given $C = 90°$, $a = 176.32$, $c = 283.14$. Determine A, B, b.

Figure 46 has been drawn employing the given data.

The three unknown parts may be found as follows:

$$\sin A = \frac{176.32}{283.14}, \qquad (1)$$

$$B = 90 - A, \qquad (2)$$

$$b = 176.32 \cot A. \qquad (3)$$

Check Formulas:

$$\sin B = \frac{b}{c}, \qquad (4)$$

$$b^2 = (c - a)(c + a). \qquad (5)$$

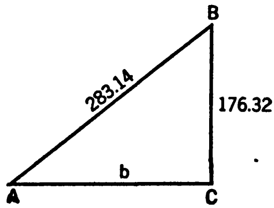


FIG. 46

Applying logarithms to Formula (1), we have

$$\log \sin A = \log 176.32 - \log 283.14.$$

$$\log 176.32 = 12.24630 - 10$$

$$(-)$$

$$\log 283.14 = 2.45200$$

$$\log \sin A = 9.79430 - 10$$

$$\therefore A = 38°30'57''.$$

By referring to relation (3),

$$\log b = \log 176.32 + \log \cot A.$$

$$\log 176.32 = 2.24630$$
$$(+)$$
$$\log \cot 38°30'57'' = 0.09915$$
$$\log b = 2.34545$$
$$\therefore b = 221.54.$$

From relation (2), $B = 90 - A = 51°29'3''$.

By reference to Check Formula (4),

$$\log \sin B = \log 221.54 - \log 283.14.$$

$$\log 221.54 = 12.34545 - 10$$
$$(-)$$
$$\log 283.14 = 2.45200$$
$$\log \sin B = 9.89345 - 10$$
$$\therefore B = 51°29'6''.$$

From Check Formula (5),

$$2 \log b \text{ must equal } \log (c - a) + \log (c + a),$$

or $$4.69090 \text{ must equal } 2.02865 + 2.66225,$$

which is true. Thus, the values of A and b are correct.

EXERCISES 20

For each of the following exercises the student should construct the figure, solve for the unknown parts, and check the solution:

1. Given $a = 268$, $b = 142$, $C = 90°$; find c, A, B.

2. Given $a = 268$, $c = 342$, $C = 90°$; find b, A, B.

3. Given $c = 361.52$, $b = 179.42$, $C = 90°$; find a, A, B.

4. Given $A = 68°27'35''$, $a = 269.12$, $C = 90°$; find b, c, B.

5. Given $B = 19°16'38''$, $a = 461.37$, $C = 90°$; find b, c, A.

6. Given $B = 29°18'45''$, $c = 23.614$, $C = 90°$; find a, b, A.

7. Find the side and area of a regular octagon (eight sides) inscribed in a circle of radius 10 in.

8. Find the side of a regular pentagon (five sides) circumscribed about a circle of radius 10 in.

9. Find the length of a chord which subtends an arc of 105° in a circle with radius 10 in.

10. Find the area of a sector of a circle with radius 10 in. if the sector is bounded by two radii and an arc which subtends an angle of 105°.

11. A segment of a circle is bounded by an arc which subtends an angle of 105° and its chord. Find the area of the segment if the radius of the circle is 10 in.

12. Derive a formula for the area of a sector of a circle in terms of its angle θ, measured in radians, and the radius r of the circle.

13. Consider a cylindrical tank 10 ft long and 5 ft in diameter placed in a horizontal position. Make a table showing the number of gallons of liquid the tank would contain at various depths. Compute volumes for depths varying at 6-in. intervals from an empty tank to a full tank.

14. To find the distance AB across a pond, a distance AC is measured 200 ft long at right angles to AB, and the angle ACB is found by measurement to be $82°50'23''$. Find AB.

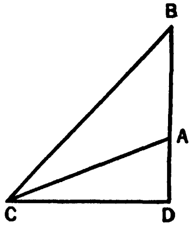


FIG. 47

15. To find the height of a vertical cliff AB (note Figure 47), the following measurements were taken: $CA = 233.16$ ft, $\angle DCA = 22°17'33''$, $\angle DCB = 48°19'52''$. Find AB.

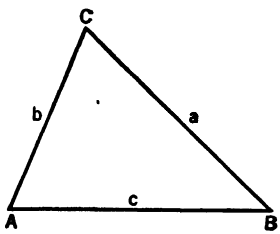


FIG. 48

16. Given $A = 68°27'35''$, $B = 45°16'27''$, $c = 292.13$ (note Figure 48). Find a, b, C.

HINT: Form right triangles by drawing a perpendicular from A to side BC.

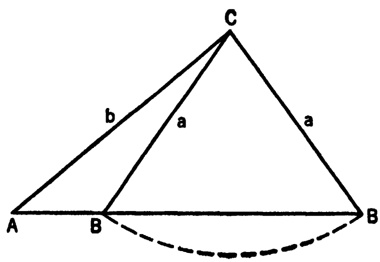


FIG. 49

17. Given $A = 41°37'25''$, $b = 476.18$, and $a = 372.96$ (note Figure 49). Find c, B, C (two possible solutions).

HINT: Form right triangles by drawing a perpendicular from C to AB.

18. Given $A = 82°27'$, $b = 271.4$, $c = 385.5$ (refer to Figure 50). Find B, C, a. Give values of the angles to the nearest minute and a to four significant figures.

HINT: Form right triangles by drawing a perpendicular from C to AB.

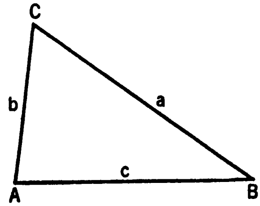


FIG. 50

19. Given $A = 68°27'35''$, $a = 965.12$, $b = 837.92$; find B, C, c.
20. Given $B = 38°16'27''$, $a = 277.19$, $c = 362.28$; find b, A, C.
21. Given $B = 138°27'52''$, $a = 277.19$, $c = 402.19$; find b, A, C.
22. Given $A = 68°27'35''$, $B = 42°16'27''$, $a = 350.52$; find C, b, c.

33. THE SOLUTION OF THE GENERAL TRIANGLE

The general triangle may be solved by means of special formulas involving the sides and functions of the angles. There are various formulas to be used, depending on the given elements of the triangle to be solved.

There are four cases to be considered, namely:

CASE 1. Given two angles and any side.

CASE 2. Given two sides and an angle opposite one of them.

Problems under Cases 1 and 2 may be solved by a formula known as the *law of sines* (Section 34).

CASE 3. Given two sides and the included angle.

CASE 4. Given three sides.

Problems under Cases 3 and 4 may be solved by a formula known as the *law of cosines* (Section 35).

34. CASES 1 AND 2: THE LAW OF SINES

We shall now develop the special formula known as the *law of sines*, which may be employed for the solution of triangles under Cases 1 and 2.

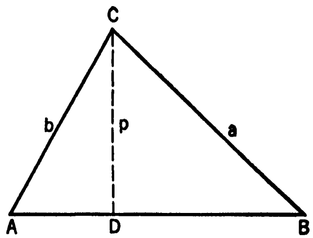


FIG. 51

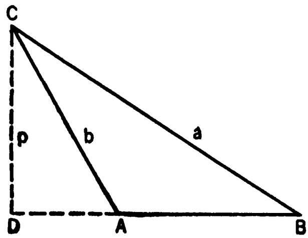


FIG. 52

In either Figure 51 or Figure 52, we have,

$$p = a \sin B,$$

and $$p = b \sin A.$$

Hence, $$a \sin B = b \sin A,$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Since ABC is any triangle, we may also write

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Therefore, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{1}$$

Equations (1) constitute the *law of sines*, which states that *any two sides of a triangle are in the same ratio as the sines of the corresponding angles opposite them.*

Equations (1) give us the three different equations

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \qquad \frac{a}{\sin A} = \frac{c}{\sin C}, \qquad \text{and} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Each involves four elements of the triangle, so that if we are given any three elements in any one of these equations, we may solve for the fourth element.

An interesting situation is encountered if we are given two sides and the angle opposite one of them, such as a, b, A. Then, from the law of sines,

$$\sin B = \frac{b \sin A}{a}.$$

If $\dfrac{b \sin A}{a} > 1$, no angle B is possible; hence, there is no solution.

If $\dfrac{b \sin A}{a} = 1$, angle $B = 90°$; hence, there is one solution. Moreover, the triangle is a right triangle.

If $\dfrac{b \sin A}{a} < 1$, there are two possible values for B. If $A \gtreqqless 90°$, however, B must be acute and only one solution is possible. If $A < 90°$, there may be two solutions.

We shall consider in detail the case when $A < 90°$, and $a < b$.

The perpendicular p dropped from angle C upon side c equals $b \sin A$; hence, our condition $\dfrac{b \sin A}{a} < 1$, means that $\dfrac{p}{a} < 1$, or $a > p$.

The added condition $a < b$ gives Figure 53, from which it is apparent there are two solutions, namely, triangles ABC and $AB'C$.

If $A < 90°$ and $a = b$, there will be only one solution, since in this case B' will coincide with A.

If $A < 90°$ and $a > b$, we have the situation depicted in Figure 54, which shows that there is one solution, the triangle ABC. The triangle $B'AC$ is not a solution since it does not contain the given $\angle A$.

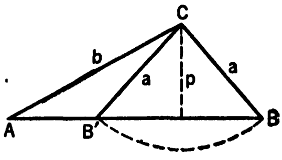


Fig. 53

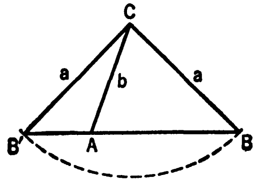


Fig. 54

35. MOLLWEIDE'S FORMULAS

It seems appropriate at this point to stop and develop formulas that are useful in checking the solution of any triangle. The formulas that we shall obtain are particularly serviceable because they contain all six elements of the triangle.

From the law of sines,

$$\frac{a}{c} = \frac{\sin A}{\sin C} \tag{1}$$

$$\frac{b}{c} = \frac{\sin B}{\sin C}. \tag{2}$$

If we add the corresponding members of Equations (1) and (2), we have

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$$

$$= \frac{2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}}{2 \sin \dfrac{C}{2} \cos \dfrac{C}{2}},$$

after employing Formulas 13 (Section 26) and 7 (Section 24).

Since $\dfrac{A+B}{2} = 90° - \dfrac{C}{2}$, and $\sin \dfrac{A+B}{2} = \cos \dfrac{C}{2}$, it follows that

$$\frac{a+b}{c} = \frac{\cos \dfrac{A-B}{2}}{\sin \dfrac{C}{2}}. \tag{3}$$

If we subtract the members of Equation (2) from the corresponding

members of (1), we have

$$\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C},$$

from which we finally obtain, by an analysis quite similar to that which preceded,

$$\frac{a-b}{c} = \frac{\sin \dfrac{A-B}{2}}{\cos \dfrac{C}{2}}. \qquad (4)$$

Formulas (3) and (4) are known as *Mollweide's formulas;* they are important check formulas, since either one contains all the elements of the triangle and may be used irrespective of what elements are given.

36. ILLUSTRATION: CASE 1

Solve the triangle ABC, where $a = 60$ ft, $A = 35°$, and $B = 85°$ (note Figure 55).

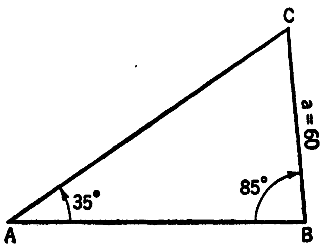


FIG. 55

By the law of sines, we have

$$b = \frac{a \sin B}{\sin A}$$
$$= \frac{60 \sin 85°}{\sin 35°} = 104.21.$$

$$\text{Also, } C = 180° - (A + B)$$
$$= 180° - (120°) = 60°.$$

Again, we may employ the law of sines to obtain

$$c = \frac{a \sin C}{\sin A}$$
$$= \frac{60 \sin 60°}{\sin 35°} = 90.592.$$

We shall use one of Mollweide's formulas for a check; namely,

$$\frac{b-a}{c} = \frac{\sin \dfrac{B-A}{2}}{\cos \dfrac{C}{2}}.$$

$$\begin{aligned} b &= 104.21 \\ a &= 60. \\ \hline b - a &= 44.21 \end{aligned} \qquad \begin{aligned} B &= 85^\circ \\ A &= 35^\circ \\ \hline B - A &= 50^\circ \end{aligned}$$

$$\sin \frac{B-A}{2} = 0.42262$$

$$c = 90.592 \qquad \cos \frac{C}{2} = 0.86603$$

$$\frac{b-a}{c} = \frac{44.21}{90.592} = 0.48801 \qquad \frac{\sin \frac{B-A}{2}}{\cos \frac{C}{2}} = 0.48800$$

The data as given do not justify carrying out the lengths of the sides to five significant figures. In practice, we would probably say that b is about 104 ft and c about 90.6 ft.

37. ILLUSTRATION: CASE 2

(1) Solve the triangle ABC, if $a = 35$ ft, $b = 45$ ft, and $A = 65°$.

It is advisable, first, to consider a triangle under Case 2 by the graphical method. This is attempted in Figure 56.

We find when the figure is drawn to scale that the side a is too short to reach the side c. Hence, there is no possible solution for a triangle possessing the given data. In fact, we have no triangle.

If we attempt to solve the same triangle by the law of sines, we have

$$\sin B = \frac{b \sin A}{a} = \frac{45 \sin 65^\circ}{35} = 1.165.$$

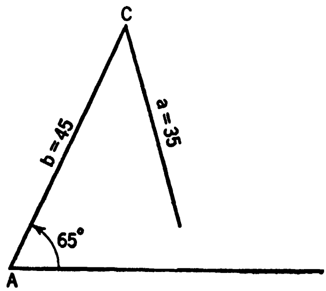


FIG. 56

Since $\sin B$ results in a number that is greater than 1, no triangle exists having parts as given.

This illustration indicates that if $\dfrac{b \sin A}{a} > 1$, there is no solution.

(2) Solve the triangle ABC if $a = 30$ ft, $b = 60$ ft, and $A = 30°$.

A sketch of this triangle indicates that it is a right triangle.

By using the law of sines, we have

$$\sin B = \frac{b \sin A}{a} = \frac{60(0.5)}{30} = 1.$$

Hence,

$$B = 90^\circ,$$

as we previously suspected.

Therefore, by the Pythagorean theorem, we obtain

$$c = \sqrt{3600 - 900} = 30\sqrt{3} = 51.963.$$

Check: $$c = 60 \cos 30° = 60(0.86603) = 51.962.$$

(3) Solve the triangle ABC if $a = 40$ ft, $b = 60$ ft, and $A = 30°$.

If we attempt a graphical solution, we find two possible triangles ABC and $AB'C$, as shown in Figure 57; therefore, there are two solutions.

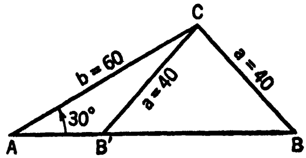


FIG. 57

Upon applying the law of sines, we have

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{60(0.5)}{40} = 0.75.$$

Note that $\dfrac{b \sin A}{a} < 1$, and since it is also true that $a < b$, there are two solutions as predicted by the graphical consideration.

From a table of trigonometric functions, we find

$$B = 48°35'25'' \qquad \text{or} \qquad 180° - (48°35'25'').$$

This second value of B is $\angle AB'C$ in Figure 57. Hence,

$$\angle AB'C = 131°24'35''.$$

To complete the solution we must find AB' in $\triangle AB'C$, and AB in $\triangle ABC$. It is readily determined that

$$\angle ACB' = 18°35'25'' \qquad \text{and} \qquad \angle ACB = 101°24'35''.$$

Thus, $$AB' = \frac{a \sin \angle ACB'}{\sin 30°} = 25.504,$$

and $$AB = \frac{a \sin \angle ACB}{\sin 30°} = 78.418.$$

Check of the solution of $\triangle AB'C$ by Mollweide's formula:

$$\frac{b - a}{c} = \frac{\sin \dfrac{B' - A}{2}}{\cos \dfrac{C}{2}}.$$

$b = 60$ $a = 40$ $b - a = 20$	$B' = 131°24'35''$ $A = 30°$ $B' - A = 101°24'35''$ $\dfrac{B' - A}{2} = 50°42'17''$
$AB' = c = 25.504$	$\dfrac{\angle ACB'}{2} = \dfrac{C}{2} = 9°17'42''$
$\dfrac{b - a}{c} = 0.78419$	$\dfrac{\sin \dfrac{B' - A}{2}}{\cos \dfrac{C}{2}} = 0.78418$

To apply Mollweide's formula to $\triangle ABC$, we have

$b = 60$ $a = 40$ $b - a = 20$	$B = 48°35'25''$ $A = 30°$ $B - A = 18°35'25''$ $\dfrac{B - A}{2} = 9°17'42''$
$AB = c = 78.418$	$\dfrac{\angle ACB}{2} = \dfrac{C}{2} = 50°42'17''$
$\dfrac{b - a}{c} = 0.25504$	$\dfrac{\sin \dfrac{B - A}{2}}{\cos \dfrac{C}{2}} = 0.25504$

EXERCISES 21

Solve the following triangles and check each solution by Mollweide's formula:

	A	B	C	a	b	c
1	......	65°13′	58°28′	768		
2	29°16′	......	70°31′	...	......	396.3
3	......	52°19′	......	385	413	
4	......	......	23°16′	308	......	273
5	......	55°16′	......	......	165	220
6	......	55°16′	......	......	180.8	220

38. CASES 3 AND 4: LAW OF COSINES

Another law, known as the *law of cosines*, will now be developed for the solution of triangles under Cases 3 and 4.

In either Figure 58 or Figure 59, $AB = c$. Let $AD = x$, and of course, $DA = -x$. We then have $DB = c - x$. By reference to $\triangle ADC$, we have

$$b^2 = x^2 + p^2. \tag{1}$$

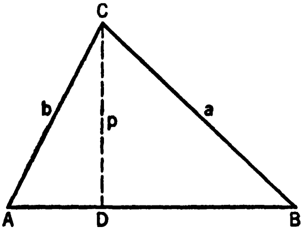


Fig. 58

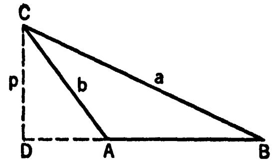


Fig. 59

From $\triangle DCB$, we obtain

$$p = a \sin B, \tag{2}$$

and $$c - x = a \cos B, \quad \text{or} \quad x = c - a \cos B. \tag{3}$$

After substituting the values for p and x from (2) and (3) into the right member of (1), we have

$$\begin{aligned} b^2 &= a^2 \sin^2 B + (c - a \cos B)^2 \\ &= a^2 \sin^2 B + c^2 - 2ac \cos B + a^2 \cos^2 B \\ &= a^2 (\sin^2 B + \cos^2 B) + c^2 - 2ac \cos B. \end{aligned}$$

Hence, $$b^2 = a^2 + c^2 - 2ac \cos B. \tag{4}$$

Similarly, we may derive

$$c^2 = a^2 + b^2 - 2ab \cos C, \tag{5}$$

and $$a^2 = b^2 + c^2 - 2bc \cos A. \tag{6}$$

Equations (4), (5), and (6) constitute the law of cosines, which may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of these two sides and the cosine of their included angle.

Under Case 4, where the three sides are given to determine the angles, we write from (5), (4), and (6), respectively,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \tag{7}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \tag{8}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \tag{9}$$

39. ILLUSTRATION: CASE 3

Solve the triangle ABC if $a = 50$, $b = 60$, and $C = 40°$.

Figure 60 contains the data for the problem. By the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Therefore,

$$\begin{aligned} c^2 &= (50)^2 + (60)^2 - 2(50)(60) \cos 40° \\ &= 2500 + 3600 - 6000(0.7660) \\ &= 6100 - 4596 \\ &= 1504. \end{aligned}$$

Hence, $\quad c = 38.8$, approximately.

We now have three sides and one angle. Each of the remaining two angles may be found by the law of sines, and the angles can be checked by the formula $A + B + C = 180°$. As an alternate method, one of the remaining angles may be found by the law of sines and the other by subtracting the sum of the two determined angles from 180°; the solution may then be checked by Mollweide's formula.

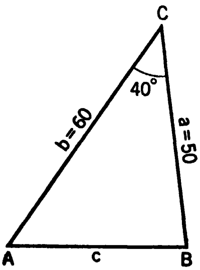


Fig. 60

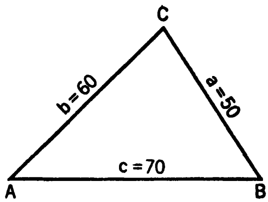


Fig. 61

40. ILLUSTRATION: CASE 4

Solve for the remaining parts of the triangle in which $a = 50$, $b = 60$, and $c = 70$ (note Figure 61).

Since

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

we have

$$\begin{aligned} \cos B &= \frac{(50)^2 + (70)^2 - (60)^2}{2(50)(70)} \\ &= \frac{3800}{7000} = 0.54285. \end{aligned}$$

Therefore, $\quad B = 57°7'19''.$

Similarly, we may find A and C. However, it is often more convenient to find the angles A and C by means of the law of sines.

EXERCISES 22

In each of the following exercises it is advisable to solve the triangle graphically before attempting a solution by formula. Check all solutions.

1. In $\triangle ABC$, $a = 360$, $b = 460$, $C = 39°17'$. Find the other parts.
2. In $\triangle ABC$, $b = 92$, $c = 84$, $A = 110°20'$. Find the other parts.
3. In $\triangle ABC$, $c = 55$, $a = 35$, $B = 90°$. Find the other parts.
4. In $\triangle ABC$, $a = 320$, $b = 410$, $c = 380$. Find the other parts.
5. Two straight roads intersect at an angle of 63.4°. Town A is located on one of the roads 86.4 miles from the intersection whereas town B is located on the other road 47.6 miles from the intersection. How far apart are the two towns? How far is town B from the other road?
6. Upon a baseball diamond, it is 60.5 ft from home plate to the pitcher's box and 90 ft from home plate to first base. If the angle is 45° at the home plate between the lines to the pitcher's box and to first base, how far is it from the pitcher's box to first base?
7. An airplane flies due south at a speed of 320 mph. Another plane flies at a speed of 284 mph in the direction 52° west of north. How far apart are the planes at the end of 15 min?
8. The two arms of a derrick are 10.2 ft and 15.6 ft, respectively. They are tied at the end by a chain so that the angle between them cannot exceed 26°30'. How long is the chain?
9. Determine the angle between the diagonal of a cube and an edge.

41. LAW OF TANGENTS

Up to this point there has been no mention of the use of logarithms in the treatment of the four cases just considered. Presumably, however, the student may have found it desirable to use logarithms in connection with solutions involving the law of sines. The law of cosines, on the other hand, involving as it does additions and subtractions, does not lend itself readily to logarithmic computation. A special formula known as the *law of tangents*, to be used under Case 3 when two sides and the included angle are given, will now be derived.

It is recalled that Mollweide's formulas are

$$\frac{a-b}{c} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}},$$

and

$$\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}.$$

If the members of the first equality are divided by the corresponding members of the second, it follows that

$$\frac{a-b}{a+b} = \frac{\sin\dfrac{A-B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A-B}{2}\cos\dfrac{C}{2}}$$

$$= \tan\frac{A-B}{2}\tan\frac{C}{2}.$$

Therefore, $$\frac{a-b}{a+b} = \frac{\tan\dfrac{(A-B)}{2}}{\tan\dfrac{(A+B)}{2}}. \quad \text{(Why?)} \tag{1}$$

Similarly, we may derive

$$\frac{a-c}{a+c} = \frac{\tan\dfrac{(A-C)}{2}}{\tan\dfrac{(A+C)}{2}}. \tag{2}$$

$$\frac{b-c}{b+c} = \frac{\tan\dfrac{(B-C)}{2}}{\tan\dfrac{(B+C)}{2}}. \tag{3}$$

This formula, in the various forms (1), (2), and (3), is the law of tangents.

42. LAW OF THE TANGENT OF HALF ANGLES

We have observed that the law of cosines may be used for the solution of triangles when the three sides are given (Case 4). But, as before noted, the law of cosines does not readily lend itself to logarithmic computation. Hence, we shall develop a special formula adaptable to logarithmic computation and useful for determining the angles of a triangle when the three sides are given. We shall designate this law as the *law of the tangent of half angles*.

From the law of cosines,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

But, $$\cos A = 2\cos^2\frac{A}{2} - 1$$

$$= 1 - 2\sin^2\frac{A}{2}.$$

Hence, $1 - 2\sin^2\frac{A}{2} = \frac{b^2 + c^2 - a^2}{2bc}$,

or
$$2\sin^2\frac{A}{2} = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$
$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}. \quad (1)$$

Similarly,
$$2\cos^2\frac{A}{2} = \frac{b^2 + c^2 - a^2}{2bc} + 1 = \frac{b^2 + c^2 - a^2 + 2bc}{2bc}$$
$$= \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c - a)(b + c + a)}{2bc}. \quad (2)$$

After dividing the members of relation (1) by the corresponding members of relation (2), we obtain
$$\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}} = \frac{(a - b + c)(a + b - c)}{(b + c - a)(b + c + a)},$$

or
$$\tan\frac{A}{2} = \sqrt{\frac{(a - b + c)(a + b - c)}{(b + c - a)(b + c + a)}}. \quad (3)$$

If we let $a + b + c = 2s$ (that is, s is one half the perimeter of the triangle), there results
$$a - b + c = 2s - 2b = 2(s - b),$$
$$a + b - c = 2s - 2c = 2(s - c),$$
$$b + c - a = 2s - 2a = 2(s - a).$$

Formula (1) may now be written
$$\sin\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}},$$

and Formula (2) may be written
$$\cos\frac{A}{2} = \sqrt{\frac{(s - a)s}{bc}}.$$

Also, Formula (3) becomes
$$\tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{(s - a)(s)}}$$
$$= \sqrt{\frac{(s - a)(s - b)(s - c)}{(s - a)^2 s}}.$$

If we let

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{4}$$

then,

$$\tan\frac{A}{2} = \frac{r}{s-a}. \tag{5}$$

Similarly,

$$\tan\frac{B}{2} = \frac{r}{s-b}, \tag{6}$$

and

$$\tan\frac{C}{2} = \frac{r}{s-c}. \tag{7}$$

Formulas (4), (5), (6), and (7) lend themselves to logarithmic computation for determining the angles of a triangle when the sides are given. After the three angles are calculated, the solution may be checked by the formula

$$A + B + C = 180°.$$

43. ILLUSTRATIONS OF LOGARITHMIC SOLUTIONS

The logarithmic solution of triangles is illustrated by the consideration of the following examples.

Illustration: Case 1. To illustrate the use of logarithms in the solution of triangles when it is possible to use the law of sines, let us solve the following triangle.

Given $c = 268$, $B = 23°16'32''$, $A = 35°29'38''$; find the remaining parts. Figure 62 illustrates the shape of the triangle.

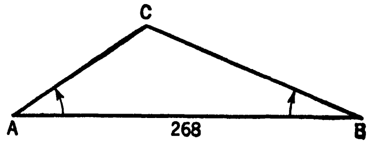


Fig. 62

Since

$$C = 180° - (A + B),$$

it follows that

$$C = 180° - (35°29'38'' + 23°16'32'') = 121°13'50''.$$

Moreover, since

$$\frac{a}{c} = \frac{\sin A}{\sin C},$$

we have

$$a = \frac{c \sin A}{\sin C} = \frac{268\ (\sin 35°29'38'')}{\sin 121°13'50''}.$$

In solving a triangle by logarithms it is important that the work be arranged systematically. The work for the solution of side a may be arranged as follows:

$$\begin{aligned} \log 268 &= 2.42813 \\ &(-) \\ \log \sin 121°13'50'' = \log \sin 58°46'10'' &= \underline{9.93201 - 10} \\ \log \frac{c}{\sin C} &= 2.49612 \\ &(+) \\ \log \sin 35°29'38'' &= \underline{9.76389 - 10} \\ \log a &= 2.26001 \\ a &= 181.97. \end{aligned}$$

Side b may be obtained by the law of sines as follows:

$$b = \frac{c \sin B}{\sin C} = \frac{268 \sin 23°16'32''}{\sin 121°13'50''}.$$

After taking advantage of the computation of $\log \dfrac{c}{\sin C}$ above, we have

$$\begin{aligned} \log \frac{c}{\sin C} &= 2.49612 \\ &(+) \\ \log \sin 23°16'32'' &= \underline{9.59677 - 10} \\ \log b &= 2.09289 \\ b &= 123.85. \end{aligned}$$

In practice, of course, these values for a and b would probably be rounded off to 182 and 124, respectively.

Check: Let us use the formula

$$\frac{a+b}{c} = \frac{\cos \dfrac{A-B}{2}}{\sin \dfrac{C}{2}}.$$

$a + b = 305.82$	$\log 305.82 = 2.48547$
$c = 268$	$(-)$
	$\log 268 = 2.42813$
	$\log \dfrac{a+b}{c} = 0.05734$
$A = 35°29'38''$	$\log \cos 6°6'33'' = 9.99754$
$B = 23°16'32''$	$(-)$
$A - B = 12°13'6''$	$\log \sin 60°36'55'' = 9.94019$
$\dfrac{A-B}{2} = 6°6'33''$	$\log \dfrac{\cos \dfrac{A-B}{2}}{\sin \dfrac{C}{2}} = 0.05735$
$\dfrac{C}{2} = 60°36'55''$	

Since the logarithms of $\dfrac{a+b}{c}$ and $\dfrac{\cos \dfrac{A-B}{2}}{\sin \dfrac{C}{2}}$ agree to four significant figures, the solutions are correct within the precision of five-place tables.

Illustration: CASE 2. Given $a = 704$, $b = 302$, and $B = 25°14'13''$; obtain the remaining parts of the triangle.

From the given data we see that $b < a$. Since

$$\sin A = \frac{a \sin B}{b} = \frac{704 \sin 25°14'13''}{302},$$

we have the following tabulated results:

$$
\begin{aligned}
\log 704 &= 2.84757 \\
&(+) \\
\log \sin 25°14'13'' &= \underline{9.62978 - 10} \\
\log a \sin B &= 12.47735 - 10 \\
&(-) \\
\log 302 \quad & \underline{\;2.48001\qquad\;} \\
\log \sin A &= 9.99734 - 10.
\end{aligned}
$$

From the fact that $\log a \sin B = 2.47735$, and $\log b = 2.48001$, we observe that $b > a \sin B$. Since $a \sin B < b < a$, there are two solutions (Section 43). Hence, by reference to a table, we have

$$A = 83°40' \quad \text{and} \quad A' = 180° - A = 96°20'.$$

From $\quad C = 180° - (A + B), \quad C = 71°5'47'';$

and from $\quad C' = 180° - (A' + B), \quad C' = 58°25'47''.$

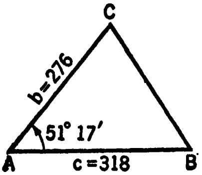


FIG. 63

As in the previous illustration, c may now be obtained by using the law of sines and employing the data for a, b, B, A, and C. Likewise, c' can be obtained from the data for a, b, B, A', and C'.

Illustration: CASE 3. Given $b = 276$, $c = 318$, and $A = 51°17'$; find the remaining parts of the triangle. Note Figure 63.

Since we expect to use the law of tangents, we determine

$$\frac{B+C}{2} = \frac{180° - A}{2} = \frac{180° - 51°17'}{2}$$

$$= \frac{128°43'}{2} = 64°21'30''.$$

Since c is larger than b, to avoid negative numbers in the logarithmic computation, we will write the law of tangents in the form

$$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\tan\frac{C+B}{2}$$

$$\begin{aligned} c &= 318 \\ b &= 276 \\ \hline c - b &= 42 \\ c + b &= 594 \\ \frac{C+B}{2} &= 64°21'30'' \end{aligned}$$

$$\begin{aligned} \log 42 &= 11.62325 - 10 \\ &(-) \\ \log 594 &= \underline{2.77379} \\ \log\frac{c-b}{c+b} &= 8.84946 - 10 \\ &(+) \\ \log\tan 64°21'30'' &= \underline{0.31874} \\ \log\tan\frac{C-B}{2} &= 9.16820 - 10 \\ \frac{C-B}{2} &= 8°22'46'' \end{aligned}$$

To complete the solution, we must solve the following system of equations in C and B:

$$\begin{aligned} \frac{C+B}{2} &= 64°21'30'' \\ \frac{C-B}{2} &= 8°22'46'' \\ \hline C &= 72°44'16'' \\ B &= 55°58'44'' \end{aligned}$$

To find a, we shall use the law of sines. Thus,

$$a = \frac{b \sin A}{\sin B} \quad \text{and} \quad a = \frac{c \sin A}{\sin C}.$$

By using both formulas, we not only obtain the value of a but have a check on our work. The logarithmic computation for the first formula follows:

$$\begin{array}{rl}
\log 276 = & 2.44091 \\
(+) & \\
\log \sin 51°17' = & 9.89223 - 10 \\ \hline
\log b \sin A = & 12.33314 - 10 \\
(-) & \\
\log \sin 55°58'44'' = & 9.91846 - 10 \\ \hline
\log a = & 2.41468 \\
a = & 259.82.
\end{array}$$

The logarithmic computation for the second formula will now be written down.

$$\begin{array}{rl}
\log 318 = & 2.50243 \\
(+) & \\
\log \sin 51°17' = & 9.89223 - 10 \\ \hline
\log c \sin A = & 12.39466 - 10 \\
(-) & \\
\sin 72°44'16'' = & 9.97998 - 10 \\ \hline
\log a = & 2.41468 \\
a = & 259.82.
\end{array}$$

Illustration: CASE 4. In triangle ABC, $a = 324.1$, $b = 395.7$, $c = 409.8$. Find the angles of the triangle by using the half-angle formulas.

The formulas in order of use are

$$2s = a + b + c,$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan \frac{A}{2} = \frac{r}{s-a},$$

$$\tan \frac{B}{2} = \frac{r}{s-b},$$

$$\tan \frac{C}{2} = \frac{r}{s-c}.$$

Check Formula: $A + B + C = 180°$.

Solution:

$$\begin{aligned}
a &= 324.1\\
b &= 395.7\\
c &= 409.8\\
\hline
2s &= 1129.6\\[6pt]
s &= 564.8\\
s - a &= 240.7\\
s - b &= 169.1\\
s - c &= 155.0\\
\hline
\textit{Check: } 2s &= 1129.6\\[6pt]
\log (s - a) &= 2.38148\\
\log (s - b) &= 2.22814\\
\log (s - c) &= 2.19033\\
\hline
&\ 6.79995\\
\log s &= 2.75189\\
\hline
\log r^2 &= 4.04806\\[6pt]
\log r &= 2.02403
\end{aligned}$$

$$\begin{aligned}
\log r &= 12.02403 - 10\\
\log (s - a) &= 2.38148\\
\hline
\log \tan \frac{A}{2} &= 9.64255 - 10\\[6pt]
\log r &= 12.02403 - 10\\
\log (s - b) &= 2.22814\\
\hline
\log \tan \frac{B}{2} &= 9.79589 - 10\\[6pt]
\log r &= 12.02403 - 10\\
\log (s - c) &= 2.19033\\
\hline
\log \tan \frac{C}{2} &= 9.83370 - 10\\[6pt]
\frac{A}{2} &= 23°42'21''\\
\frac{B}{2} &= 32°0'22''\\
\frac{C}{2} &= 34°17'20''\\[6pt]
A &= 47°24'42''\\
B &= 64°0'44''\\
C &= 68°34'40''\\
\hline
\textit{Check: } A + B + C &= 180°0'6''
\end{aligned}$$

EXERCISES 23

Solve the following triangles by the use of logarithms, and check the solutions:

1. $a = 438.30$, $A = 43°50'24''$, $B = 69°30'12''$
2. $A = 64°35'$, $C = 73°49'$, $a = 213.47$
3. $B = 51°41'48''$, $C = 93°46'6''$, $b = 0.19740$
4. $a = 374$, $b = 412$, $C = 58°28'$
5. $a = 238.5$, $b = 197.3$, $c = 205.0$
6. $B = 65°13'$, $C = 58°28'$, $a = 768.0$
7. $a = 732.5$, $b = 968.3$, $C = 80°25'$
8. $a = 10.05$, $b = 19.03$, $c = 15.98$
9. $a = 695$, $b = 345$, $B = 21°14'25''$
10. $b = 113.07$, $c = 120.55$, $A = 100°50'48''$
11. $a = 103.21$, $b = 152.37$, $A = 15°32'42''$
12. $a = 148.60$, $b = 121.78$, $A = 69°20'10''$

13. $a = 0.9686$, $c = 1.0073$, $B = 41°17'18''$

14. $a = 1.4595$, $b = 1.6072$, $c = 1.8278$

15. Two planes leave an airport at the same time. One flies a course of 46°35′ measured east of north, and the other a course of 72°18′ measured in the same manner. If the planes fly 250 and 300 mph, respectively, how far apart are they at the end of 2 hr?

16. A triangular lot ABC has $AB = 130$ rd, $BC = 165$ rd, and $AC = 172$ rd in length. How far is it from A to the mid-point of BC?

17. In order to find the distance AB across a pond, the distances from A and B to a third point C were measured and found to be 327 rd and 247 rd, respectively. It was also found that $\angle ABC = 57.3°$. Find the distance AB.

18. From a boat which is 4.2 miles from one end of an island and 6.3 miles from the other end, the island subtends an angle of 36°45′. How long is the island?

19. The longer base of an isosceles trapezoid is 11.2 in., while the nonparallel sides are 6.4 in. long. If each base angle is 68°36′, how long is each diagonal?

20. From a cliff 316 ft high, a boat is observed to be sailing toward the cliff. If the angle of depression of the boat is 7.3°, and 2 min later is 13.4°, how fast is the boat sailing?

NOTE: The angle of depression is the angle between the horizontal and the line of sight from the observer to the boat.

44. AREA OF A TRIANGLE

There are two cases to consider when finding the area of a triangle; namely, when two sides and the included angle are given, and when three sides are given.

CASE 1. Given b, c, and A; find the area.

In plane geometry we learn that the area K is given by the formula $K = \frac{1}{2}pc$, where p is the perpendicular from C to AB (note Figure 64). Since

$$p = b \sin A,$$

it follows that

$$K = \tfrac{1}{2}bc \sin A. \tag{1}$$

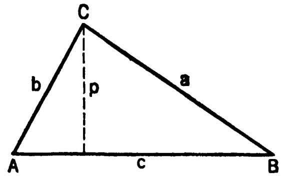


FIG. 64

Similarly,

$$K = \tfrac{1}{2}ac \sin B,$$

and

$$K = \tfrac{1}{2}ab \sin C.$$

The law just derived may be stated as follows: *The area of a triangle is equal to one half the product of any two sides and the sine of their included angle.*

CASE 2. Given the three sides a, b, c; find the area.

It has already been shown that

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and

$$\cos\frac{A}{2} = \sqrt{\frac{(s-a)s}{bc}}, \qquad \text{where } s = \frac{a+b+c}{2}.$$

Since

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2},$$

it follows that

$$\begin{aligned} K &= \tfrac{1}{2}bc\sin A \\ &= \frac{1}{2}bc\,(2)\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)s}{bc}} \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned} \tag{2}$$

It is recalled that r has already been defined by the relation

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence,

$$K = rs. \tag{3}$$

In the next section it is shown that r is actually the radius of the circle inscribed in the given triangle. Thus, Formula (3) may be stated as follows: *The area of a triangle is equal to the product of half the perimeter and the radius of the inscribed circle.*

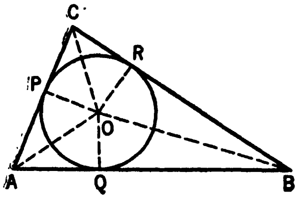


FIG. 65

45. THE RADIUS OF THE INSCRIBED CIRCLE

Let Figure 65 represent a triangle ABC with inscribed circle of radius r. From plane geometry it is known that the center of the inscribed circle is the intersection of the bisectors of the angles of the triangle. Let OQ, OP, and OR be radii of the circle that have been drawn to the points of tangency Q, P, and R, respectively. Then from plane geometry $OQ \perp AB$, $OP \perp AC$, and $OR \perp BC$.

It is apparent that the area K of triangle ABC is given by

$$\begin{aligned} K &= \tfrac{1}{2}\overline{OQ}\cdot\overline{AB} + \tfrac{1}{2}\overline{OP}\cdot\overline{AC} + \tfrac{1}{2}\overline{OR}\cdot\overline{BC} \\ &= \frac{r}{2}(a+b+c) = rs. \end{aligned}$$

By comparing this result with Formula (3) of the previous section, it is

observed that the radius of the inscribed circle is the same r that was defined by

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

In fact, it was for this very reason that the right member of the preceding formula was designated by r.

EXERCISES 24

Definitions: An angle in a vertical plane between a horizontal line and the line from the eye to some object is defined as the angle of elevation of the object if the object is above the eye, and as the angle of depression if the object is below the eye.

Solve each of the six triangles that follow; check your solution; and find the area.

1. Given $A = 82°17'23''$, $b = 384.23$, $c = 416.52$

2. Given $C = 97°28'45''$, $a = 36.244$, $b = 21.765$

3. Given $C = 50°20'38''$, $B = 42°54'7''$, $a = 1027.6$

4. Given $a = 0.1027$, $b = 0.1562$, $c = 0.1398$

5. Given $C = 62°15'35''$, $c = 816.51$, $a = 458.19$

6. Given $a = 3$, $b = 4$, $c = 6$

7. Find the radius of the circle inscribed in the triangle of Exercise 6.

8. Find the radius of the circle circumscribed about the triangle of Exercise 6.

9. In the triangle ABC, $a = 352.4$, $B = 36°17'$, and $C = 65°20'$. Find the radii of the inscribed and the circumscribed circles.

10. A triangular lot is 230 ft on one side; the angles of the lot at the extremities of this side are $38°27'$ and $47°42'$, respectively. Find the value of the lot at \$2 per sq ft.

11. The diagonals of a parallelogram are 13.5 ft and 20.4 ft, and one side is 12 ft. Find the angles of the parallelogram and its area.

12. The bases of a trapezoid are 58.25 and 94.75 ft. The angles at the ends of the longer base are $68°52'$ and $55°27'$. Find the lengths of the other two sides.

13. Two sides of a parallelogram are 180 and 255 ft, and the included angle is $40°17'$. Find the length of the diagonals and the area.

14. One diagonal of a parallelogram is 6291.3 ft, and the sides of the parallelogram make angles of $25°10'30''$ and $35°14'50''$ with the diagonal. Find the length of the sides of the parallelogram.

15. Observations to find the height of a mountain are taken at two points A and B on the same side of the mountain. The points are 3521.0 ft apart, at the same level, and in the same vertical plane with the top. The angle of elevation of the top at A is $54°50'35''$ and at B is $37°19'43''$. Find the height of the mountain.

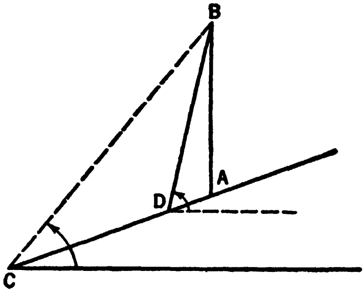


FIG. 66

16. Figure 66 represents a tower AB on the side of a hill CDA. At point C the angle of elevation of the top of the tower is $51°16'$. At point D, in the same vertical plane as C and the tower, the elevation of the top is $73°15'$. The hill from C through D to A is inclined $20°$ with the horizontal, and the distance $CD = 54$ ft. Find the height of the tower.

17. In order to find the distance AB across a river, certain measurements were made. The straight line AC along one bank was found to be 500 ft, and the angles BAC and BCA were found to be $88°33'0''$ and $73°48'30''$, respectively. Find the distance AB.

18. In a survey it is required to continue a straight line AB past an obstacle. A line BD, 100 yd long, is measured at right angles to AB. From D the line DP is established at an angle of $46°$ with the line BD. Find the length DP and angle DPQ so that the points P and Q will fall on the extension of AB, Q being the greater distance from B.

19. A surveyor measured a triangular piece of land which we shall designate as ABC. His notes gave $AB = 538$ ft, $BC = 237$ ft, and the angle $CAB = 31°27'$. Show that there must have been some error in the notes.

20. An engineer wanted to build a horizontal bridge across a valley from A to B, as shown in Figure 67. The bridge was to be supported by a pier at C. From A the angle of depression of C is $28°20'15''$, and from B the angle of depression of C is $47°11'45''$. The distance AB is 250 ft. Find the height of the pier.

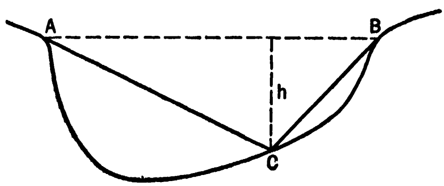


FIG. 67

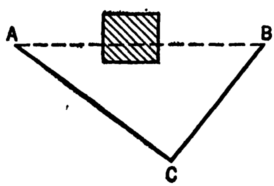


FIG. 68

21. To find the distance between two points A and B not visible from each other, a third point C is selected from which A and B are visible (note Figure 68). The distance $CA = 444.38$ ft, $CB = 322.76$ ft, and the angle $ACB = 87°17'36''$. Compute AB.

22. To find the distance from a point A to another point B as shown in Figure 69, point B being inaccessible and invisible from A, two points C and D are selected so that C, A, and D will be in the same straight line. A and B are both visible from C and from D. By measurement it is found that $CA = 456.72$ ft, $AD = 490.74$ ft, $\angle BCD = 71°22'35''$, and $\angle BDC = 36°19'24''$. Find AB.

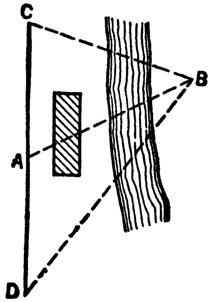


Fig. 69

23. To find the distance between two inaccessible points A and B (Figure 70), two points C and D are selected from which both A and B can be seen. The following measurements were made: $CD = 456.32$ ft, $\gamma = 35°16'24''$, $\alpha = 30°40'30''$, $\delta = 56°47'30''$, $\beta = 40°14'50''$. Compute AB.

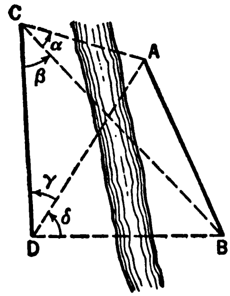


Fig. 70

24. In measuring the line from A to B (Figure 71) whose direction was known, it was necessary to pass an obstacle at F. A distance $CD = 144.31$ ft was measured, making an angle $\alpha = 39°35'24''$ with AB, and the angle $\beta = 102°10'20''$ was laid off. Compute DE, CE, and angle DEB in order that AC may be continued.

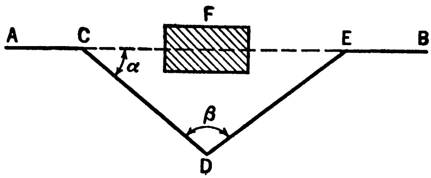


Fig. 71

25. In a survey of the field portrayed in Figure 72, a thick wood prevented the measurement of the angle ABD and of the distance BD. The angle $ABC = 70°14'30''$ was measured, a line BC was run 700 ft, the angle BCD was found to be $65°18'23''$, and the distance CD was found to be 925.2 ft. Compute $\angle ABD$ and the distance BD.

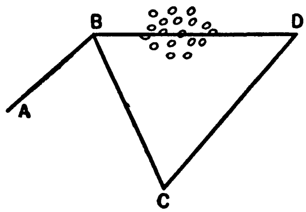


Fig. 72

26. From one corner of a triangular lot the other corners are found to be, respectively, 122 ft in a direction 78°45′ east of north and 157 ft in the direction 11°15′ west of south. Find the area of the lot.

27. From the top of a lighthouse 100 ft high, standing on a cliff, the angle of depression of a ship was 3°10′, and at the bottom of the lighthouse the angle of depression for the same ship was 2°20′. Find the horizontal distance to the ship and the height of the cliff.

28. A surveyor observed two inaccessible headlands, A and B. A was north 48°20′ west, and B was north 35°25′ east. He went 20 miles north, where he noted that the headlands were south 62°30′ west and south 11°15′ east, respectively. How far is A from B?

29. From an airplane 4 miles above the earth, the dip of the horizon is 2°33′. Compute the approximate radius of the earth and the distance from the airplane to the horizon.

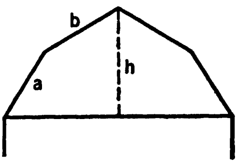


Fig. 73

30. A barn 50 ft wide has a gambrel roof. Note Figure 73. The lower rafter a makes an angle of 60° with the horizontal, and the upper rafter b makes an angle of 60° with the vertical. If the lower and upper rafters are equal in length, find their length and the height h of the ridge.

31. Find the number of square feet in a conical tent with a circular base if an element of the cone is inclined 50° with the horizontal, and the center pole is 14 ft high.

32. Given $M = \sin i/\sin i'$. Find i' when $M = 1\frac{1}{8}$ and $i = 23°15'$.

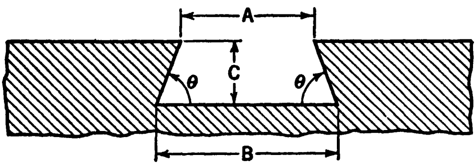


Fig. 74

33. Figure 74 represents a machine dovetail for guiding sliding parts. In order to make one of these dovetails, B, C, and the angle θ are given, and A must be computed. If $B = 8$ in., $C = 3$ in., and $\theta = 70°$, find the length of A.

34. An airplane is flying a straight, horizontal course at the rate of 280 mph. A person directly below the path of the plane observes it just after it has passed overhead. Its angle of elevation is 84°30′. Twenty seconds later its angle of elevation is 35°40′. At what height is it flying?

35. The pilot of an airplane flying over an island observes one extremity of the island, which is behind him, to have an angle of depression equal to 46°42′. The extremity of the island in front of him has an angle of depression equal to 62°37′. The plane's altimeter reads 8270 ft. How long is the island?

36. A ladder leaning against a building makes an angle of 47°30′ with the horizontal. When its foot is moved 18 ft nearer the building, the ladder makes an angle of 68°20′ with the horizontal. How much higher does it reach in the second position than it did in the first?

37. The area of a triangular lot is 7248 sq ft. One side of the lot is 123 ft, and an angle adjacent to that side is 74°18.6′. Find the remaining sides and angles.

38. In the measurement of a distance between two points with a 100-ft steel tape, one end was held 3 ft out of line. What error would this produce in the measurement per 100 ft?

39. The curb lines of two streets that cross would make an angle of 101°27′ with each other if they were extended to a point of intersection. A rounded curb line with a radius of 18 ft is built in at the corner. How far from the point of intersection will the curve start?

40. A tapering hole is to be drilled into a piece of steel 3 in. thick (note Figure 75). The small diameter of the hole must be 1 in. and the taper $\theta = 5°55'$. To test the size of the hole, a ball is frequently used, and the distance C is measured. If the diameter D of the ball is 1.3 in., find C.

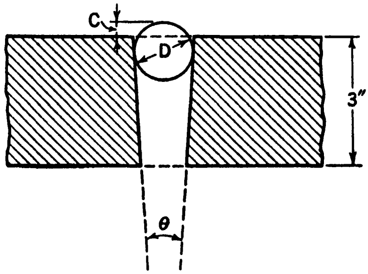


FIG. 75

4 Complex Numbers

46. COMPLEX NUMBERS

Complex numbers have already been considered briefly. Now, by means of the trigonometric functions and their properties, we are in a position to make a more thorough study of this important kind of number.

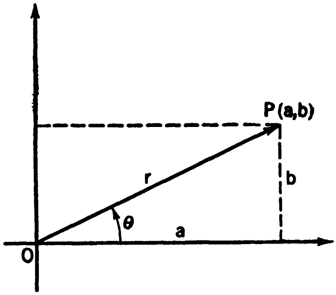


FIG. 76

Every real number corresponds to a unique point on a straight line; similarly, every complex number may be represented by a definite point in a plane. In Figure 76 the point P, designated by the coordinates (a, b), represents the complex number $a + bi$ in the plane. It is noted that the point $P(a, b)$ is located by means of the two real numbers a and b, where a is the abscissa and b is the ordinate relative to two axes. Since a denotes the real part of the given complex number, the horizontal axis is usually designated as the *axis of reals;* since b is the coefficient of the imaginary unit i, the vertical axis is known as the *axis of imaginaries.*

The distance $OP = r$ is called the *absolute value,* or *modulus,* of the complex number and is always considered positive. The angle $\theta = \tan^{-1} \dfrac{b}{a}$ is called the *amplitude,* or *argument,* of $a + bi$.

From a consideration of Figure 76, it is apparent that

$$a = r \cos \theta, \tag{1}$$

$$b = r \sin \theta, \tag{2}$$

$$r = \sqrt{a^2 + b^2}. \tag{3}$$

The complex number of which $a + ib$ is the *rectangular,* or *algebraic, form* may, by the use of relations (1) and (2), be expressed in the *polar,* or *trigonometric, form:*

$$r(\cos \theta + i \sin \theta). \tag{4}$$

47. CONJUGATE COMPLEX NUMBERS

The complex numbers

$$a + ib \quad \text{and} \quad a - ib$$

are called *conjugate complex numbers*. In their polar forms, these two conjugate complex numbers would be written

$$r(\cos\theta + i\sin\theta)$$

and

$$r(\cos\theta - i\sin\theta).$$

48. FUNDAMENTAL THEOREMS ON COMPLEX NUMBERS

The derivations of the theorems that follow are based on the definition that $i^2 = -1$ and on the assumption that the operations of addition, subtraction, multiplication, and division, as employed in the algebra of real numbers, are likewise applicable to complex numbers. Moreover, we shall assume the fundamental principle

If $\quad a + ib = 0, \quad$ then $\quad a = 0$ and $b = 0$.

The desirability of this latter assumption is seen from the fact that if a and b were not zero, we would have

$$a = -ib;$$

that is, a real number would equal an imaginary number, which is contrary to our purpose.

Theorem 1. If $a_1 + ib_1 = a_2 + ib_2$, then $a_1 = a_2$ and $b_1 = b_2$.

Proof: If

$$a_1 + ib_1 = a_2 + ib_2,$$

then

$$(a_1 - a_2) + i(b_1 - b_2) = 0.$$

Hence, by the fundamental principle stated above,

$$a_1 - a_2 = 0 \quad \text{or} \quad a_1 = a_2,$$

and

$$b_1 - b_2 = 0 \quad \text{or} \quad b_1 = b_2.$$

Theorem 2. The sum, difference, product, and quotient of two complex numbers is a complex number.

Proof: For the sum,

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2),$$

which is obviously another complex number.

For the difference,

$$(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2),$$

which is also a complex number.

For the product,

$$(a_1 + ib_1)(a_2 + ib_2) = [a_1a_2 + i(a_1b_2 + a_2b_1) + i^2b_1b_2].$$

Since $i^2 = -1$, this result may be written

$$(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1),$$

which is a complex number.

For the quotient, we consider

$$\frac{a_1 + ib_1}{a_2 + ib_2}.$$

After multiplying the numerator and the denominator by the conjugate of the denominator, we have

$$\frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}.$$

$$= \frac{[a_1a_2 + i(b_1a_2 - a_1b_2) - i^2b_1b_2]}{a_2^2 - i^2b_2^2}.$$

Again, since $i^2 = -1$, we have

$$\frac{a_1 + ib_1}{a_2 + ib_2} = \left(\frac{a_1a_2 + b_1b_2}{a_2 + b_2^2}\right) + i\left(\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}\right),$$

which is a complex number.

EXERCISES 25

1. Write each of the following complex numbers in polar form. Restrict each angle to less than 360°.

(a) $1 + \sqrt{3}i$ (b) $\sqrt{3} - i$
(c) $-2 - 2i$ (d) $2i$
(e) 3 (f) $-i$
(g) $3 - 5i$ (h) $1 - i$

2. Change each of the following complex numbers to the equivalent rectangular form:

(a) $5(\cos 60° + i \sin 60°)$ (b) $4(\cos 45° - i \sin 45°)$
(c) $2(\cos 90° + i \sin 90°)$ (d) $(\cos 42°17' + i \sin 42°17')$
(e) $4(\cos 225° + i \sin 225°)$ (f) $5(\cos 90° + i \sin 90°)$

3. (a) Find the algebraic sum of $2 - 3i$ and $1 + 4i$.
(b) Subtract $1 + 4i$ from $2 - 3i$.
(c) Find the product of $2 - 3i$ and $1 + 4i$, and express the result in the form $a + bi$.
(d) Divide $2 - 3i$ by $1 + 4i$, and express the quotient in the form $a + bi$.

4. Find the sum and product of $a + bi$ and its conjugate.

5. Prove that if the number i multiplies a complex number $a + bi$, it rotates the line joining the point $a + bi$ to the origin through an angle of 90°, but does not alter the absolute value of the complex number.

49. PRODUCTS, QUOTIENTS, POWERS, ROOTS

We have just found that the product or the quotient of two complex numbers is itself a complex number. The actual process of multiplication and division of complex numbers is considerably simplified if the numbers are written in polar form. This will be seen from the discussion that follows.

Product of Complex Numbers. The product of the two numbers

$$\alpha_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and

$$\alpha_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

is

$$\alpha_1\alpha_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)],$$

or

$$\alpha_1\alpha_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \tag{1}$$

From relation (1) we note that the absolute value of the product of two complex numbers is the product of their absolute values, and the argument of the product is the sum of their arguments.

Quotient of Complex Numbers. We shall now find the quotient of the same two complex numbers.

We have

$$\frac{\alpha_1}{\alpha_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}.$$

After multiplying the numerator and the denominator of the right member by $(\cos\theta_2 - i\sin\theta_2)$, the conjugate of $\cos\theta_2 + i\sin\theta_2$, we have

$$\begin{aligned}\frac{\alpha_1}{\alpha_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos^2\theta_2 - i^2\sin^2\theta_2)}\\ &= \frac{r_1}{r_2}\frac{[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{\cos^2\theta_2 + \sin^2\theta_2}\\ &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \end{aligned} \tag{2}$$

From relation (2) we note that the absolute value of the quotient of two complex numbers is equal to the absolute value of the numerator divided by the absolute value of the denominator, and the argument of the quotient is equal to the argument of the numerator minus the argument of the denominator.

50. DE MOIVRE'S THEOREM

This important theorem is as follows:

The absolute value of the n*th power of a number is equal to the* n*th power of its absolute value, and the argument of the* n*th power of a number is equal to* n *times the argument of the number.*

In symbolic form, the theorem is

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta),$$

where n is a positive integer.

We shall prove this theorem, when n is a positive integer, by induction (Book I, Chapter XV).

When $n = 1$, the theorem is obviously true. When $n = 2$, by employing relation (1) in the previous section, we have

$$\begin{aligned}[r(\cos\theta + i\sin\theta)]^2 &= r^2[\cos(\theta+\theta) + i\sin(\theta+\theta)]\\ &= r^2(\cos 2\theta + i\sin 2\theta).\end{aligned}$$

Hence, the theorem is true when $n = 2$.

We now assume that

$$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta),$$

where k is an arbitrary positive integer. After multiplying both members by $r(\cos\theta + i\sin\theta)$, we have as a result

$$r^{k+1}[\cos(k\theta+\theta) + i\sin(k\theta+\theta)]$$

or

$$r^{k+1}(\cos\overline{k+1}\,\theta + i\sin\overline{k+1}\,\theta).$$

Since the law has been verified for $k = 2$, the above demonstration shows that it is true for $k = 3$, and by continuing the application of this reasoning, it is true for any positive integer.

It may be shown that De Moivre's theorem holds also for any real value of n. We shall assume this generalization without proof.

De Moivre's theorem has many important applications. As an illustration of its use, we shall consider the determination of the roots of a complex number; this includes any real number.

51. ROOTS OF A COMPLEX NUMBER

To find the nth roots of a *real* number a, we solve the equation $x^n = a$ for x. Similarly, to find the nth roots of a complex number α, we solve the equation $z^n = \alpha$ for z; that is, we are to determine

$$z = \sqrt[n]{\alpha}. \tag{1}$$

Let

$$z = r_1(\cos\phi + i\sin\phi), \tag{2}$$

and let

$$\alpha = r_2(\cos\theta + i\sin\theta). \tag{3}$$

Then, by De Moivre's theorem,

$$z^n = r_1^n(\cos n\phi + i\sin n\phi), \tag{4}$$

and hence, since $z^n = \alpha$, it follows that

$$r_1^n(\cos n\phi + i\sin n\phi) = r_2(\cos\theta + i\sin\theta). \tag{5}$$

To satisfy equality (5),

$$r_1^n = r_2 \quad \text{or} \quad r_1 = \sqrt[n]{r_2},$$

and

$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n},$$

where k is zero or any real integer.

Hence, substituting $\sqrt[n]{r_2}$ for r_1 and $\dfrac{\theta + 2k\pi}{n}$ for ϕ, relation (2) becomes

$$z = \sqrt[n]{r_2}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right). \tag{6}$$

If we use the values $k = 0, 1, 2, 3, \cdots, (n - 1)$, we shall obtain all the n nth roots of α. If further values of k are used, the values for z are repeated.

The geometrical arrangement of these roots in the plane is interesting. Since the absolute values of all the roots are equal, they will lie on a circle of radius $\sqrt[n]{r_2}$. The n values will be equally spaced around this circle, the first or principal root z_0 being on the line $\phi = \dfrac{\theta}{n}\cdot$

To illustrate the use of result (6), we shall find the three cube roots of $8(\cos 30° + i\sin 30°)$. In this case $n = 3$.

If $k = 0$,

$$z_0 = \sqrt[3]{8}\left(\cos\frac{30° + 0°}{3} + i\sin\frac{30° + 0°}{3}\right) = 2(\cos 10° + i\sin 10°).$$

If $k = 1$,

$$z_1 = \sqrt[3]{8}\left(\cos\frac{30° + 360°}{3} + i\sin\frac{30° + 360°}{3}\right) = 2(\cos 130° + i\sin 130°).$$

If $k = 2$,

$$z_2 = \sqrt[3]{8}\left(\cos\frac{30° + 720°}{3} + i\sin\frac{30° + 720°}{3}\right) = 2(\cos 250° + i\sin 250°).$$

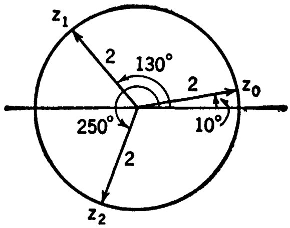


FIG. 77

The three cube roots z_0, z_1, and z_2 are displayed in Figure 77.

EXERCISES 26

1. Find the product of $2(\cos 20° + i \sin 20°)$ and $3(\cos 40° + i \sin 40°)$.
2. Find the product of $3(\cos 50° + i \sin 50°)$ and $5(\cos 70° + i \sin 70°)$.
3. Show that $\dfrac{1}{r(\cos\theta + i\sin\theta)} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$.
4. Divide $6(\cos 120° + i \sin 120°)$ by $3(\cos 30° + i \sin 30°)$.
5. Express the quotient $\dfrac{3(\cos 120° + i \sin 120°)}{5(\cos 30° + i \sin 30°)}$ in the form $a + ib$.
6. Find $[5(\cos 45° + i \sin 45°)]^2$. Write the result in the form $a + ib$.
7. Find $[2(\cos 20° + i \sin 20°)]^3$. Write the result in the form $a + ib$.
8. Find the value of

$$\frac{[3(\cos 30° + i \sin 30°)][5(\cos 60° + i \sin 60°)]}{6(\cos 120° + i \sin 120°)}.$$

9. Change each complex number to polar form and find the product of $1 - \sqrt{3}i$ and $1 + i$.

10. Change each complex number to polar form and divide $3 - 5i$ by $2 - i$. Check your result by finding the quotient of the numbers in the form as given and then changing the result to polar form.

11. (*a*) Find the two square roots of 1.

SUGGESTION: First change 1 to its equivalent polar form, $1(\cos 0° + i \sin 0°)$.

(*b*) Determine the three cube roots of 1.
(*c*) Find the four fourth roots of 1.

12. Write i in the polar form. Show that $i^4 = 1$ by using the polar form of i.

52. GRAPHICAL REPRESENTATION OF $z = z_1 + z_2$, $z = z_1 - z_2$, $z = z_1z_2$, and $z = z_1/z_2$

In this discussion, the letter z is used to denote a complex number; that is common in advanced mathematics. The rectangular forms for the complex numbers z_1 and z_2 are well adapted to the study of the graphical representation of $z = z_1 + z_2$ and $z = z_1 - z_2$, while the polar forms for the complex numbers are better adapted to the consideration of the graphical representation of $z = z_1z_2$ and $z = \dfrac{z_1}{z_2}$.

The point that corresponds to the complex number z, where $z = z_1 + z_2$, and where

$$z_1 = a_1 + ib_1 \quad \text{and} \quad z_2 = a_2 + ib_2,$$

is P, as located in Figure 78. This fact is easily established after the triangles OAP_2 and P_1BP are proved congruent. It is then readily shown that the line OP is the diagonal of the parallelogram determined by drawing P_1P both equal and parallel to OP_2.

To locate a point P corresponding to $z = z_1 - z_2$, Figure 79 is appropriate. The line OP is the diagonal of the parallelogram determined by

drawing P_1P equal and parallel to OP_2', where OP_2' is equal but having a direction opposite to that of OP_2.

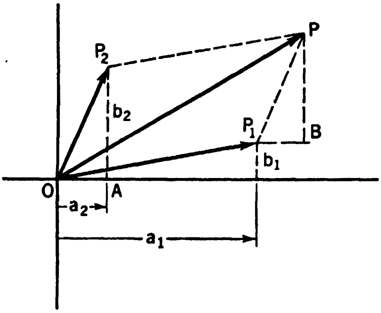


FIG. 78

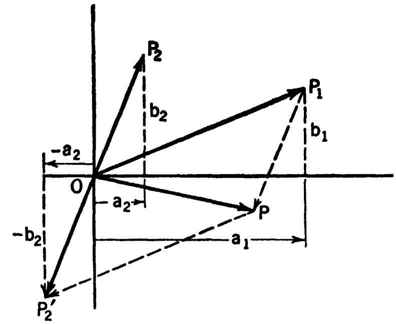


FIG. 79

To locate the point corresponding to z, where $z = z_1z_2$ and where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we may refer to

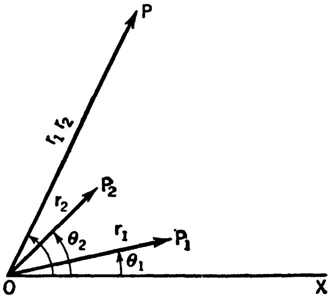


FIG. 80

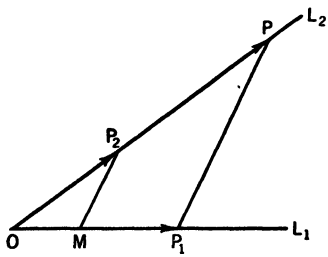


FIG. 81

Figure 80. The desired point P is the representation of $r(\cos\theta + i\sin\theta)$, where $r = r_1r_2$ and $\theta = \theta_1 + \theta_2$.

The magnitude $OP = r_1r_2$ may be constructed graphically as follows: Draw any two intersecting lines L_1 and L_2, as in Figure 81. On line L_1 let $r_1 = OP_1$ and $OM = 1$ unit. On L_2 let $r_2 = OP_2$. Draw MP_2, and construct $P_1P \parallel MP_2$. Then, $OP = r_1r_2$. The proof is left as an exercise for the student.

To locate the point corresponding to z, where $z = \dfrac{z_1}{z_2}$ and where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we may refer to Figure 82. The desired point P is the representation of the quotient $r(\cos\theta + i\sin\theta)$, where $r = \dfrac{r_1}{r_2}$ and $\theta = \theta_1 - \theta_2$.

The magnitude $OP = \dfrac{r_1}{r_2}$ may be constructed graphically as follows: Draw any two intersecting lines L_1 and L_2, as in Figure 83. On L_2 let $r_2 = OP_2$ and $OM = 1$. On L_1 let $r_1 = OP_1$. Draw P_1P_2, and construct $MP \parallel P_1P_2$. Then $OP = \dfrac{r_1}{r_2}$. The proof is left as an exercise for the student.

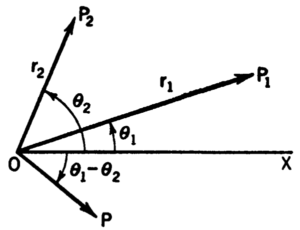


FIG. 82

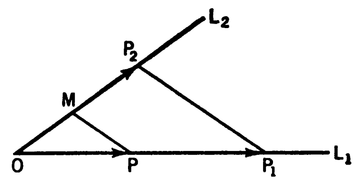


FIG. 83

It may be noted that if k is a positive real number other than 1 and $z = r(\cos\theta + i\sin\theta)$, then $kz = kr(\cos\theta + i\sin\theta)$. Hence the positive real number k, as a multiplier of the complex number $r(\cos\theta + i\sin\theta)$, changes the absolute value r to kr, leaving the argument θ unchanged.

If k is a negative real number other than -1, then k as a multiplier of $r(\cos\theta + i\sin\theta)$ changes the absolute value of r to $|k|r$ and rotates through 180° the line joining the origin to the point representing the given complex number. This fact is readily seen to be true when it is realized that the negative number k may be written in the polar form $|k|(\cos 180° + i\sin 180°)$. Of course, the argument may be $-180°$ as well as 180°.

It may also be noted that if $z = r(\cos\theta + i\sin\theta)$, and since $i = 1(\cos 90° + i\sin 90°)$ and $-i = 1[\cos(-90°) + i\sin(-90°)]$, then $iz = r[\cos(\theta+90°) + i\sin(\theta+90°)]$ and $-iz = r[\cos(\theta-90°) + i\sin(\theta-90°)]$. Thus, the imaginary number i as a multiplier leaves the absolute value of a complex number unchanged, but causes rotation through 90°, while the imaginary number $-i$ as a multiplier leaves the absolute value unchanged and causes a rotation through $-90°$.

EXERCISES 27

1. By algebraic addition find the sum of $3 + 2i$ and $1 - 3i$. Graph the two numbers and their sum.

2. Find $z_1 - z_2$, where $z_1 = 3 + 2i$ and $z_2 = 1 - 3i$. Graph the two numbers and the difference.

3. Find algebraically the product $(3 - 2i)(1 - 3i)$. Construct the diagram for obtaining their product.

4. Find the reciprocal of $(1 - 3i)$, and express the result in the form $a + ib$ by rationalizing the denominator.

5. Find the quotient $(3 + 2i) \div (1 - 3i)$, and express the result in the form $a + ib$. Draw the diagram for obtaining the quotient.

6. Locate the points representing the complex numbers $z_1 = 5$, $z_2 = -4 - i$, and their sum z. Find the absolute values of the three numbers, and note that the absolute value of a sum of two complex numbers may be less than the absolute value of either one.

7. Solve the quadratic equation $z^2 - z + 4 = 0$. Using one of the roots, locate the points z^2, $-z$, and $+4$, and find the sum graphically of these three complex numbers. Do the same for the other root.

8. Solve the quadratic equation $z^2 + bz + c = 0$ for z. What is the absolute value of z^2?

9. Find the product of the roots of the equation in Exercise 7, and locate the point representing the product. Note that it coincides with the point representing the coefficient of the last term. Do the same for the equation of Exercise 8.

10. Plot the following complex numbers: $3(\cos 60° + i \sin 60°)$, $2(\cos 90° + i \sin 90°)$, $4(\cos 30° + i \sin 30°)$, $5(\cos 120° + i \sin 120°)$.

(a) Find their sum graphically.

(b) Find their product graphically.

(c) Subtract the sum of the first two from the sum of the last two and locate the point representing the difference.

(d) Divide the product of the first two by the product of the last two.

(e) Divide the sum of the first two by the sum of the last two.

11. Write the numbers $-2 + 3i$ and $3 - 4i$ in the polar form, and write their product; also locate the points representing the numbers, and find their product graphically.

12. Find the product of $1.336 - 2.550i$ and $2.774 + 0.550i$, and express the result in the polar form.

13. Find the quotient $(-2 + 3i) \div (3 - 4i)$ graphically.

14. Find the quotient $100(\cos 90° + i \sin 90°) \div 5(\cos 30° + i \sin 30°)$. Write the result in the form $a + ib$.

15. Express the reciprocal of each of the following numbers in the same form in which it is given: $5(\cos 30° + i \sin 30°)$, $3 + 4i$, $10(\cos 45° - i \sin 45°)$.

16. Express the square of each of the complex numbers of Exercise 15 in the same form in which it is given.

17. Express the square root of each of the complex numbers of Exercise 15 in the same form in which it is given.

18. Express the product of the three complex numbers of Exercise 15 in the polar form.

19. Find all the square roots of -4. Show them graphically.

20. Find all the square roots of $3 - 4i$. Show them graphically.

21. Find the two square roots of $1.336 - 2.550i$.

22. Find the three cube roots of 8. Show them graphically.

23. Find the three cube roots of -8. Show them graphically.

24. Find all the cube roots of $1.336 - 2.550i$.

25. Find the cube roots of $-4 + 3i$.

26. Given $x^5 + 1 = 0$; find all the roots.

27. Given $x^5 - 32 = 0$; find all the roots.

28. Given $x^6 - 27 = 0$; find all the roots.

29. In an electric circuit, two a-c impedances may be represented by the

complex quantities $Z_1 = R_1 + iX_1$ and $Z_2 = R_2 + iX_2$. The combined impedance of the two in series is the sum $Z = Z_1 + Z_2$. Draw the diagram showing Z_1, Z_2, and Z.

30. Two impedances $Z_1 = 3 + 5i$ ohms and $Z_2 = 2 - 3i$ ohms are combined in series. Find the impedance Z of the combination (note Exercise 29).

31. Express the impedances Z_1, Z_2, and Z of Exercise 30 in polar form.

32. Two impedances $Z_1 = R_1 + iX_1$ and $Z_2 = R_2 + iX_2$, when connected in parallel, give a joint impedance of $Z = Z_1Z_2/(Z_1 + Z_2)$, where the products and sum are taken in the complex sense. Draw a diagram displaying Z_1, Z_2, and Z, when $R_1 = 1$, $X_1 = 2$, $R_2 = 3$, $X_2 = 4$.

33. Find the joint impedance of the combination $3 + 2i$ ohms and $1 - 3i$ ohms connected in parallel (see Exercise 32).

34. Find the joint impedance of $3 + 5i$ ohms and $2 - 3i$ ohms when connected in series. Find the joint impedance if these impedances are connected in parallel (see Exercise 29 and Exercise 32).

35. The two complex numbers of Exercise 12 represent two impedances. Find their joint impedance when connected in series. Find their joint impedance when connected in parallel. Express the results in the polar form.

36. An alternator producing an emf of $E = 100$ vector volts has in its circuit an impedance of $3 + 4i$ vector ohms. How much current (I vector amperes) will flow? ($I = E/Z$.) Express in the polar form.

REVIEW EXERCISES 28

1. Draw diagrams and write the six trigonometric functions of 150°, 225°, 330°, $+\dfrac{7\pi}{4}$, $-\dfrac{2\pi}{3}$.

2. Given $\cos x = \frac{8}{17}$; construct all possible values of x less than 360°, and find the other functions of x.

3. Given $\tan x = -\frac{3}{4}$ and $\cos x$ positive; construct x and find the other functions.

4. Given a circle with radius 6 ft; find the length of the arc of the circle and the chord intercepted by the sides of a central angle of 105°32′.

5. Find the side of a regular octagon inscribed in a circle of radius 6 ft.

6. Find the area of the segment bounded by the arc and chord in Exercise 4.

7. From the top of a lighthouse 150 ft above sea level, the angle of depression of a buoy was 12°10′ and that of the distant shore measured in the same vertical plane with the buoy was 62°14′. Find the distance of the buoy from the shore in feet.

8. Find the radius and the length of an arc of 1° of a parallel of latitude at a place whose latitude is 43°20′, the earth being regarded as a sphere whose radius is 3963 miles.

9. Write the functions of the following angles in terms of some positive acute angle: 101°16′, 194°7′, 265°5′, 328°16′, −27°10′, −137°21′.

10. A pendulum 12 in. long is displaced through an angle of 43°15′ with the vertical. In the vertical position, the pendulum bob is 42 in. from the floor. How high is it from the floor at the point of maximum displacement?

11. Transform the first member into the second in each of the following:

(*a*) $\sec x \csc x(\cos^2 x - \sin^2 x) = \cot x - \tan x$.

(b) $\sin^2 x(\tan^2 x - 1) + \cos^2 x(\cot^2 x - 1) = \dfrac{(1 - 2\cos^2 x)^2 \sec^4 x}{\tan^2 x}$.

(c) $\tan^2 x - \sin^2 x \cos^2 x = \dfrac{(\sec^2 x + 1)(\sec^2 x - 1)}{\sec^4 x}$.

(d) $\dfrac{\cos x \cot x - \sin x \tan x}{\csc x - \sec x} = 1 + \sin x \cos x$.

(e) $\dfrac{\sec x + \csc x}{\sec x - \csc x} = \dfrac{\tan x + 1}{\tan x - 1} = \dfrac{1 + \cot x}{1 - \cot x}$.

12. Find the value of $\dfrac{\sec x + \tan x}{\csc x - \cos x}$ when $\cot x = -\frac{1}{2}$, if x is in the second quadrant.

13. Solve the following equations for all values of θ less than 360°:

(a) $2 \sin\theta \tan\theta + 2\sin\theta - \tan\theta - 1 = 0$.

(b) $16\cos^2\theta + 8\sin\theta - 13 = 0$.

14. Determine the angle between the diagonal of a cube and an adjacent edge.

15. If $\sin A = \frac{5}{13}$ and $\cos B = \frac{8}{17}$, where $A < 90°$ and $B < 90°$, find $\sin(A + B)$, $\cos(A - B)$, $\cos 2A$, and $\sin\dfrac{A}{2}$.

16. By inspection find one value of x satisfying each equation that follows:

(a) $\sin(n - 1)A \cos A + \cos(n - 1)A \sin A = \sin x$.

(b) $\cos 45° \cos(90° - \theta) - \sin 45° \sin(90° - \theta) = \cos x$.

17. Transform the first member into the second:

(a) $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$.

(b) $\cos^2 x + \cos^2 y - 2\cos x \cos y \cos w = \sin^2 w$, when $w = x + y$.

(c) $\tan\theta + \dfrac{\tan\phi \sec\theta}{\cos\theta - \tan\phi \sin\theta} = \tan(\theta + \phi)$.

(d) $\dfrac{1 - \tan x}{1 + \tan x} = \tan(45° - x)$.

(e) Express $\cos^4\theta$ in terms of cosines of multiple angles but with no power higher than the first.

18. If $y = \tan^{-1} m + \tan^{-1} n$, find $\tan y$ in terms of m and n.

19. If $y = \sin^{-1}\frac{1}{2} + \tan^{-1}\frac{3}{4}$, find $\tan y$.

20. If $m = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$, find m in degrees.

21. Show that $\tan^{-1} m = \frac{1}{2}\tan^{-1}\dfrac{2m}{1 - m^2}$.

22. Show that $\cos^{-1} m = \frac{1}{2}\cos^{-1}(2m^2 - 1)$.

23. Find all values of θ less than 360° satisfying each of the following:

(a) $\cot 2\theta + \tan\theta = -\frac{2}{3}\sqrt{3}$.

(b) $\sin 4\theta - 2\sin 2\theta = 0$.

24. Given

$$x = a\cos\theta,$$

$$y = b\sin\theta;$$

eliminate θ.

25. Given

$$x = \cos\theta,$$
$$y = \sin 2\theta;$$

eliminate θ.

26. Given

$$x = a(2\cos t - \cos 2t),$$
$$y = a(2\sin t - \sin 2t);$$

eliminate t.

27. Draw the graph of each of the following:

(*a*) $y = \sin x + \sin 2x + \sin 3x$.

(*b*) $y = 2\sin x + \sin 2x$.

(*c*) $y = \sin x + \sin\left(2x + \frac{\pi}{3}\right)$.

(*d*) $y = \sin^{-1} 3x$.

(*e*) $y = 2\sin^{-1} 2x - \frac{\pi}{4}$.

(*f*) $y = \cos^{-1}(3x - 2)$.

(*g*) $y = \frac{1}{2}\cos^{-1}\left(x - \frac{\pi}{6}\right)$.

28. Find the sum, the difference, the product, and the quotient of $3 - 5i$ and $-4 + i$.

29. Express the number $8 - 15i$ in polar form, and find its cube roots.

30. Solve the equation $x^6 + 1 = 0$ for all six values of x.

Book III : ANALYTIC GEOMETRY

1 Points; Line Segments

1. ANALYTIC GEOMETRY

In general, there are two broad approaches to the study of geometry, namely, with or without the use of an axis system. The geometry of high school, which is in the Euclidean tradition, did not use an axis system; geometry considered in such a manner is described as *synthetic*. In this book, our study of geometry will be facilitated by using an axis system; such an approach is said to be *analytic*.

2. PROJECTION OF A LINE SEGMENT

The projection of the directed line segment AB upon a line L, by definition, is the line segment of L from the foot C of the perpendicular from A to L to the foot D of the perpendicular from B to L (see Figure 1). Thus, CD is the projection of AB upon L, whereas DC is said to be the projection of BA upon L.

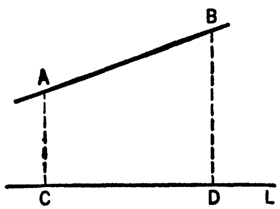


FIG. 1

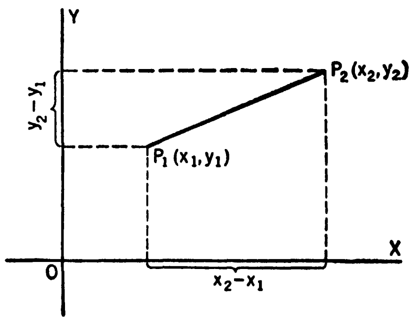


FIG. 2

In Figure 2, where P_1 has the coordinates (x_1, y_1) and P_2 has the coordinates (x_2, y_2), the projections of P_1P_2 upon the x axis and y axis, respectively, are $x_2 - x_1$ and $y_2 - y_1$, respectively.

3. LENGTH OF A LINE SEGMENT

If in Figure 2 we designate the distance between P_1 and P_2 by d, and note that we are now speaking of the magnitude of the segment P_1P_2

irrespective of direction, we have from elementary geometry

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is known as the *distance formula* for plane analytic geometry, and it has many applications in the work that follows. Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, it is apparent that the order in which the two points are chosen is immaterial insofar as the application of the distance formula is concerned.

If $y_1 = y_2$, then $d = |x_2 - x_1|$, where the bars denote the absolute value; that is, if $x_2 - x_1$ is negative, d must be taken equal to the positive value $x_1 - x_2$. Similarly, if $x_1 = x_2$, $d = |y_2 - y_1|$.

Illustration: Show that the points $P_1(1, -2)$, $P_2(4, 2)$, and $P_3(-3, -5)$ are the vertices of an isosceles triangle.

From the distance formula,

$$P_1P_2 = \sqrt{(4 - 1)^2 + (2 + 2)^2} = 5$$

and

$$P_1P_3 = \sqrt{(-3 - 1)^2 + (-5 + 2)^2} = 5.$$

Since two sides of the triangle are equal, the triangle is isosceles.

EXERCISES 1

1. Take a point $A(x_1, y_1)$ in the second quadrant and the point $B(x_2, y_2)$ in the third quadrant. Draw the figure and derive the formula for the length of AB.

2. Find the length of the line segment joining $(1, -6)$ and $(-4, -3)$.

3. Find the length of the sides, the altitude upon AB, and the area of the triangle having the vertices $A(1, 0)$, $B(10, 0)$, $C(3, 9)$.

4. Show that the triangle of vertices $A(10, 2)$, $B(20, 6)$, $C(6, 12)$ is isosceles.

5. Show that the triangle whose vertices are the points $(-1, -6)$, $(7, 0)$, and $(1, 8)$ is an isosceles triangle.

6. Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are the vertices of a parallelogram.

7. Prove analytically that the diagonals of a rectangle are equal.

Suggestion: Any rectangle may be located on an axis system so that it has the coordinates $(a, 0)$, (a, b), $(0, b)$ and $(0, 0)$.

8. Write an equation expressing the fact that the point (x, y) is 5 units from the point $(2, -3)$.

9. Write an equation expressing the fact that the point (x, y) is equidistant from the points $(2, 3)$ and $(-1, 5)$.

10. Find the area of the quadrilateral formed by connecting the points $(3, 4)$, $(-2, 6)$, $(-4, -5)$, $(4, -9)$, and $(3, 4)$ in the order given.

Hint: Draw lines through the vertices parallel to the axes, thereby forming parallelograms and right triangles: then make use of projections to find areas.

4. COORDINATES OF A POINT WHICH DIVIDES A LINE SEGMENT IN A GIVEN RATIO

If P is a point which divides the line segment P_1P_2 in a given ratio r, we mean that $P_1P/P_1P_2 = r$ (note Figures 3, 4, and 5). In view of our

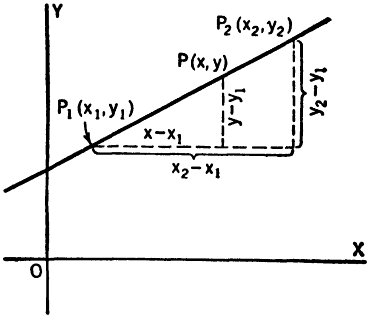


FIG. 3

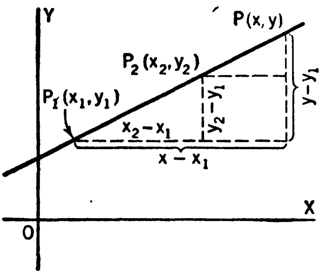


FIG. 4

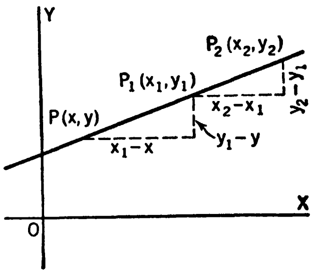


FIG. 5

definition that $r = P_1P/P_1P_2$, it follows that r is positive in Figures 3 and 4, since P_1P and P_1P_2 have the same direction; whereas r is negative in Figure 5, since in this figure P_1P and P_1P_2 have opposite directions.

In each of the three figures,

$$\frac{x - x_1}{x_2 - x_1} = r \qquad \text{or} \qquad x = x_1 + r(x_2 - x_1),$$

and

$$\frac{y - y_1}{y_2 - y_1} = r \qquad \text{or} \qquad y = y_1 + r(y_2 - y_1).$$

In the particular case where P is the mid-point of P_1P_2, $r = \frac{1}{2}$. Hence,

$$x = x_1 + \tfrac{1}{2}(x_2 - x_1) = \frac{x_1 + x_2}{2},$$

and

$$y = y_1 + \tfrac{1}{2}(y_2 - y_1) = \frac{y_1 + y_2}{2}.$$

EXERCISES 2

1. Find the coordinates of the point that bisects the line segment joining the point $(-2, -7)$ to $(3, 4)$.

2. How far is it from the origin to the mid-point of the segment from $(2, 3)$ to $(6, 9)$?

3. Find the coordinates of the two points which trisect the segment joining $(1, -6)$ and $(-4, -3)$.

4. Given the points $A(-3, 5)$ and $B(6, -2)$:

(*a*) Find the coordinates of the mid-point of AB.

(*b*) Find the coordinates of the points that trisect AB.

(*c*) Find the coordinates of the points that divide AB into four equal parts.

(*d*) Find the coordinates of the point that divides AB in the ratio $-\frac{2}{3}$.

5. Given a triangle with vertices at the points $A(-2, -10)$, $B(3, 5)$, and $C(1, -8)$:

(*a*) Find the length of each side.

(*b*) Find the length of each median.

(*c*) Find the length of the line joining the mid-points of sides AB and BC.

(*d*) Find the coordinates of the points that are two thirds of the distance from each vertex to the mid-point of the opposite side of the triangle.

6. Prove that the quadrilateral whose vertices are $(6, 3)$, $(16, -3)$, $(-9, -12)$, and $(-19, -6)$ is a parallelogram, and that the quadrilateral formed by joining the mid-points of the sides is also a parallelogram.

7. Prove analytically that the mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

HINT: The triangle may be placed in some convenient position with respect to the axis system. For instance, the vertices might be chosen as $(0, a)$, $(0, 0)$, and $(b, 0)$.

8. Show analytically that the line joining the mid-points of two sides of a triangle is equal to one half the third side.

9. Prove analytically that the figure formed by joining the mid-points of any quadrilateral is a parallelogram.

10. Determine the area of the isosceles triangle having the vertices $(10, 2)$, $(20, 6)$, and $(6, 12)$.

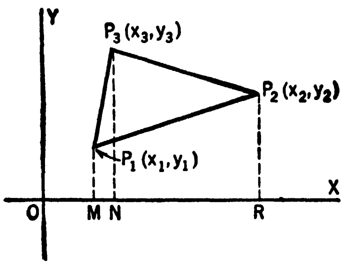


FIG. 6

5. AREA OF A TRIANGLE IN TERMS OF THE COORDINATES OF ITS VERTICES

Let P_1, P_2, P_3 be the vertices of a triangle such as the one shown in Figure 6.

By reference to the figure, we observe that

$$\begin{aligned}\text{Area of triangle } P_1P_2P_3 &= \text{area of trapezoid } MNP_3P_1\\ &\quad + \text{area of trapezoid } NRP_2P_3\\ &\quad - \text{area of trapezoid } MRP_2P_1\\ &= \frac{(y_1+y_3)(x_3-x_1)}{2} + \frac{(y_2+y_3)(x_2-x_3)}{2} - \frac{(y_2+y_1)(x_2-x_1)}{2}\\ &= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3).\end{aligned}$$

This result can be written in the form of a determinant as follows:

$$\text{Area of triangle } P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

It is readily confirmed that this result is entirely general, irrespective of the location of the points in the various quadrants.

EXERCISES 3

1. Find the area of the triangle whose vertices are the points (2, 3), (−1, 4) and (2, −5).

2. By using the formula for the area of a triangle, show that the points (2, 4), (0, −5), and (−2, −14) are on a straight line.

3. Show that the area of the quadrilateral $P_1P_4P_3P_2$ in Figure 7 is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

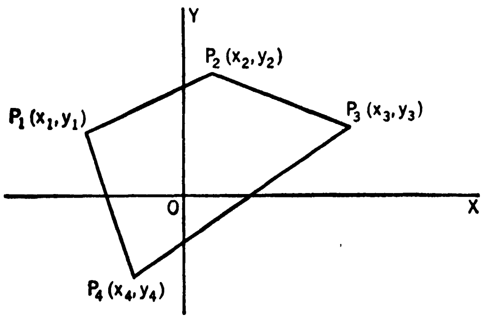


FIG. 7

4. Find the area of the quadrilateral whose vertices are (1, 0), (5, 7), (−2, 3), and (−1, −4).

5. Find the altitude upon AB of the triangle whose vertices are the points $A(1, 2)$, $B(9, -4)$, and $C(4, 7)$.

6. Show analytically that a line connecting the mid-points of two sides of a triangle forms with those sides a new triangle whose area is one fourth the area of the given triangle.

7. (*a*) Show that the quadrilateral whose vertices are $A(-6, 1)$, $B(4, -3)$, $C(9, 1)$, and $D(-1, 5)$ is a parallelogram.
(*b*) Find the area of the quadrilateral.
(*c*) Find the length of each diagonal.
(*d*) Find the length of AB.
(*e*) Find the length of the altitude from D to the side AB.

8. A triangle has vertices at the points (a, b), (c, d), and $(0, 0)$. Show that its area is

$$\frac{1}{2}\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6. POLAR COORDINATES

Instead of locating a point by means of its distances from two fixed lines, as in Cartesian coordinates, we may locate a point by a method illustrated in Figure 8. Point P in this figure is determined by means of two coordinates, which are the measures, respectively, of the distance $OP = r$ and the angle θ. The line OX is called the *initial line* and the point O is called the *pole*. The coordinates of P are given as (r, θ). The measure of angle θ may be given in degrees or in radians.

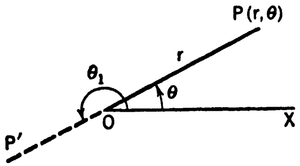


FIG. 8

The distance r is positive in the direction of the terminal line of the angle under consideration. Thus, relative to the angle θ, OP is positive and OP' is negative. Similarly, relative to the angle θ_1, OP' is positive and OP is negative.

When using polar coordinates, the same point P may be designated by pairs of coordinates in many ways. Thus, the same point P may be given by $(2, 30°)$, $(-2, -150°)$, $(-2, 210°)$, $(2, -330°)$, and so on. In spite of this fact, the system of polar coordinates is much more serviceable in certain kinds of problems than the system of rectangular coordinates.

7. RELATION BETWEEN RECTANGULAR AND POLAR COORDINATES

It is possible to establish a relation between polar and rectangular coordinates. Thus, if (x, y) are the rectangular coordinates of P, and (r, θ) are the polar coordinates of the same point, we have, by reference to Figure 9, $x = r\cos\theta$, $y = r\sin\theta$, $\theta = \tan^{-1}\dfrac{y}{x}$, and $r = \pm\sqrt{x^2 + y^2}$. An examination of these formulas reveals that they are valid irrespective of the quadrant in which θ terminates and of the sign of r; of course the sign of r depends upon the choice of θ.

By means of these relations we may translate equations involving polar coordinates into corresponding equations involving rectangular coordinates, and conversely.

Thus, the equation $x^2 - y^2 = 36$ in rectangular coordinates becomes

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 36$$

or

$$r^2 \cos 2\theta = 36$$

in polar coordinates. Conversely, the equation $r = \sin 2\theta$ in polar coordinates, which may be written $r = 2 \sin \theta \cos \theta$, becomes

$$\sqrt{x^2 + y^2} = 2\left(\frac{y}{\sqrt{x^2 + y^2}}\right)\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

or

$$(x^2 + y^2)^3 = 4x^2y^2.$$

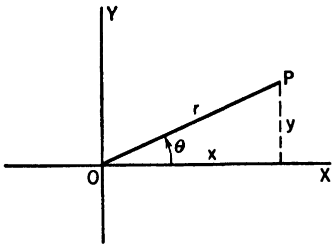


FIG. 9

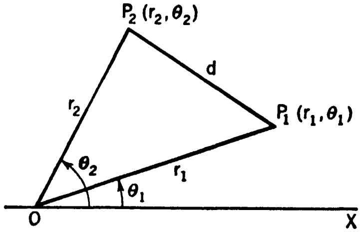


FIG. 10

8. DISTANCE BETWEEN TWO POINTS IN POLAR COORDINATES

The distance d between the two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$, as displayed in Figure 10, is determined by the law of cosines from trigonometry. Thus,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_2 - \theta_1),$$

or

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_2 - \theta_1)}.$$

EXERCISES 4

1. Plot the following points given in polar coordinates: $(2, 30°)$, $(-2, 30°)$, $\left(5, \frac{\pi}{2}\right)$, $\left(5, -\frac{3\pi}{4}\right)$.

2. Express the coordinates of the points in Exercise 1 in the Cartesian system.

3. Plot the following points given in rectangular coordinates: $(1, \sqrt{8})$, $(1, 1)$, $(-5, -7)$. Express these points in polar coordinates; in each case, use the smallest positive value for θ.

4. Find the equation for the line $y = 3x + 5$ in polar coordinates; for the line $x = 5$.

5. Express the equation $x^2 + y^2 = 25$ in polar coordinates.

6. Express $y^2 = 8x$ in polar coordinates.

7. Express the equation $r = \sin \theta$ in rectangular coordinates. Construct the curve.

8. Express the equation $r = \sin 2\theta$ in rectangular coordinates.

9. Express $r = \dfrac{2}{1 - \cos\theta}$ in rectangular coordinates.

10. Express $r = 2\sin\left(\dfrac{\pi}{4} - \theta\right)$ in rectangular coordinates.

11. Find the distance between the two points $P_1(5, 30°)$ and $P_2(10, 45°)$.

12. Find the distance between the two points $P_1(5, 0°)$ and $P_2\left(10, \dfrac{3\pi}{4}\right)$.

2 Graphs of Certain Equations

9. GRAPHS

In Book I, we considered the graphs of first-degree equations in x and y, which are straight lines, as well as the graphs of certain second-degree equations, which led to a brief consideration of the circle, the ellipse, the parabola, and the hyperbola. In Book II, we considered the graphs of certain transcendental equations in x and y, that is, the graphs of such equations as $y = \sin x$ and $y = \sin^{-1} x$.

In general, the curve consisting of all the points corresponding to every pair of values of x and y that satisfy a given equation in x and y, and those points only, is said to be the *locus*, or *graph*, of the equation. The term *curve* is a general word that is applied to any locus, including a straight line.

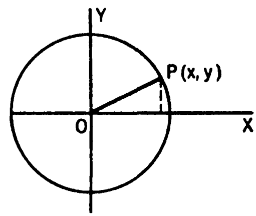


FIG. 11

In practice, we usually draw a sketch of a desired graph by obtaining a sufficient number of points close enough to each other so that we may draw a smooth approximate curve through these points. As an illustration of an equation that determines a curve that may be constructed without employing such an approximation device, note the equation $x^2 + y^2 = 36$. Obviously, each pair of numbers (x, y) that satisfies the equation $x^2 + y^2 = 36$ determines a point that is 6 units from the origin. In other words, the equation $x^2 + y^2 = 36$ has for its graph a circle of radius 6, with its center at the origin (see Figure 11). Furthermore, the coordinates of any point not on the circle do not satisfy the equation $x^2 + y^2 = 36$. Hence, the circle is the complete graph of the equation.

It is evident that the line corresponding to any first-degree equation in x and y is completely determined by two points which satisfy the equation, although a line, being unlimited in extent, cannot be drawn in its entirety.

The usual sketching process may be illustrated by considering in some detail the construction of the graph of the equation $x^2 - y^2 = 36$. In Book I, the graph of an equation of the form $x^2 - y^2 = 36$ has been designated as a *hyperbola*.

If we solve the equation $x^2 - y^2 = 36$ for y, we obtain

$$y = \pm\sqrt{x^2 - 36}.$$

From this equation we see that y is imaginary for $-6 < x < 6$. Hence, the graph exists only for $x \geqq 6$ and $x \leqq -6$. Below are tabulated a few pairs of values of (x, y) which satisfy the equation.

x	y
6	0
6.5	± 2.5
7	$\pm\sqrt{13} = \pm 3.60$
8	$\pm\sqrt{28} = \pm 5.29$
9	$\pm\sqrt{45} = \pm 6.71$
10	± 8

x	y
-6	0
-6.5	± 2.5
-7	$\pm\sqrt{13} = \pm 3.60$
-8	$\pm\sqrt{28} = \pm 5.29$
-9	$\pm\sqrt{45} = \pm 6.71$
-10	± 8

The points corresponding to these tabulated values are shown in Figure 12.

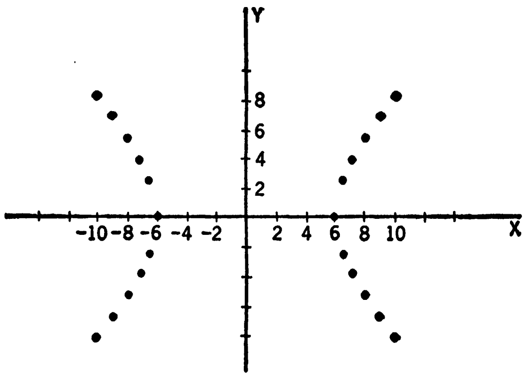


FIG. 12

Since y is real for any value of x greater than 6 or any value of x less than -6, the curve is not limited in extent when we move to the right of $x = 6$ and to the left of $x = -6$. We also see that y increases without limit as x increases without limit. Hence, the curve is not limited in extent above and below the x axis.

Another interesting property of the behavior of the points may also be

observed. From the equation $y = \sqrt{x^2 - 36}$, we obtain

$$\begin{aligned} y - x &= \sqrt{x^2 - 36} - x \\ &= \frac{(\sqrt{x^2 - 36} - x)(\sqrt{x^2 - 36} + x)}{\sqrt{x^2 - 36} + x} \\ &= \frac{(x^2 - 36 - x^2)}{\sqrt{x^2 - 36} + x} = -\frac{36}{\sqrt{x^2 - 36} + x}. \end{aligned}$$

From this result it is seen that y is less than x, and as x becomes larger and larger the right member of the equation becomes smaller and smaller. Hence, $y - x$ approaches zero; that is, y approaches x. Similarly y approaches $-x$ when $y = -\sqrt{x^2 - 36}$.

Thus, it is seen that the curve approaches nearer and nearer to the lines $y = x$ and $y = -x$, but the curve is confined within these lines. In Figure 13 we have drawn a curve through the points determined by the

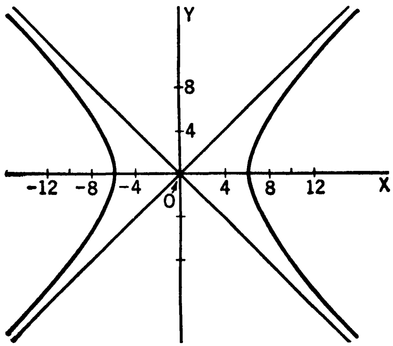


FIG. 13

tabulated values of x and y, and as guide lines we have also drawn the lines $y = x$ and $y = -x$. Obviously, the lines $y = x$ and $y = -x$ are not part of the locus of the equation. The curve consists of two separated branches, the one on the right and the one on the left; they are not connected.

The guide lines $y = x$ and $y = -x$ are designated as the *asymptotes* of the curve. It will be discovered later that the existence of asymptotes is part of the characterization of the behavior of any hyperbola. Of course, other curves also have asymptotes.

In higher mathematics asymptotes are given an analytical and geometrical definition. We shall, however, limit ourselves to a few additional illustrations to convey a conception of a line which we designate as an

asymptote. Let us consider an equation of the form

$$y = \frac{f(x)}{\phi(x)},$$

such as

$$y = \frac{2 + x}{x - 3}.$$

From the form of the latter equation we see that y cannot have a value if $x = 3$, since division by zero is impossible. Furthermore, if $x > 3$, y is positive, and as x gets nearer and nearer to 3, but is always greater than 3, y becomes larger and larger.

On the other hand, as x gets nearer and nearer to 3, but is always less than 3, y becomes larger and larger numerically but is negative.

Thus, the line having the equation $x = 3$, that is, the line parallel to the y axis and 3 units to the right of that axis, separates the curve into two branches. This line $x = 3$ is designated as an asymptote of the curve.

If we solve the equation

$$y = \frac{2 + x}{x - 3}$$

for x, we obtain

$$x = \frac{2 + 3y}{y - 1}.$$

From the form of this equation we see that x cannot have a value if $y = 1$. Furthermore, if $y > 1$, x is positive, and as y gets nearer and nearer to 1, but is always greater than 1, x becomes larger and larger.

On the other hand, as y gets nearer and nearer to 1, but is always less than 1, x becomes larger and larger numerically but is negative.

Thus, the line having the equation $y = 1$, that is, the line parallel to the x axis and 1 unit above it, separates the curve into two branches. So the line $y = 1$ is also designated as an asymptote.

If we now tabulate a few values of x and y that satisfy the equation, locate the corresponding points, and draw a smooth curve through these points, employing the asymptotes as guide lines, we shall have a fairly good sketch of the locus of the equation. Such a sketch is shown in Figure 14.

In general, if an algebraic equation can be put in the form $y = f(x)/\phi(x)$, and if the real roots of $\phi(x) = 0$ are $r_1, r_2, \cdots, r_n$, then the lines parallel to the y axis, of which the respective equations are $x = r_1$, $x = r_2$, $\cdots$, $x = r_n$, are designated as *vertical asymptotes.*

Similarly, if an algebraic equation can be put in the form $x = f(y)/\phi(y)$, and if the real roots of $\phi(y) = 0$ are $r_1, r_2, \cdots, r_n$, then the lines parallel

x	y
−20	0.8
−10	0.6
−5	0.37
−2	0
0	−0.67
1	−1.5
2	−4
2.5	−9
2.6	−11.5
2.9	−49
3.1	51
3.4	13.5
4	6
5	3.5
6	2.7
10	1.7

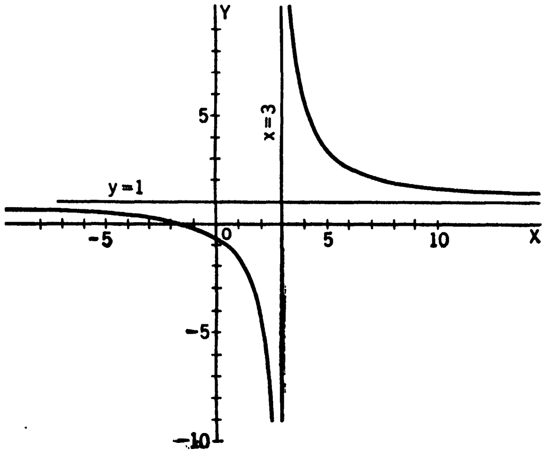


FIG. 14

to the x axis, of which the respective equations are $y = r_1$, $y = r_2$, $\cdots$, $y = r_n$, are designated as *horizontal asymptotes.*

As an illustration, the curve representing the equation $y = 1/x^2$ has the line $x = 0$ as a vertical asymptote. Since we may transform this equation into the form $x = \pm 1/\sqrt{y}$, we see that $y = 0$ is a horizontal asymptote. The curve of this equation is drawn in Figure 15.

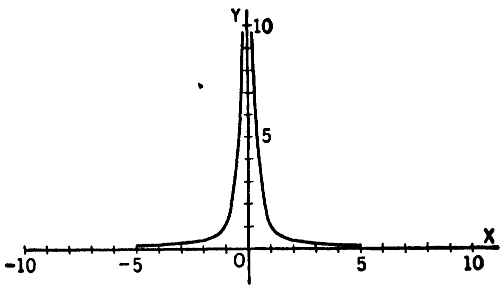


FIG. 15

EXERCISES 5

Determine any vertical and horizontal asymptotes that each of the following curves possess; then make a rough sketch of each curve:

1. $y^2x = 2$
2. $(x - 2)y = 3$
3. $x(y^2 - 4) = 6$
4. $y = 5 - \dfrac{2}{x - 1}$

5. $y = \dfrac{x+3}{2x-3}$

6. $(x^2 - 1)(y + 3) = 5$

7. $x = \dfrac{y-2}{2y+1}$

8. $x = 6 + \dfrac{y-1}{y+3}$

9. $(x + 3)(2y - 5) = 6$

10. $xy = x + 1$

10. EQUATIONS OF THE FORM $f(x, y)\phi(x, y) = 0$

From the relation between an equation and its curve, it follows that if $f(x, y)$ and $\phi(x, y)$ are functions of x and y, then the equation $f(x, y)\phi(x, y) = 0$ has for its graph the graphs of both $f(x, y) = 0$ and $\phi(x, y) = 0$. This follows readily from the fact that the coordinates of any point (x_1, y_1) satisfying the equation $f(x, y) = 0$, if $\phi(x_1, y_1)$ exists, must also satisfy $f(x, y)\phi(x, y) = 0$. A similar statement may also be made about $\phi(x, y) = 0$. Thus, the equation $x^2 - y^2 = 0$ has for its locus the two lines $x - y = 0$ and $x + y = 0$.

Similarly, the equation $y^3 - x^3 + x^2y - xy^2 - x^2 - y^2 = 0$ may be written $(x^2 + y^2)(y - x - 1) = 0$; hence, its graph consists of the graph of $x^2 + y^2 = 0$ and the graph of $y - x - 1 = 0$. The graph of $x^2 + y^2 = 0$ is merely the point $(0, 0)$, and the graph of $y - x - 1 = 0$ is a straight line.

11. SYMMETRY

A curve is said to be symmetrical with respect to a certain line if the line bisects every chord of the curve that is perpendicular to the line.

As an illustration, the curve $x^2 - y^2 = 36$ is symmetrical with respect to both the x and y axes. To examine the curve for symmetry with respect to the x axis, we may write the equation in the form

$$y = \pm\sqrt{x^2 - 36},$$

whereupon we see that for each value of x there are two values of y, numerically equal, but opposite in sign.

If we transform the equation to the form

$$x = \pm\sqrt{y^2 + 36},$$

we see in a similar manner that the curve is symmetrical with respect to the y axis.

From a popular point of view, we may also express the concept of symmetry with respect to a line as follows: A curve is symmetrical with respect to a certain line if the curve on one side of the line coincides with the curve on the other side of the line when the paper on which the curve is drawn is folded along the line.

A little reflection shows that for a general equation in x and y, symbolized by $f(x, y) = 0$, the x axis is a line of symmetry if $f(x, y) = 0$ is identical with $f(x, -y) = 0$, since the same values of x are determined for a negative y as for a numerically equal but positive y. Hence, if an algebraic equation

contains no odd powers of y, the curve is symmetrical with respect to the x axis.

Similarly, if $f(-x, y) = 0$ is identical with $f(x, y) = 0$, the curve of $f(x, y) = 0$ is symmetrical with respect to the y axis. So, if an algebraic equation contains no odd powers of x, the curve is symmetrical with respect to the y axis.

A curve is said to be symmetrical with respect to a certain point if the point bisects every chord drawn through it. If $f(-x, -y) = 0$ is identical with $f(x, y) = 0$, the curve of $f(x, y) = 0$ is symmetrical with respect to the origin; for, if a pair of numbers (x_1, y_1) satisfies $f(x, y) = 0$, the pair of numbers $(-x_1, -y_1)$ will also satisfy $f(x, y) = 0$. Hence, the points of the curve are located on a chord through the origin and are equidistant from the origin. It is apparent that symmetry with respect to both the x and y axes implies symmetry with respect to the origin.

12. INTERCEPTS

If we have a curve defined by the equation $f(x, y) = 0$, then the values of x satisfying the equation $f(x, 0) = 0$ give the points of intersection of the curve with the x axis, and these values of x are defined as the x intercepts.

Similarly, the values of y satisfying the equation $f(0, y) = 0$ give the points of intersection of the curve with the y axis, and these values of y are defined as y intercepts.

Thus, the x intercepts of the curve $4x^2 + 9y^2 = 36$ are $x = \pm 3$, and y intercepts are $y = \pm 2$.

EXERCISES 6

Discuss the curves defined by the following equations. Specify any x and y intercepts, any obvious properties of symmetry, and any limitations upon the extent of the curve. Determine which of the equations have graphs with asymptotes parallel to the x and y axes; find the equations of the asymptotes; and draw the asymptotes as guide lines for the curves. Determine several points on each curve, and sketch it.

1. $3x - 5y = 15$

2. $y = 8x^2$

3. $y = 2x^2 - 3x + 1$

4. $x = 10y^2$

5. $25x^2 + 9y^2 = 225$

6. $y = 2x^3 - 5x + 3$

7. $y = 2 - \dfrac{3}{x}$

8. $y = \dfrac{2 - 3x}{4 + x}$

9. $y = \dfrac{x - 4}{2x - 1}$

10. $x = 5 - \dfrac{3}{y}$

11. $y = 2 - \dfrac{3}{x} + \dfrac{5}{x^2}$

12. $y = 5x^3$

13. $y = 2 \cdot 3^x$

14. $y = 3 \cdot 5^{-x}$

15. $x^3 + y^3 = 1$

16. $(x - 2)(y - 3) = 12$

17. $(x - 2)(y - 3) = -12$

18. $x(y^2 - 4) = 4$

19. $y(x^2 - 4) = 4$

20. $x(y^2 - 4) = y$

21. $y(x^2 - 1) = x$

22. $y = \dfrac{5x}{x^2 + 1}$

23. $y^2 = \dfrac{x^3}{4 - x}$

24. $y^2 = x(x - 2)^2$

13. GRAPHS OF POLAR COORDINATE EQUATIONS

A graph that represents an equation involving polar coordinates is sketched by drawing a smooth curve through the points corresponding to the various pairs of values of r and θ that satisfy the equation. It is necessary to draw the curve through the points in order of the magnitude of θ, and it is desirable to have points located for small intervals of θ.

Illustration 1: Thus if $r = \sin\theta$, we tabulate related pairs of values of r and θ, as shown in the following table, and plot the corresponding points.

θ	0	30°	45°	60°	90°	120°	135°
r	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$

θ	150°	180°	210°	240°	270°	300°	330°	360°
r	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0

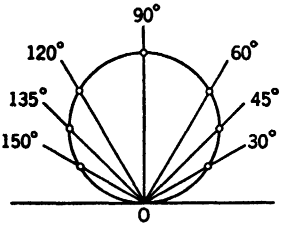


FIG. 16

Then, if we draw a smooth curve through the points corresponding to these pairs of values, we obtain the curve shown in Figure 16. It is not necessary in this case to go beyond $\theta = 180°$, as additional pairs of values merely duplicate points already located on the curve.

The curve representing the relation $r = \sin\theta$ is a circle with its center at $(\frac{1}{2}, 90°)$; its radius is $\frac{1}{2}$. The correctness of this conclusion may be shown by finding the distance between $(\frac{1}{2}, 90°)$ and (r, θ), where (r, θ) is any point on the circle. Thus,

$$d^2 = r^2 + \tfrac{1}{4} - 2(\tfrac{1}{2})r\cos(90° - \theta),$$

or

$$d^2 = r^2 + \tfrac{1}{4} - r\sin\theta.$$

But since $r = \sin\theta$, it follows that

$$d^2 = r^2 + \tfrac{1}{4} - r^2 = \tfrac{1}{4},$$

or

$$d = \tfrac{1}{2}.$$

Thus, the distance from the point $(\frac{1}{2}, 90°)$ to any point on the curve is a constant $\frac{1}{2}$.

Illustration 2: Obtain the graphical representation of $r = \sin 2\theta$.

Although it is possible to examine this equation for various properties that will assist in determining the nature of the curve, we shall still use the point-by-point method employed in the previous illustration. We first assign values to θ, then find 2θ; after that, r is determined from the given relation. We tabulate the results as in the following table. The curve that is obtained appears as Figure 17.

θ	2θ	r
0	0	0
15	30	$\frac{1}{2}$
30	60	$\sqrt{3}/2$
45	90	1
60	120	$\sqrt{3}/2$
75	150	$\frac{1}{2}$
90	180	0
105	210	$-\frac{1}{2}$
120	240	$-\sqrt{3}/2$
135	270	-1
150	300	$-\sqrt{3}/2$
165	330	$-\frac{1}{2}$
180	360	0

θ	2θ	r
195	390	$\frac{1}{2}$
210	420	$\sqrt{3}/2$
225	450	1
240	480	$\sqrt{3}/2$
255	510	$\frac{1}{2}$
270	540	0
285	570	$-\frac{1}{2}$
300	600	$-\sqrt{3}/2$
315	630	-1
330	660	$-\sqrt{3}/2$
345	690	$-\frac{1}{2}$
360	720	0

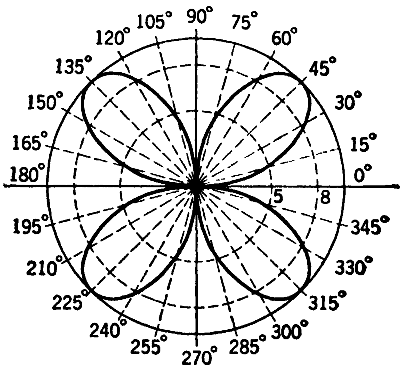


FIG. 17

In the case of this curve it is necessary to take values of θ from 0° to 360° in order to complete the entire curve. Values of θ beyond 360° will merely duplicate points already obtained.

The Cartesian equation for this curve is

$$(x^2 + y^2)^3 = 4x^2y^2.$$

This is an equation of the sixth degree. It would be quite a task to graph the curve from its Cartesian equation.

14. SKETCHING POLAR EQUATIONS

The following suggestions will be found helpful in tracing polar equations.

(1) **Intercepts:** Let us consider the equation

$$f(r, \theta) = 0.$$

If we let $\theta = 0$, we have $f(r, 0) = 0$. Sometimes it is possible to solve this latter equation for r. The values of r thus found are intercepts on the polar axis.

If we let $\theta = \pi$, we have $f(r, \pi) = 0$. After solving this equation for r, if possible, we have other intercepts on the polar axis.

(2) **Symmetry:** If $f(r, \theta)$ is identical with $f(r, -\theta)$ or with $f(-r, \pi - \theta)$, it is apparent that the polar axis is a line of symmetry. If $f(r, \theta)$ is identical with $f(r, \pi - \theta)$ or with $f(-r, -\theta)$, the line perpendicular to the polar axis at the pole is a line of symmetry.

If $f(r, \theta)$ is identical with $f(-r, \theta)$, or with $f(r, \pi + \theta)$, the curve is symmetrical with respect to the pole.

(3) **Extent:** In studying the extent of polar curves, the problem is essentially that of finding the extent of r; in particular, finding whether r becomes infinite for certain values of θ.

Illustration 1: Consider the properties of the curve

$$r - \sin\theta = 0.$$

The function $r - \sin\theta$ is identical with $r - \sin(\pi - \theta)$; hence, the line perpendicular to the polar axis at the pole is a line of symmetry.

Since $-1 \leq \sin\theta \leq 1$, then $-1 \leq r \leq 1$.

Also, since $\sin(-\theta) = -\sin\theta = -r$, we see that the curve is always above the polar axis.

When $\theta = 0$ and when $\theta = \pi$, $r = 0$.

When $\theta = \pi/2$, $r = 1$.

After locating a very few points, then, it is possible to sketch the curve of Figure 16.

Illustration 2: Study the properties of the curve,

$$r = \sin 2\theta.$$

For both $\theta = 0$ and $\theta = \pi$, we have $r = 0$. Hence, the curve passes through the pole.

Since the maximum value of $\sin 2\theta = 1$ and the minimum value of $\sin 2\theta = -1$, the curve must lie within a circle of radius 1 having its center at the pole.

Maximum values of r occur when $\sin 2\theta = 1$, that is, when $2\theta = 90°$ and $450°$, or when $\theta = 45°$ and $225°$.

Minimum values of r occur when $\sin 2\theta = -1$, that is, when $2\theta = 270°$ and $630°$, or when $\theta = 135°$ and $315°$.

In this illustration $f(r, \theta)$ is not identical with $f(r, -\theta)$; however, $f(r, \theta)$ is identical with $f(-r, \pi - \theta)$. Hence, the curve is symmetrical with respect to the polar axis.

Also, $f(r, \theta)$ is identical with $f(-r, -\theta)$. Hence, the curve is symmetrical with respect to the line perpendicular to the axis at the pole.

Moreover, $f(r, \theta)$ is identical with $f(r, \pi + \theta)$; hence, the curve is symmetrical with respect to the pole.

EXERCISES 7

Each of the following curves is to be graphed. The student will find it desirable to obtain polar graphing paper.

1. $r = 5 \cos \theta$
2. $r = 5 \cos 2\theta$
3. $r = 5(1 - \cos \theta)$
4. $r = \dfrac{6}{1 - \cos \theta}$
5. $r \cos \theta = 5$
6. $r \sin \theta = -5$
7. $r = 5(1 + \cos \theta)$
8. $r = 5(1 + \sin \theta)$
9. $r = \dfrac{8}{1 + \cos \theta}$
10. $r = 5(2 + \cos \theta)$
11. $r = 4(1 - 2 \cos \theta)$
12. $r = 3(3 - 2 \cos \theta)$
13. $r = \sin^2 \dfrac{\theta}{2}$
14. $r^2 = a^2 \cos 2\theta$
15. $r^2 = a^2 \sin 2\theta$
16. $r = \dfrac{8}{1 + 2 \cos \theta}$
17. $r = \dfrac{8}{2 + \cos \theta}$
18. $r = \dfrac{8}{2 - \cos \theta}$
19. $r = \theta$
20. $r = 2^\theta$
21. $r\theta = k$
22. $r = a \sin 3\theta$
23. $r = 10 \cos 3\theta$
24. $r = k \sin 4\theta$
25. $r = k \cos 4\theta$
26. $r = a (\cos \theta + \sin \theta)$
27. $r = a (\cos 2\theta + \sin 2\theta)$
28. $r = a \sec \left(\theta - \dfrac{\pi}{4}\right)$
29. $r = a \sec^2 \dfrac{\theta}{2}$
30. $r = a - \cos \theta;\ |a| < 1$

15. INTERSECTIONS OF CURVES

In Cartesian coordinates a point lies on two curves if, and only if, the coordinates of the point satisfy the equations of both curves. Hence, the coordinates of the points of intersection of two curves are found by solving the system composed of the two equations. If there are no real solutions, the curves do not intersect.

The situation is not so simple when dealing with polar coordinates, owing to the fact that the same point may be expressed in the form (r, θ)

in many different ways. In polar coordinates a point is said to be the intersection of two curves if the coordinates of the point in some mode of representation (r, θ) satisfy the equations of both curves; but it does not follow that the coordinates of the point in every mode of representation, or that the same coordinates in any mode of representation, will necessarily satisfy both equations.

Thus the points of intersection of the curves represented by the equations $r = \sin\theta$ and $r = 2\cos\theta + 1$ are given by those values of θ and r satisfying the relations, $\cos\theta = 0$, $r = 1$; and $\cos\theta = -\frac{4}{5}$, $\sin\theta = -\frac{3}{5}$, $r = -\frac{3}{5}$.

But an actual examination of the two curves reveals that they also intersect at the pole, since $(0, 0°)$ satisfies $r = \sin\theta$, and $(0, 120°)$ satisfies $r = 2\cos\theta + 1$; but neither $(0, 0°)$ nor $(0, 120°)$ satisfies both equations. Of course, $(0, 0°)$ and $(0, 120°)$ denote the same point, but in different modes of representation.

It is therefore evident that the algebraic solution of systems of polar equations may not give all the intersections. Careful graphs may be drawn to assist in determining the intersections not given by the algebraic solution of equations in polar coordinates.

Illustration: Let us find the intersections of the following curves:

$$y = x^2 - 3x + 4, \tag{1}$$

$$2x - 3y = -4. \tag{2}$$

Solving this system, we have

$$x^2 - 3x + 4 = \frac{2x + 4}{3},$$

or
$$3x^2 - 11x + 8 = 0,$$
$$(x - 1)(3x - 8) = 0,$$
$$x = 1 \quad \text{or} \quad \tfrac{8}{3}.$$

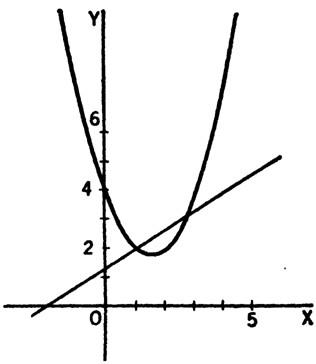


FIG. 18

Hence, the coordinates of the points of intersection are $(1, 2)$ and $(\frac{8}{3}, \frac{28}{9})$. The situation is displayed in Figure 18.

If $u = 0$ and $v = 0$ are both functions of x and y, then the equation $u + kv = 0$, where k is a constant, represents a curve through the intersections of the curves of $u = 0$ and $v = 0$. This is evident from the fact that the coordinates of the intersections of the loci of $u = 0$ and $v = 0$ satisfy the equation $u + kv = 0$. Hence, the points of intersection lie on the curve of $u + kv = 0$.

Thus, as an illustration, if we consider the two curves represented by the equations $x - y + 1 = 0$ and $x^2 + y^2 - 25 = 0$, then

$$(x - y + 1) + k(x^2 + y^2 - 25) = 0,$$

where k is a constant, represents a curve through the intersections of the line of $x - y + 1 = 0$ and the circle of $x^2 + y^2 = 25$.

EXERCISES 8

Draw the graphs of each of the following pairs of equations, and find the coordinates of the points of intersection by solving algebraically.

1. $3x - 2y = 6$
$y = 3$

2. $3x - 2y = 6$
$5x - y = 4$

3. $x + 2y = 10$
$y^2 = 8x$

4. $6x - 2y - 3 = 0$
$y^2 = 8x - 3$

5. $y^2 = 8x$
$3y + 2x + 9 = 0$

6. $y^2 = x(x + 5)^2$
$x - y + 5 = 0$

7. $y^2 = 6x$
$x^2 + y^2 = 16$

8. $x^2 + y^2 = 4$
$9x^2 + 16y^2 = 144$

9. $xy = 20$
$x + y = 12$

10. $y = 4x - x^2$
$2x + y = 5$

11. $y^2 = 4x + 4$
$y^2 = -2x + 4$

12. $y = x^3$
$y = 2x - x^2$

13. $y = x^3 - x^2$
$y = x^2$

14. $y = \sin x$
$y = \cos x$

15. $y = \sin\left(2x - \dfrac{\pi}{3}\right)$
$y = \sin\left(2x + \dfrac{\pi}{3}\right)$

16. The equation $x^2 + y^2 - 16 + k(y^2 - 6x) = 0$ represents a system of curves through the intersections of $x^2 + y^2 - 16 = 0$ and $y^2 - 6x = 0$. Draw the graphs $x^2+y^2-16=0$ and $y^2-6x=0$, and also $x^2+y^2-16+k(y^2-6x)=0$, for $k = 1$, $k = -1$, $k = 2$, and $k = -2$.

The following equations are expressed in terms of polar coordinates. Draw the graphs of each pair, and find the coordinates of the points of intersection.

17. $r \sin\theta = 10$
$r \cos\theta = 10$

18. $r \sin\theta = 5$
$r = 10 \sin\theta$

19. $r = 1 - \sin\theta$
$r = \dfrac{1}{1 - \sin\theta}$

20. $r = \dfrac{4}{2 + \cos\theta}$
$r = 4$

21. $r^2 = \cos 2\theta$
$r = \cos\theta$

22. $r = 2\cos\theta$
$r = 2\sin\theta\tan\theta$

23. $\left.\begin{aligned} r &= a\cos 2\theta \\ 2r &= a \end{aligned}\right\}$ where a is a constant

3 Equations of Loci

16. EQUATIONS OF LOCI

In Chapter 2 we considered briefly some fundamental topics pertaining to curves that represent given equations. In this chapter we shall consider the determination of the equations of the loci of points satisfying certain given conditions. The equation that is satisfied by those coordinates which correspond to all points on the locus, but not by coordinates of any points not on the locus, is the equation of the locus determined by the given conditions.

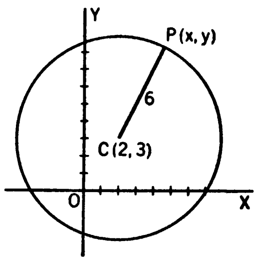


FIG. 19

In thinking of the locus of all points satisfying certain given conditions it is frequently convenient to speak of the locus as generated by a moving point which satisfies the given conditions in every position.

Illustration 1: Find the equation of the locus of points that are 6 units from the point (2, 3).

Solution: Let $P(x, y)$ be any point on the locus, as shown in Figure 19; then we have $CP = 6$. It is evident that the locus must be a circle with $C(2, 3)$ as its center.

From the distance formula, we have

$$\sqrt{(x-2)^2 + (y-3)^2} = 6. \tag{1}$$

Hence, relation (1) is the required equation of the locus, namely, the circle of radius 6 and center at (2, 3).

Though by squaring the two members the equation may be expressed in the form

$$x^2 + y^2 - 4x - 6y - 23 = 0,$$

it is frequently regarded as preferable to leave the result in the first form, from which we may recognize that the center is at (2, 3) and the radius is 6.

We shall now show that every point whose coordinates satisfy the equation $x^2 + y^2 - 4x - 6y - 23 = 0$ will satisfy

$$\sqrt{(x-2)^2 + (y-3)^2} = 6.$$

The equation $x^2 + y^2 - 4x - 6y - 23 = 0$ is obtained by squaring either of the two equations

$$\sqrt{(x-2)^2 + (y-3)^2} = +6$$

or

$$\sqrt{(x-2)^2 + (y-3)^2} = -6.$$

This last equation, however, states that a positive quantity, or zero, equals a negative quantity, which is impossible. Hence, every pair of real coordinates which satisfies the equation

$$x^2 + y^2 - 4x - 6y - 23 = 0$$

will satisfy

$$\sqrt{(x-2)^2 + (y-3)^2} = 6.$$

Illustration 2: Find the equation of the locus generated by a point subject to the condition that in every position the sum of the squares of its distances from $(-5, 0)$ and $(5, 0)$ is 100.

Solution: Let $P(x, y)$ be any point on the locus shown in Figure 20.

If we designate the points whose coordinates are $(-5, 0)$ and $(5, 0)$ by P_1 and P_2, respectively, and if we designate P_1P by d_1 and P_2P by d_2, then by the distance formula

$$d_1^2 = (x+5)^2 + y^2$$

and

$$d_2^2 = (x-5)^2 + y^2.$$

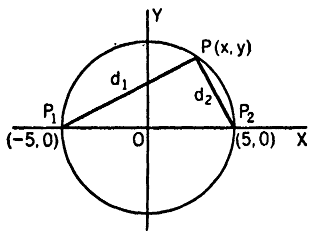


FIG. 20

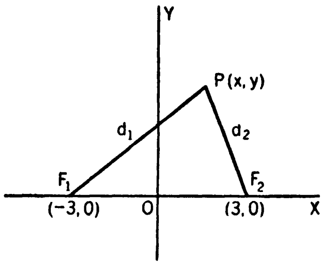


FIG. 21

Hence, the equation of the locus determined by the given conditions is

$$(x+5)^2 + y^2 + (x-5)^2 + y^2 = 100,$$

which may be simplified to

$$x^2 + y^2 = 25.$$

Evidently the locus is a circle.

Illustration 3: Determine the equation of the locus generated by a point moving in such a manner that in every position the sum of its distances from $(-3, 0)$ and $(3, 0)$ is 10.

Solution: Let $P(x, y)$ be any point on the locus (note Figure 21).

If we designate the points whose coordinates are $(-3, 0)$ and $(3, 0)$ by F_1 and F_2, respectively, and if we designate F_1P by d_1 and F_2P by d_2, then by the distance formula,

$$d_1 = \sqrt{(x+3)^2 + y^2} \quad \text{and} \quad d_2 = \sqrt{(x-3)^2 + y^2}.$$

Hence, the equation of the locus determined by the given condition is

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10,$$

which may be simplified to

$$16x^2 + 25y^2 = 400.$$

We shall now show that every point whose coordinates satisfy $16x^2 + 25y^2 = 400$ will satisfy

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10.$$

The equation $16x^2 + 25y^2 = 400$ is obtained by rationalizing any one of the four equations

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10, \tag{1}$$

$$-\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10, \tag{2}$$

$$\sqrt{(x+3)^2 + y^2} - \sqrt{(x-3)^2 + y^2} = 10, \tag{3}$$

or

$$-\sqrt{(x+3)^2 + y^2} - \sqrt{(x-3)^2 + y^2} = 10. \tag{4}$$

Obviously (4) cannot be satisfied by any real values of x and y. Equation (2) or (3) requires the difference of the distances d_1 and d_2 to equal 10; but 10 is greater than F_1F_2. Thus, (2) or (3) requires that the difference of two sides of a triangle be greater than the third side, which is impossible.

Hence, every pair of real coordinates that satisfies the equation

$$16x^2 + 25y^2 = 400$$

will satisfy

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10.$$

If the rationalized form of the equation is transformed to

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1,$$

we have the typical equation of an ellipse.

EXERCISES 9

1. Find the equation of the perpendicular bisector of the line joining the points $(2, 1)$ and $(3, -5)$.

2. A point moves so that in every position the sum of the squares of its distances from the points $(-5, 0)$ and $(5, 0)$ is 75. Find the equation of its path.

3. A point moves so that in every position the sum of the squares of its distances from the points (3, 4) and $(-1, 1)$ is 77. Find the equation of its path.

4. A point moves so that in every position its distance from the x axis is equal to its distance from the y axis. Find the equation of its path.

5. A point moves so that in every position it is as far from the y axis as from the point (4, 0). Find the equation of its path.

6. A point moves so that in every position it is as far from the y axis as from the point (2, 3). Find the equation of its path.

7. A point moves so that in every position it is as far from the x axis as from the point (2, 3). Find the equation of its path.

8. A point moves so that in every position the square of its abscissa is always eight times its ordinate. Find the equation of its path.

9. Find the equation of the path of a point that moves so that in every position the sum of its distances from the points (7, 3) and $(-2, 2)$ is 10.

10. A point moves so that it is equidistant from the point (3, 0) and the line 3 units to the left of and parallel to the y axis. Find the equation of its locus.

11. Find the equation of the path of a point that moves so that in every position the difference of its distances from the points $(-3, 0)$ and (3, 0) is equal to 5.

12. Find the equation of the path of a point that moves so that in every position it is twice as far from the point (5, 0) as it is from the y axis.

13. Find the equation of the path of a point that moves so that in every position it is twice as far from the y axis as it is from the point (5, 0).

14. Find the equation of the path of a point that moves so that in every position it is as far from the line $x = -1$ as it is from the point (1, 0).

15. A point moves so that in every position it is 5 units from the point $(5, 0°)$. Find its equation in polar coordinates.

16. A point moves so that in every position it is 5 units from the point $(5, \pi/2)$. Find the equation of its path in polar coordinates.

17. A point moves so that in every position it is 10 units from the point $(4, 30°)$. Find the equation of its path in polar coordinates.

18. A point moves so that in every position it is as far from the point (3, 0) as it is from the point $(2, 2\pi/3)$. Find the polar equation of its path.

4 The Straight Line

17. THE STRAIGHT LINE

Although we have considered the straight line in Chapter 4, Book I, we shall now give a more systematic treatment of the subject. In our study, we may consider three cases:

(1) A straight line parallel to the y axis;

(2) A straight line parallel to the x axis;

(3) A straight line not parallel to either axis.

CASE 1. As an illustration, let the line be parallel to the y axis and 5 units to the right. The corresponding equation is readily seen to be $x = 5$ for all values of y. Since the x coordinate of every point on the line is 5 and any point not on the line will have an x coordinate other than 5, we merely say $x = 5$ is the equation of the given line, the phrase "for all values of y" being implied. By analogy, any line parallel to the y axis will have the equation $x = a$, where a is some constant, positive or negative.

CASE 2. As an illustration, let the line be parallel to the x axis and 3 units above it. The corresponding equation is $y = 3$. In general, any line parallel to the x axis will have the equation $y = b$, where b is some constant, positive or negative.

CASE 3. If the line is not parallel to either axis, it may be determined by various given conditions. The more common cases are considered in the paragraphs that follow.

Line Determined by Two Given Points. It is learned in elementary geometry that two points determine a straight line. To examine the implications of this statement in analytic geometry, let a straight line be determined by the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, where $x_1 \neq x_2$, and let $P(x, y)$ be any point on the line P_1P_2. From a study of the similar triangles appearing in Figure 22, we have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

This form of the equation of a straight line, determined as it is by two points, is known as the *two-point form.*

It is left as an exercise for the student to show that Equation (1) may

be written in the form of an equation involving a determinant; namely,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

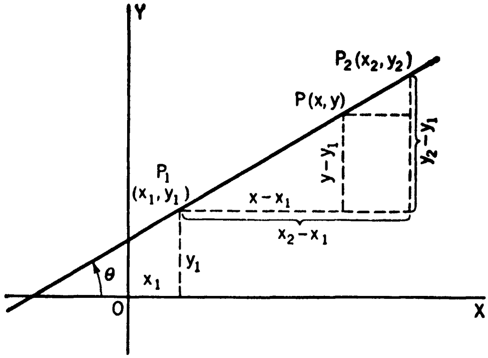


FIG. 22

If θ is a positive angle less than 180°, measured from the positive x axis to the given line, then the ratio

$$\tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$

is defined as the slope of the line through (x_1, y_1) and (x_2, y_2). The slope is usually designated by m.

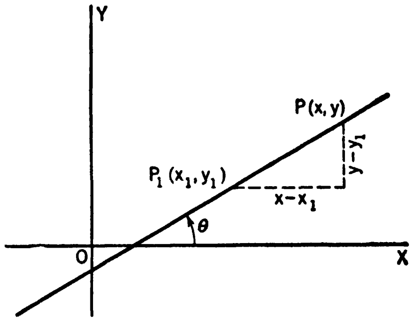


FIG. 23

Line Determined by a Given Point and a Given Slope. Let a straight line be determined by the given point $P_1(x_1, y_1)$ and the given slope m, and let $P(x, y)$ be any point on this line; then, from Figure 23,

$$\frac{y - y_1}{x - x_1} = \tan \theta = m,$$

or
$$y - y_1 = m(x - x_1). \tag{2}$$

This form of the equation of a straight line, determined as it is by one point and the slope, is known as the *point-slope form.*

The equation of a line parallel to the y axis cannot be written in form (2), since $\tan\theta$ does not exist for $\theta = 90°$.

Line Determined by a Given Slope and a Given y Intercept. Let us next consider a line determined by the y intercept b and the slope m. Since we are given the y intercept b, we actually know the coordinates of one point on the line, namely, $(0, b)$. Hence, employing Equation (2), we have

$$y - b = m(x - 0) \quad \text{or} \quad y = mx + b. \tag{3}$$

This equation is known as the **slope-intercept form.**

The equation of a line parallel to the y axis cannot be written in Form (3), since $\tan 90°$ does not exist.

Line Determined by a Given x Intercept and a Given y Intercept. Let the line now under consideration be determined by the x intercept a, $a \neq 0$, and the y intercept b, $b \neq 0$. In other words, the line is determined by the points $(0, b)$ and $(a, 0)$. Hence, employing Equation (1), we have

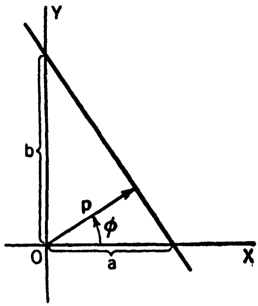


FIG. 24

$$\frac{y - b}{x - 0} = \frac{0 - b}{a - 0},$$

which may be written

$$ay - ab = -bx$$

or
$$\frac{x}{a} + \frac{y}{b} = 1. \tag{4}$$

This equation is known as the **intercept form.**

The equation of a line through the origin cannot be written in Form (4).

Line Determined by Its Perpendicular Distance from the Origin and the Positive Angle from the x Axis to the Perpendicular. If we designate the perpendicular distance from the origin to the given line by p and the given angle by ϕ, we have, by reference to Figure 24,

$$a = \frac{p}{\cos\phi} \quad \text{and} \quad b = \frac{p}{\sin\phi}.$$

Since we now know the intercepts, we may apply Equation (4) and obtain

$$\frac{x}{\dfrac{p}{\cos\phi}} + \frac{y}{\dfrac{p}{\sin\phi}} = 1,$$

or
$$x \cos \phi + y \sin \phi - p = 0. \tag{5}$$

This equation is known as the *normal form*. In general, ϕ may vary from 0° to 360°. However, if $p = 0$, we may limit ϕ from 0° to 180°.

EXERCISES 10

Find the equations of the lines satisfying the following conditions:

1. Parallel to the x axis and 3 units below it.

2. Parallel to the y axis and 10 units to the right of it.

3. Through the points (2, 5) and (−1, 7).

HINT: Use Formula (1). Note that it is immaterial which point is called P_1 and which is called P_2.

4. Through the points (−2, −5) and (3, −1).

5. Through the points (0, 2) and (5, 0).

6. Through the point (3, 1) and with $\theta = 30°$.

7. Through the point (2, −3) and making an angle of 60° with the positive direction of the x axis.

8. Making an angle of 45° with the positive direction of the x axis and cutting the y axis 2 units above the origin.

9. Making an angle of 135° with the positive direction of the x axis and passing through the point (0, 2).

10. Cutting the x axis 4 units to the right of the origin and the y axis 6 units below the origin.

11. Intersecting the x axis 5 units to the right of the origin and making an angle of 150° with the positive direction of the x axis.

12. Six units from the origin and cutting the x axis and the y axis at equal distances from the origin. Find all solutions.

13. Sketch the lines determined by each of the following pairs of conditions:

(*a*) Having the slope 2 and passing through the point (3, 4).
(*b*) Having the slope −2 and passing through the point (3, 4).
(*c*) Having the slope $\frac{1}{3}$ and passing through the point (2, 1).
(*d*) Having the slope $-\frac{1}{3}$ and passing through the point (0, 4).
(*e*) Three units from the origin and with $\phi = 30°$.

14. Write the equation of each of the lines in Exercise 13.

15. The vertices of a triangle are the points $A(2, 1)$, $B(4, -3)$, and $C(-1, -4)$.

(*a*) Find the equation of each side.
(*b*) Find the equation of each median.
(*c*) Find the length of BC.
(*d*) Find the area of the triangle.

18. THE GENERAL EQUATION OF THE FIRST DEGREE

The general equation of the first degree may be written in the form
$$Ax + By + C = 0, \tag{1}$$
where A, B, and C are constants; that is, every equation of first degree may be obtained from this form by properly choosing A, B, and C.

If $A = 0$, Equation (1) becomes $By + C = 0$ or

$$y = -\frac{C}{B}, \tag{2}$$

which represents a line parallel to the x axis.

If $B = 0$, Equation (1) becomes $Ax + C = 0$ or

$$x = -\frac{C}{A}, \tag{3}$$

which represents a line parallel to the y axis.

It is impossible for A and B to be zero simultaneously in Equation (1) unless $C = 0$, and in that case the equation reduces to $0 = 0$.

If neither A nor B is zero, Equation (1) may be written

$$By = -Ax - C$$

or

$$y = -\frac{A}{B}x - \frac{C}{B}. \tag{4}$$

A comparison of Equation (4) with the slope-intercept form shows that Equation (1) represents a straight line with the slope $-A/B$, and with a y intercept $-C/B$.

If neither A, B, nor C is zero, Equation (1) may also be written in the form

$$\frac{Ax}{-C} + \frac{By}{-C} = 1$$

or

$$\frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1. \tag{5}$$

A comparison of Equation (5) with the intercept form of the straight line shows that Equation (1) represents a straight line whose x intercept is $-C/A$ and whose y intercept is $-C/B$.

If neither A nor B is zero, Equation (1) may also be written in the form

$$KAx + KBy + KC = 0, \tag{6}$$

where K is to be determined in such a way that the coefficients of Equation (6) may be equated to the corresponding coefficients of the equation

$$x \cos \phi + y \sin \phi - p = 0.$$

Hence, $KA = \cos \phi$, $KB = \sin \phi$, and $KC = -p$. From these equations we have

$$K^2A^2 = \cos^2 \phi,$$

and

$$K^2B^2 = \sin^2 \phi.$$

After adding, we obtain

$$K^2(A^2 + B^2) = \sin^2 \phi + \cos^2 \phi = 1$$

or

$$K = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

It is desirable for p to be positive. Hence, from $KC = -p$, it is seen that if C is positive, K must be chosen negative; and if C is negative, K must be chosen positive.

If it should happen that $C = 0$, then $p = 0$, and if ϕ is considered only from 0° to 180°, the sign of K is determined from $KB = \sin \phi$. Since $\sin \phi$ is always positive, K must have the same sign as B.

Thus, Equation (1) may be written as

$$\frac{Ax}{\pm\sqrt{A^2 + B^2}} + \frac{By}{\pm\sqrt{A^2 + B^2}} + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0, \tag{7}$$

where

$$\cos \phi = \frac{A}{\pm\sqrt{A^2 + B^2}},$$

$$\sin \phi = \frac{B}{\pm\sqrt{A^2 + B^2}},$$

$$p = -\frac{C}{\pm\sqrt{A^2 + B^2}},$$

and where only one sign of the radical is selected in accordance with the rules given above.

19. THE DISTANCE BETWEEN A LINE AND A POINT

In case a given line is parallel to either axis of reference, the distance between the line and a given point may be obtained by inspection. Thus, if the equation of the line is $x = 5$ and the given point is (7, 2), the distance between the line and the point is obviously 2 units. If the equation of the given line is $x = -3$ and the point is (5, 7), the distance between the line and point is 8 units. Similarly, if the given line is $y = 6$, and the point is (3, 1), the distance between the line and the point is 5 units.

In each of these cases, it is observed, we have determined the absolute value of the distance between the line and the point.

If the given line is not parallel to either axis, the problem is to find the numerical value of the distance d between the line $Ax + By + C = 0$ and some point (h, k) (note Figure 25).

If we write the given equation

$$Ax + By + C = 0 \tag{1}$$

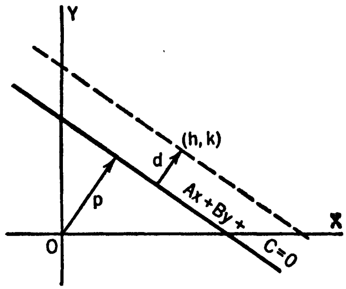


FIG. 25

in the form

$$x \cos \phi + y \sin \phi - p = 0; \qquad (2)$$

we may write the equation of a line through (h, k), parallel to (1), in the form

$$x \cos \phi + y \sin \phi - (p + d) = 0, \quad (3$$

where it is apparent that we are considering d to be positive or negative, according as the given point is on the opposite side or on the same side of the line with respect to the origin.

From Relation (3), we obtain

$$d = x \cos \phi + y \sin \phi - p. \qquad (4)$$

Now, since (h, k) satisfies (4), we have

$$d = h \cos \phi + k \sin \phi - p. \qquad (5)$$

By reference to the previous section, we note that

$$\cos \phi = \frac{A}{\pm\sqrt{A^2 + B^2}},$$

$$\sin \phi = \frac{B}{\pm\sqrt{A^2 + B^2}},$$

and

$$p = \frac{-C}{\pm\sqrt{A^2 + B^2}}.$$

Hence, Relation (5) may be written as

$$d = \frac{hA + kB + C}{\pm\sqrt{A^2 + B^2}}. \qquad (6)$$

A consideration of Formula (6) reveals that the distance between a line $Ax + By + C = 0$ and the point (h, k) is found by substituting h and k for x and y, respectively, in the expression

$$\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}}.$$

This expression is the left member of Equation (1) written in the normal form. If the sign of the radical is determined in accordance with the principles of the previous section, d may be either positive or negative. However, the result obtained is consistent with the statement of signs previously given; that is, if the origin and the given point are on the same side of the line, d will be negative; otherwise d will be positive.

Illustration 1: Find the distance from the line $3x + 4y - 7 = 0$ to the point (5, 1) (note Figure 26).

By Formula (6) above, we have

$$d = \frac{3(5) + 4(1) - 7}{+\sqrt{9 + 16}}$$
$$= +\frac{12}{5}.$$

Illustration 2: Find the distance from the line $3x + 4y - 7 = 0$ to the point (1, −4) (note Figure 26). This time

$$d = \frac{3(1) + 4(-4) - 7}{5} = -4.$$

The negative sign merely indicates, of course, that the origin and the given point are on the same side of the line.

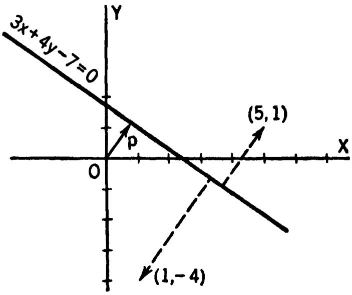


FIG. 26

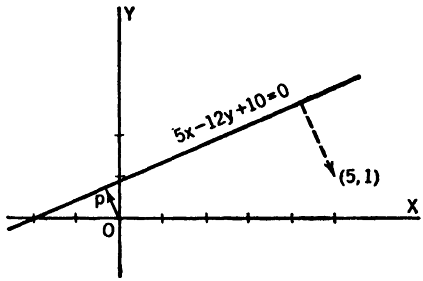


FIG. 27

Illustration 3: Find the distance from the line $5x - 12y + 10 = 0$ to the point (5, 1) (see Figure 27).

In this case,

$$d = \frac{5(5) - 12(1) + 10}{-13} = -\frac{23}{13}.$$

EXERCISES 11

1. Write the equation $3x - 4y = 10$ in the slope-intercept form and indicate its y intercept.

2. Write the equation $3x - 4y = 10$ in the intercept form and indicate both the x and y intercepts.

3. Write $3x - 4y = 10$ in the normal form, and find its distance from the origin.

4. Find the distance from the line $3x + 5y = 15$ (*a*) to the point (2, 7); (*b*) to the point (−1, 2).

5. The vertices of a triangle are $A(0, 0)$, $B(5, 0)$, and $C(2, 7)$. Find the three altitudes of the triangle and the area of the triangle.

6. Find the area of the triangle $A(0, 0)$, $B(5, 2)$, and $C(1, 6)$ by two methods. (HINT: Draw the figure and find the length of AB and the altitude from C to AB.)

7. Draw the line $y - 3 = 2(x - 1)$, and find the angle that the line makes with the positive direction of the x axis.

(HINT: Let θ equal the positive angle from the x axis to the line. Therefore, $\tan\theta = 2$. From a trigonometric table find θ.)

8. Given the equation $2x - 3y = 6$; sketch the line and find θ.

9. Given the line through the two points $(-1, 4)$ and $(3, -2)$; find angle θ.

10. If the line AB cuts the x axis and the y axis, respectively, at the points $(3, 0)$ and $(0, 7)$, find the equation of the line and angle θ. Find also the distance from the line to the origin.

11. Find the equation of a line through the point $(3, 7)$ and having the same slope as the line $3x + 2y = 10$.

12. Find the distance from the line $3x + 2y = 10$ to the point $(3, 7)$.

13. Find the distance from the intersection of the lines $x - y = 7$ and $2x + 5y = 21$ to the line $5x - 13y = 20$.

14. The equations of the sides of a triangle are $2x - y + 2 = 0$, $5x + 4y - 21 = 0$, and $x + 6y + 1 = 0$.

(*a*) Find the coordinates of the vertices.
(*b*) Find the length of each altitude.
(*c*) Find the slope of each side.
(*d*) Find the length of each side.
(*e*) Find the mid-point of each side.
(*f*) Find the lengths of the medians.
(*g*) Find the coordinates of a point $\frac{2}{3}$ of the distance from each vertex of the triangle to the mid-point of the opposite side.
(*h*) Find the area of the triangle by two methods.

20. PARALLEL LINES

If two lines are parallel, and if θ_1 and θ_2 are the positive angles from the x axis to the lines, respectively, then $\theta_1 = \theta_2$ (note Figure 28). Hence, $\tan\theta_1 = \tan\theta_2$, and the slopes m_1 and m_2 of the lines are equal.

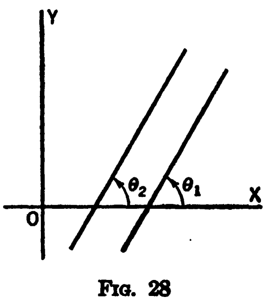


FIG. 28

Conversely, if the slopes of the lines are equal, the lines are parallel; for, if $\tan\theta_1 = \tan\theta_2$, then for angles between 0° and 180°, $\theta_1 = \theta_2$. Hence, the lines are parallel.

If two lines are each written in the slope-intercept form, namely, $y = mx + b$, it is immediately possible to compare their slopes and decide whether the lines are parallel. Thus, it is apparent that the lines $y = 3x + 2$ and $y = 3x - 5$ are parallel. If the equations of the lines are written in the more general form $Ax + By + C = 0$, the slopes are readily compared by realizing that the general form of the equation may be re-

written as

$$y = -\frac{A}{B}x - \frac{C}{B},$$

if $B \neq 0$; that is, the slope of the line is $-A/B$.

Thus, the slope of the line $3x + 2y - 7 = 0$ is $-\frac{3}{2}$, and the slope of $6x + 4y + 10 = 0$ is $-\frac{6}{4}$ or $-\frac{3}{2}$. So the slopes are equal, and the lines are parallel.

21. PERPENDICULAR LINES

If two lines are perpendicular, as in Figure 29, it follows that $\theta_2 = \theta_1 + 90°$, or that

$$\tan\theta_2 = \tan(\theta_1 + 90°) = -\cot\theta_1 = -\frac{1}{\tan\theta_1},$$

which means that

$$m_2 = -\frac{1}{m_1}.$$

Conversely, if $m_2 = -\dfrac{1}{m_1}$, the lines are perpendicular. The proof is left as an exercise for the student.

Hence, by determining the slopes of two given lines and comparing them, it is possible to ascertain whether the lines are perpendicular. For example, the slope of $5x + 7y - 8 = 0$ is $-\frac{5}{7}$, and the slope of $14x - 10y + 9 = 0$ is $\frac{14}{10}$ or $\frac{7}{5}$. Therefore, one slope is the negative reciprocal of the other, which shows that the lines are perpendicular.

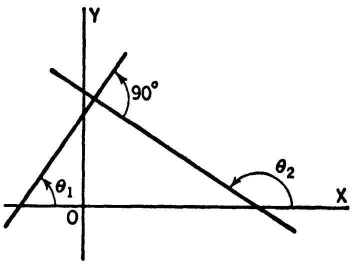


FIG. 29

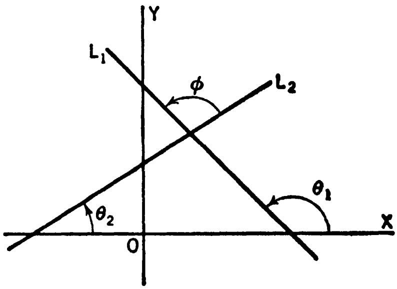


FIG. 30

22. ANGLE BETWEEN TWO LINES

To find the angle that the line L_1 makes with the line L_2, let ϕ be the angle from line L_2 to line L_1 measured counterclockwise. Then, by reference to Figure 30, we have

$$\theta_1 = \theta_2 + \phi$$

or

$$\phi = \theta_1 - \theta_2.$$

Consequently,

$$\tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}. \tag{1}$$

If we represent the slope of L_1 by m_1 and the slope of L_2 by m_2, then

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}. \tag{2}$$

Thus, to find the angle from the line $3x - 5y = 10$ to the line $2x + y = 4$, we have $m_2 = \frac{3}{5}$ and $m_1 = -2$. Hence,

$$\tan\phi = \frac{-2 - \dfrac{3}{5}}{1 - \dfrac{6}{5}} = 13,$$

or

$$\phi = \tan^{-1} 13.$$

EXERCISES 12

1. Take a point $A(x_1, y_1)$ in the second quadrant and a point $B(x_2, y_2)$ in the third quadrant. Draw the figure and derive the formula for the length of AB.

2. Find the length of the line segment joining $(1, -6)$ and $(-4, -3)$.

3. Find the coordinates of the points that trisect the line segment joining $(1, -6)$ and $(-4, -3)$.

4. Which of the following lines are parallel, and which are perpendicular?

(*a*) $2x - 3y - 10 = 0$.
(*b*) $4x - 6y - 6 = 0$.
(*c*) $3x - 2y + 10 = 0$.
(*d*) $6x + 4y - 20 = 0$.
(*e*) $x - \frac{2}{3}y + 8 = 0$.

5. Find the angle that the line $x - y - 7 = 0$ makes with the line $x + y + 1 = 0$.

6. Find the angle that the line $x + y + 1 = 0$ makes with the line $3x - 4y - 5 = 0$.

7. The vertices of a triangle ABC are $A(0, 0)$, $B(5, 0)$, and $C(1, 3)$.

(*a*) Find the lengths of the sides.
(*b*) Find the equations of the sides.
(*c*) Find the three altitudes.
(*d*) Find the three angles.
(*e*) Find the equations of the altitudes.
(*f*) Find the common intersection of the altitudes.
(*g*) Find the equations of the medians.
(*h*) Find the area of the triangle.

8. Find the equation of the perpendicular bisector of the line that joins the points $(-2, -5)$ and $(6, 7)$.

9. Find the equation of the line through the intersection of the lines $3x - y = 10$ and $2x + 3y + 8 = 0$ and whose slope is $-\frac{1}{2}$.

10. Find the distance from the intersection of the lines $3x - y = 10$ and $x + y = 2$ to the line $x - 7y = 28$.

11. Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are the vertices of a parallelogram. Employ two methods.

12. Show analytically that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one half the third side.

13. Prove analytically that the diagonals of a rhombus bisect each other at right angles.

14. Write an equation expressing the fact that the point (x, y) is 5 units from the point $(2, -3)$.

15. Prove analytically that the medians of a triangle intersect in a point one third of the distance from a side to the opposite vertex.

16. Suppose that the sides of a triangle are as follows:
AB: $2x - 3y = 12$; AC: $x + y = 10$; BC: $15x + 3y = 10$.

(*a*) Find the number of degrees in angle A.

(*b*) Find the coordinates of A.

17. Find the altitudes of the triangle whose vertices are $(1, 1)$, $(5, 2)$, and $(3, 7)$.

18. Find the area of the triangle of Exercise 17.

19. Determine the equations of two straight lines perpendicular to the line $x - 2y = 3$ and at a distance of 5 from the origin.

20. How far apart are the two lines $2x + 3y = 7$ and $4x + 6y = 9$?

5 The Circle

23. THE CIRCLE

Although we have considered briefly the circle, the ellipse, the parabola, and the hyperbola in Book I, we shall now treat these curves in more detail.

The circle is a curve possessing the property that every point of the curve is equidistant from a fixed point called the *center*.

Let $C(h, k)$ be the center of a circle, and let its radius be r; then, if $P(x, y)$ is any point on the circle, we have by use of the distance formula

$$r = \sqrt{(x - h)^2 + (y - k)^2},$$

or

$$(x - h)^2 + (y - k)^2 = r^2. \tag{1}$$

In particular, if $h = 0$ and $k = 0$, then the equation of a circle of radius r is

$$x^2 + y^2 = r^2. \tag{2}$$

The general equation

$$Ax^2 + Ay^2 + 2Dx + 2Ey + F = 0 \qquad (A > 0), \tag{3}$$

may be transformed to

$$\left(x^2 + \frac{2D}{A}x + \frac{D^2}{A^2}\right) + \left(y^2 + \frac{2E}{A}y + \frac{E^2}{A^2}\right) = -\frac{F}{A} + \frac{D^2}{A^2} + \frac{E^2}{A^2}$$

or

$$\left(x + \frac{D}{A}\right)^2 + \left(y + \frac{E}{A}\right)^2 = \frac{D^2 + E^2 - AF}{A^2}. \tag{4}$$

If $D^2 + E^2 - AF > 0$, we may compare Equations (1) and (4), and note that the form is that of a circle. In fact,

$$h = -\frac{D}{A}, \qquad k = -\frac{E}{A}, \qquad \text{and} \qquad r = \frac{\sqrt{D^2 + E^2 - AF}}{A}.$$

Hence, if $D^2 + E^2 - AF > 0$, Equation (3) is the equation of a circle with its center at $(-D/A, -E/A)$ and its radius equal to

$$\frac{\sqrt{D^2 + E^2 - AF}}{A}.$$

If $D^2 + E^2 - AF = 0$, the locus of Equation (3) is merely the point $(-D/A, -E/A)$.

If $D^2 + E^2 - AF < 0$, Equation (3) does not represent a real locus.

Equation (3) may always be transformed to the form

$$x^2 + y^2 + ax + by + c = 0, \tag{5}$$

where $a = \dfrac{2D}{A}$, $b = \dfrac{2E}{A}$, and $c = \dfrac{F}{A}$.

From this equation we see that if we are given sufficient conditions to determine a, b, and c, the equation of the circle is determined.

Illustration 1: Suppose we are given three points $P_1(0, 0)$, $P_2(0, 3)$, and $P_3(2, 1)$, which are not all on the same straight line. To determine the equation of a circle through the given points, we proceed as follows:

After substituting the coordinates of P_1, that is, $x = 0$ and $y = 0$, in (5), we obtain

$$c = 0.$$

After substituting the coordinates of P_2, that is, $x = 0$ and $y = 3$, in (5), we obtain

$$9 + 3b + c = 0.$$

After substituting the coordinates of P_3, that is, $x = 2$, $y = 1$, in (5), we obtain

$$4 + 1 + 2a + b + c = 0.$$

If we solve this system of three equations involving a, b, and c, we obtain

$$c = 0, \qquad b = -3, \qquad a = -1.$$

Hence, the required equation is

$$x^2 + y^2 - x - 3y = 0,$$

which may be written in the form

$$(x^2 - x + \tfrac{1}{4}) + (y^2 - 3y + \tfrac{9}{4}) = \tfrac{10}{4},$$

or
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{\sqrt{10}}{2}\right)^2.$$

Thus, we note that $h = \dfrac{1}{2}$, $k = \dfrac{3}{2}$, $r = \dfrac{\sqrt{10}}{2}$. So the circle has its center at $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$, and its radius is $\dfrac{\sqrt{10}}{2}$.

In the case of this problem it would have been equally satisfactory to substitute the coordinates of the given points in Equation (1) and obtain

the three equations

$$h^2 + k^2 = r^2,$$

$$h^2 + (3 - k)^2 = r^2,$$

and

$$(2 - h)^2 + (1 - k)^2 = r^2.$$

The solution of this system of equations also gives, of course, $k = \frac{3}{2}$, $h = \frac{1}{2}$, and $r = \frac{\sqrt{10}}{2}$.

Illustration 2: The equation

$$3x^2 + 3y^2 - 7x - 8 = 0$$

is of the form (3), where $A = 3$, $2D = -7$, $E = 0$, and $F = -8$.

Since

$$D^2 + E^2 - AF = \frac{49}{4} + 24 = \frac{145}{4},$$

the curve represents a circle with

$$r = \frac{\sqrt{D^2 + E^2 - AF}}{A} = \frac{1}{3}\sqrt{\frac{145}{4}} = \frac{1}{6}\sqrt{145},$$

$$h = -\frac{D}{A} = \frac{7}{6}, \qquad \text{and} \qquad k = -\frac{E}{A} = 0.$$

It is usually regarded as preferable, however, to proceed as follows:

$$3x^2 + 3y^2 - 7x - 8 = 0,$$

$$x^2 - \frac{7}{3}x + y^2 = \frac{8}{3},$$

$$\left(x^2 - \frac{7}{3}x + \frac{49}{36}\right) + y^2 = \frac{8}{3} + \frac{49}{36} = \frac{145}{36},$$

or

$$\left(x - \frac{7}{6}\right)^2 + y^2 = \left(\frac{\sqrt{145}}{6}\right)^2.$$

Comparing this equation with (1), we observe that it is the equation of a circle with its center at $(\frac{7}{6}, 0)$ and having the radius $\sqrt{145}/6$.

EXERCISES 13

1. Find the equation of a circle with its center at $(2, -3)$ and having the radius 6.

2. Find the equation of a circle with its center at $(-2, -5)$ and having the radius 7.

3. Find the equation of a circle with its center at $(-3, 4)$ and having the radius 5.

4. Find the equation of a circle with its center at (5, 0) and passing through the origin.

5. Find the equation of a circle with its center at (0, 5) and passing through the origin.

6. Show that the equation $x^2 + y^2 + 6x - 8y = 11$ is the equation of a circle, and find its radius and the coordinates of the center.

7. Find the equation of a circle through the points (1, 0), (2, 4), and (−1, 3). Find its radius and the coordinates of the center.

8. Find the equation of a circle with its center at the point (2, 1) and passing through the point (5, −3).

9. Find the equation of a circle with its center at the point (9, −3) and tangent to the y axis.

10. Find the equation of a circle with its center at the y intercept of the line $3x + 7y = 14$ and tangent to the x axis.

11. Find the equation of a circle with its center at the point (−7, 2) and tangent to the line $5x - 8y = 20$.

12. Find the radius and the coordinates of the center of each of the following circles; sketch the circle:
(*a*) $x^2 + y^2 + 6x = 0$;
(*b*) $x^2 + y^2 - 4y = 0$;
(*c*) $x^2 + y^2 - 8x + 6y = 0$;
(*d*) $x^2 + y^2 - 2ax = 0$;
(*e*) $x^2 + y^2 - 2ax - 2by = 0$.

13. Find the equation of the locus of a point that moves so that the sum of the squares of its distances from the points (−4, 0) and (4, 0) is equal to 64.

14. Find the equation of the locus of a point that moves so that the sum of the squares of its distances from the points $A(-3, 5)$ and $B(5, -2)$ is 74. Show that this locus is a circle with its center at the mid-point of AB.

15. Find the equation of the locus of a point that moves so that the sum of the squares of its distances from the points $A(a, b)$ and $B(c, d)$ is equal to k. Show that if the locus is real, it is a circle with its center at the mid-point of AB.

16. Find the equation of a circle with its center at the intersection of $5x - y = 17$ and $3x + 2y = 5$ and passing through the point (−1, 1).

17. Find the equation of the circle circumscribed about the triangle having the following equations as sides: $8x + 7y - 12 = 0$, $x + 2y - 6 = 0$, and $3x - 2y - 27 = 0$.

18. Find the equation of the circle that passes through the points (2, 3) and (7, −5) and has its center on the line $2x + 3y + 6 = 0$.

19. Find the distance between the centers of the two circles $x^2 + y^2 + 4x = 17$ and $x^2 + y^2 - 8x + 32y = 5$.

20. Find the equation of the circle that is tangent to both axes and passes through the point (6, 6).

24. THE EQUATION OF A CIRCLE IN POLAR COORDINATES

In Figure 31, let (r_1, θ_1) be the center of a circle and (r, θ) be any point on the circle, and let a be the radius.

From the law of cosines in trigonometry, we have

$$a^2 = r_1^2 + r^2 - 2r_1 r \cos(\theta_1 - \theta),$$

which is the required equation.

If, as a particular case, $r_1 = 0$, the equation merely becomes

$$r^2 = a^2 \qquad \text{or} \qquad r = a.$$

If the circle should pass through the pole, it follows that $r_1 = a$. Consequently, the equation becomes

$$a^2 = a^2 + r^2 - 2ar\cos(\theta_1 - \theta),$$

or

$$r = 2a\cos(\theta_1 - \theta).$$

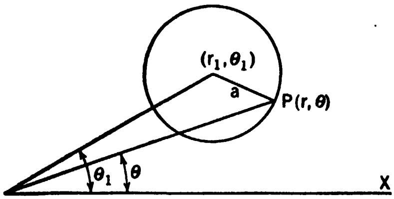


FIG. 31

Thus, the equation $r = 3$ represents a circle with center at the pole and radius equal to 3. The equation $r = 8\cos\theta$ represents a circle passing through the pole; its radius is 4, and its center is at $(4, 0°)$.

The equation $r = 4\sin\theta$ may be written $r = 4\cos(\pi/2 - \theta)$; hence, the equation represents a circle of radius 2 and with its center at $(2, \pi/2)$.

The equation $r = \cos\theta + \sin\theta$ may be written

$$r = 2\cos\frac{\pi}{4}\cos\left(\theta - \frac{\pi}{4}\right)$$

or

$$r = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right).$$

Hence, the equation represents a circle of radius $\frac{\sqrt{2}}{2}$ and with its center at $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$.

EXERCISES 14

1. Find the equation in polar coordinates of the circle with its center at $(5, 0°)$ and with a radius of 5.

2. Find the equation in polar coordinates of the circle with its center at $(5, \pi/2)$ and with a radius of 5.

3. Find the equation in polar coordinates of the circle with its center at $(5, -\pi/2)$ and with a radius of 10.

4. Write in polar coordinates the equation of the circle $x^2 + y^2 - 6x = 0$.

5. Determine the center and the radius of each of the following circles:

(*a*) $r = 7$.

(*b*) $r = 6 \cos \theta$.

(*c*) $r = 10 \sin \theta$.

(*d*) $r = 4 \cos \left(\frac{\pi}{4} - \theta \right)$.

(*e*) $r = \cos \left(\theta - \frac{\pi}{6} \right)$.

(*f*) $r = 12 \sin \left(\frac{\pi}{2} + \theta \right)$.

(*g*) $r = 3 \cos \theta + 4 \sin \theta$.

6 The Ellipse

25. THE ELLIPSE

In Book I, we defined the curve that represents the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

as an ellipse with its center at (h, k).

We shall now define an ellipse as the locus of a point that moves so that in every position the ratio of its distance from a fixed point, called the *focus*, to its distance from a fixed line, called the *directrix*, is a constant that is less than 1. The constant is called the *eccentricity* of the ellipse and is designated by e. Other important varieties of curves result when the eccentricity e, defined in the same manner, is equal to 1 or is greater than 1. These latter cases will be treated in succeeding chapters.

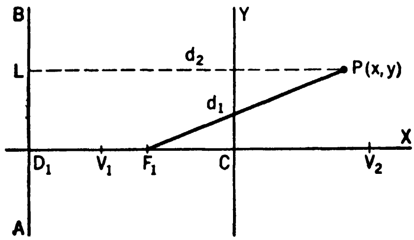


FIG. 32

In Figure 32, we have chosen F_1 as the focus and AB as the directrix. As a convenience, the focus F_1 has been located on the x axis, and the directrix AB has been taken parallel to the y axis; this does not destroy the generality of the approach, for, irrespective of the given positions of the directrix and focus, an axis system may be inserted so that we have the conditions involved in the figure.

Let $P(x, y)$ be any point on the locus. Moreover, if V_1 is a point that divides D_1F_1 so that $V_1F_1/D_1V_1 = e$, then V_1 is a point on the locus. Point V_2 is also on the locus, for it has been located so that $F_1V_2/D_1V_2 = e$.

Let $V_1V_2 = 2a$, and let C be the mid-point of V_1V_2; in other words,

the y axis has been inserted so that it bisects V_1V_2. From the previous ratios, we have

$$V_1F_1 = eD_1V_1 \tag{1}$$

and

$$F_1V_2 = eD_1V_2. \tag{2}$$

After adding the members of Equations (1) and (2), we obtain

$$V_1F_1 + F_1V_2 = e(D_1V_1 + D_1V_2). \tag{3}$$

From the figure,

$$V_1F_1 + F_1V_2 = 2a,$$

$$D_1V_1 = D_1C - a,$$

$$D_1V_2 = D_1C + a.$$

Substituting these values in Equation (3), we have

$$2a = e(D_1C - a + D_1C + a) = 2eD_1C.$$

Hence,

$$D_1C = a/e. \tag{4}$$

After subtracting the members of Equation (1) from those of (2), we obtain

$$F_1V_2 - V_1F_1 = e(D_1V_2 - D_1V_1). \tag{5}$$

From the figure,

$$F_1V_2 = F_1C + a,$$

$$V_1F_1 = a - F_1C,$$

$$D_1V_2 - D_1V_1 = 2a.$$

Substituting these values in Equation (5), we obtain

$$2F_1C = 2ae,$$

or

$$F_1C = ae. \tag{6}$$

As a result of these considerations, it is apparent that the coordinates of F_1 are $(-ae, 0)$, and the directrix AB has the equation $x = -a/e$. Hence, by using the distance formula,

$$d_1 = F_1P = \sqrt{(x + ae)^2 + y^2}, \qquad \text{and} \qquad d_2 = LP = x + \frac{a}{e}.$$

Since $d_1/d_2 = e$, we have

$$d_1 = ed_2$$

or

$$\sqrt{(x + ae)^2 + y^2} = e\left(x + \frac{a}{e}\right) = ex + a.$$

The square of each member of this equation yields

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2,$$

or

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

After dividing each member by the right member, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{7}$$

If $y = 0$, $x = \pm a$, the x intercepts; and if $x = 0$, $y = \pm a\sqrt{1 - e^2}$, the y intercepts.

Let us designate the y intercepts by $\pm b$; that is, we shall let

$$b^2 = a^2(1 - e^2). \tag{8}$$

An immediate consequence of this relation is that $b < a$. With this change, Equation (7) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{9}$$

This is said to be the standard form of the equation of an ellipse.

From Equation (9), we see that the curve is symmetrical with respect to both axes and to the origin. Moreover, if we write the equation in the forms

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \tag{10}$$

and

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}, \tag{11}$$

we observe that the curve is within $-a \leqq x \leqq a$, $-b \leqq y \leqq b$.

The shape of the curve is displayed in Figure 33.

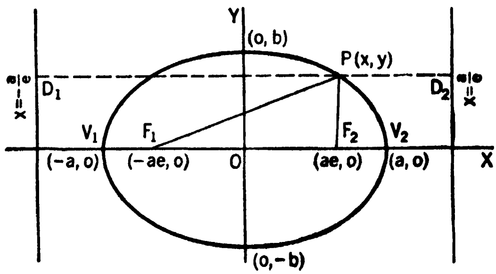


Fig. 33

From the symmetry of the curve, we note that the same graph is obtained when the focus is located at $(ae, 0)$ and its corresponding directrix is taken as the line $x = a/e$. The possible second focus $(ae, 0)$ will be designated as F_2.

Definitions: The chord V_1V_2 through the foci is called the *major axis.* The chord through the center, perpendicular to the major axis, is called the *minor axis.* The lines F_1P and F_2P, when P is any point on the ellipse,

are called *focal radii*. The chord through a focus, perpendicular to the major axis, is called a *latus rectum*.

The presumption throughout the analysis has been that the major axis is the longer of the two axes. If, however, the major axis is on the y axis, we shall still represent our curve by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

but in this case the semimajor axis is b, and the semiminor axis is a. Hence, under this circumstance, equation (8) is replaced by

$$a^2 = b^2(1 - e^2).$$

Moreover, the equations of the directrices are $y = \pm b/e$, and the coordinates of the foci are $(0, -be)$ and $(0, be)$.

Properties of the Ellipse: (1) The length of the latus rectum is $2b^2/a$, if $a > b$.

If we let $x = ae$ in Equation (7), we have

$$\frac{a^2e^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

from which we obtain

$$y^2 = a^2(1 - e^2)^2 = \frac{b^4}{a^2}$$

or

$$y = \pm\frac{b^2}{a}.$$

Hence, $|2y|$, the actual length of the latus rectum, is $2b^2/a$.

If $a < b$, the length of the latus rectum is $2a^2/b$.

(2) The sum of the two focal radii to any point on the ellipse is $2a$, if $a > b$.

From the definition of an ellipse,

$$\frac{F_1P}{D_1P} = e \quad \text{and} \quad \frac{F_2P}{PD_2} = e.$$

Therefore, $\quad F_1P = eD_1P \quad$ and $\quad F_2P = ePD_2.$

Adding, we have

$$F_1P + F_2P = e(D_1P + PD_2) = eD_1D_2 = e\left(\frac{2a}{e}\right) = 2a.$$

If $a < b$, the sum of the two focal radii is $2b$.

Illustration 1: Consider the equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

It is observed immediately that $a = 5$ and $b = 4$. So the equation represents an ellipse of major axis 10 and minor axis 8. Since $b^2 = a^2(1 - e^2)$, it follows that $16 = 25(1 - e^2)$, and $e = \frac{3}{5}$. Thus, $ae = 3$, and F_1 is located at $(-3, 0)$ and F_2 is at $(3, 0)$. The directrices have the equations $x = \pm\frac{25}{3}$. The curve is shown in Figure 34.

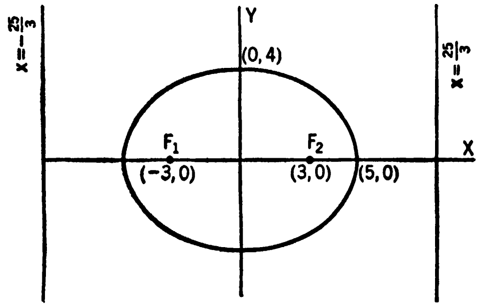


FIG. 34

Illustration 2: Study the equation

$$\frac{x^2}{16} + \frac{y^2}{25} = 1.$$

This equation represents an ellipse with its major axis on the y axis; in fact, $a = 4$ and $b = 5$. Hence, $a^2 = b^2(1 - e^2)$, which becomes $16 = 25(1 - e^2)$; so, $e = \frac{3}{5}$.

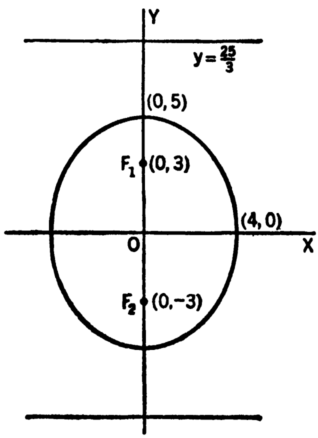


FIG. 35

The foci are at $(0, 3)$ and $(0, -3)$, and the equations of the directrices are $y = \pm\frac{25}{3}$ (note Figure 35).

Illustration 3: If $e = \frac{1}{3}$, the major axis (located on the x axis) is 12, and the center is at $(0, 0)$, it follows that

$$b^2 = 36(1 - \tfrac{1}{9}) = 32.$$

Hence, the desired equation is

$$\frac{x^2}{36} + \frac{y^2}{32} = 1.$$

Illustration 4: If $e = \frac{1}{3}$, and the major axis (located on the y axis) is 12, and the center is at $(0, 0)$, the desired equation is

$$\frac{x^2}{32} + \frac{y^2}{36} = 1.$$

EXERCISES 15

Find the equation of the ellipse, with center at the origin, that satisfies each of the following sets of conditions. Sketch each curve.

1. $e = \frac{1}{2}$, and the equations of the directrices are $x = \pm 8$.
2. $e = \frac{3}{4}$, foci on the y axis, and the ellipse passes through the point $(3, 4)$.
3. One vertex is at $(-6, 0)$ and the corresponding focus is at $(-4, 0)$.
4. One focus is at $(4, 0)$, and length of the latus rectum through this focus is 3.6.
5. Major axis is 16 units, and the coordinates of one focus are $(5, 0)$.
6. Major axis is 20 units, and the coordinates of one focus are $(0, -8)$.
7. Minor axis is $2\sqrt{5}$, and the coordinates of one focus are $(0, \sqrt{3})$.
8. One focus is at $(0, 2\sqrt{2})$, and the equations of the directrices are $y = \pm 6\sqrt{6}$.
9. Find the eccentricity, the equations of the directrices, and the coordinates of the foci for each of the following ellipses:

$$(a)\ \frac{x^2}{36} + \frac{y^2}{16} = 1; \quad (b)\ \frac{x^2}{9} + \frac{y^2}{25} = 1; \quad (c)\ 5x^2 + y^2 = 25.$$

10. The vertical dimension of a rectangle inscribed in the ellipse $x^2/25 + y^2/9 = 1$ is $2\sqrt{5}$. Determine the area of the rectangle.
11. How large a square can be inscribed in the ellipse $x^2/9 + y^2/25 = 1$?
12. If, as in Figure 36, we take a line $AP = a$ and a point B a distance b from P and then revolve the line in the plane so that A slides on the y axis and B slides on the x axis, P will describe a curve. Prove that the locus of P is the ellipse $x^2/a^2 + y^2/b^2 = 1$.

NOTE: An ellipse may be constructed mechanically by this method.

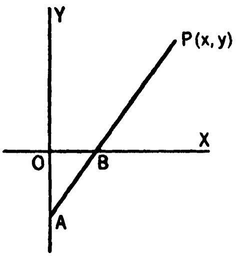


FIG. 36

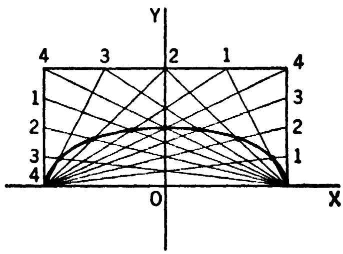


FIG. 37

13. If we have a rectangle whose base is $2a$ and whose altitude is $2b$, and if we divide the sides into the same number of equal parts, as in Figure 37, prove that the intersections of lines through corresponding points of division lie on the ellipse $x^2/a^2 + y^2/b^2 = 1$. The axes are to be taken as shown in Figure 37.
14. Describe a method of drawing an ellipse by use of Property (2), using a string $2a$ units long and two thumb tacks. Use your method to construct an ellipse with major axis 10 and minor axis 6.
15. Show that if $a = b$ in the equation $x^2/a^2 + y^2/b^2 = 1$, the locus is a circle. By employing the formula $b^2 = a^2(1 - e^2)$, show that the eccentricity of the circle is zero.

16. Find the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in polar coordinates.

17. The area of an ellipse is given by the formula πab. Determine the area of the ellipse having the major axis 10 and the eccentricity $\frac{4}{5}$.

18. A sound emanating from one focus within an ellipsoidal chamber is reflected from the wall in such a manner that it passes through the other focus. If an ellipsoidal chamber is generated by revolving the ellipse $x^2/625 + y^2/144 = 1$ about the x axis, and if a sound ray emanates from one focus, how far will it travel before it returns to the same focus?

26. THE ELLIPSE AND THE GENERAL QUADRATIC EQUATION

The equation

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0 \qquad (A > 0,\ C > 0),$$

may be written in the form

$$A\left(x^2 + \frac{2D}{A}x + \frac{D^2}{A^2}\right) + C\left(y^2 + \frac{2E}{C}y + \frac{E^2}{C^2}\right) = -F + \frac{D^2}{A} + \frac{E^2}{C}.$$

If

$$-F + \frac{D^2}{A} + \frac{E^2}{C} = \frac{D^2C + E^2A - FAC}{AC} = G \neq 0,$$

the equation may be written in the form

$$\frac{\left(x + \frac{D}{A}\right)^2}{\frac{G}{A}} + \frac{\left(y + \frac{E}{C}\right)^2}{\frac{G}{C}} = 1. \tag{1}$$

If we make the transformations

$$x' = x + \frac{D}{A} \qquad \text{and} \qquad y' = y + \frac{E}{C},$$

it is equivalent to setting up a new axis system in terms of x' and y'. In fact, the origin in the new system has the coordinates $(-D/a, -E/C)$ relative to the old axes. With respect to the new axis system, Equation (1) becomes

$$\frac{x'^2}{\frac{G}{A}} + \frac{y'^2}{\frac{G}{C}} = 1. \tag{2}$$

Therefore, if the numerator of G, namely, $D^2C + E^2A - FAC > 0$, Equation (1) represents an ellipse with center at $(-D/A, -E/C)$.

If $D^2C + E^2A - FAC = 0$, Equation (1) merely represents the point $(-D/A, -E/C)$.

If $D^2C + E^2A - FAC < 0$, Equation (1) does not represent a real locus.

If $D^2C + E^2A - FAC > 0$, we may let

$$\frac{G}{A} = a^2,$$

$$\frac{G}{C} = b^2,$$

and

$$-\frac{D}{A} = h \quad \text{and} \quad -\frac{E}{C} = k.$$

Then Equation (1) may be written

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{3}$$

This equation, then, is the typical form of the equation of an ellipse with semiaxes a and b and with its center at (h, k).

Illustration: The equation

$$x^2 + 2x + 4y^2 + 8y - 31 = 0$$

may be written

$$(x^2 + 2x + 1) + 4(y^2 + 2y + 1) = 36$$

or

$$(x + 1)^2 + 4(y + 1)^2 = 36.$$

After dividing each member of this equation by 36, there results

$$\frac{(x+1)^2}{36} + \frac{(y+1)^2}{9} = 1.$$

So we see that the curve of $x^2 + 2x + 4y^2 + 8y - 31 = 0$ is an ellipse, with its center $0'$ at $(-1, -1)$. The major axis is 12 units long and is parallel to the x axis; the minor axis is 6 units long and is parallel to the y axis A sketch of the curve appears as Figure 38.

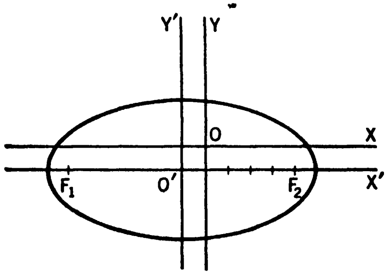


FIG. 38

Obviously, the eccentricity of the ellipse is independent of the location of the axes. Since $9 = 36(1 - e^2)$, it follows that

$$e = \frac{\sqrt{3}}{2}.$$

The foci F_1 and F_2 are on the major axis a distance $ae = 6(\sqrt{3}/2) = 3\sqrt{3}$ to the left and to the right of $0'$. So the coordinates of F_1 are $(-3\sqrt{3} - 1, -1)$ and the coordinates of F_2 are $(3\sqrt{3} - 1, -1)$. The directrices are $x = \pm 4\sqrt{3} - 1$.

EXERCISES 16

Find the equation of the ellipse determined by each of the following sets of conditions. Sketch each curve.

1. Center is at $(5, -3)$, one focus is at $(8, -3)$, and the eccentricity is $\frac{1}{2}$.

2. Center is at $(5, -3)$, one vertex is at $(5, 2)$, and the eccentricity is $\frac{3}{5}$.

3. Foci are at $(2, 12)$ and $(2, 6)$, and one vertex is at $(2, 4)$.

4. Eccentricity is $\frac{1}{2}$, center is at $(3, 4)$, and major axis is parallel to the x axis and equal to 10.

5. Eccentricity is $\frac{1}{2}$, center is $(3, 4)$, and major axis is parallel to the y axis and equal to 10.

6. One focus is at the origin, the equation of the corresponding directrix is $x = 9$, and the eccentricity is $\frac{1}{2}$.

7. Center is at $(5, 0)$, the origin is at a vertex, and the curve passes through the point $(2, 1)$. Find e, the coordinates of the foci, and the length of the latus rectum.

Show that each of the following equations represents an ellipse. For each curve, find the center, the semimajor and semiminor axes, the coordinates of the foci, and the equations of the directrices.

8. $4x^2 + y^2 - 8x + 4y + 7 = 0$.

9. $x^2 + 5y^2 - 10y = 20$.

10. $16x^2 + y^2 - 64x + 4y + 19 = 0$.

11. $2x^2 - 12x + 4y^2 + 8y - 78 = 0$.

12. $4x^2 + 16x + y^2 - 2y = 83$.

13. $6x^2 + 2y^2 - 12y - 270 = 0$.

14. Find the locus of a point that moves so that in every position the ratio of its distance from the point $(-1, 0)$ to its distance from the line $x = 5$ is $\frac{3}{5}$. Show that the locus is an ellipse, and find its center and semiaxes.

15. (*a*) Derive the equation of an ellipse in polar coordinates if one focus is at the pole $e = \frac{1}{2}$ and the directrix is 6 units to the right of the pole.
(*b*) Derive the equation of an ellipse in polar coordinates if one focus is at the pole $e = \frac{1}{2}$ and the directrix is 6 units to the left of the pole.
(*c*) Derive the equation of an ellipse in polar coordinates if one focus is at the pole $e = \frac{1}{2}$ and the directrix is 6 units above the pole.

16. Change the equation obtained in Exercise 15*a* to rectangular coordinates.

17. Find the equation of an ellipse in polar coordinates if one focus is at the pole, the eccentricity is e, and the directrix is perpendicular to the horizontal axis and at a distance k to the left of the focus.

18. Determine the distance from the point $(0, b)$ to either focus of the ellipse $x^2/a^2 + y^2/b^2 = 1$, where a is the semimajor axis.

7 The Hyperbola

27. THE HYPERBOLA

In Book I, we described the curves representing the equations

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{and} \quad -\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

as hyperbolas; in each case the center is located at (h, k).

To be more thorough in our study, we shall define a hyperbola as the locus of a point that moves so that in every position the ratio of its distance from a fixed point, called the *focus*, to its distance from a fixed line, called the *directrix*, is a constant greater than 1. As in the case of the ellipse, the constant is called the *eccentricity* and is designated by *e*.

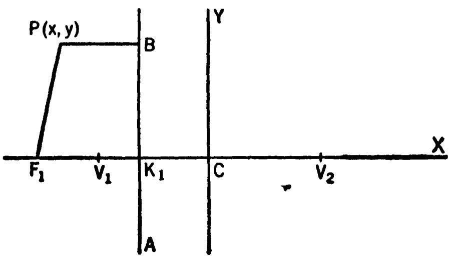


FIG. 39

The derivation of the equation of the hyperbola, based on the definition just given, closely resembles the derivation of the equation of the ellipse. Our study will be based on Figure 39. Let the line AB be the directrix, F_1 the focus, and $P(x, y)$ any point on the locus. An x axis will be inserted through F_1 and perpendicular to AB at the point K_1. Then it is evident that there are two points V_1 and V_2 on the x axis such that $F_1V_1/V_1K_1 = e$ and $F_1V_2/K_1V_2 = e$; that is, V_1 and V_2 are on the locus of the hyperbola. Let us designate V_1V_2 by $2a$ and the mid-point of V_1V_2 by C; then, $V_1C = a$, and $CV_2 = a$. Through the point C we construct the y axis perpendicular to the x axis.

From the discussion of the previous paragraph, we have

$$F_1V_1 = eV_1K_1 \tag{1}$$

and

$$F_1V_2 = eK_1V_2. \tag{2}$$

After subtracting the members of Equation (1) from those of (2), we have

$$F_1V_2 - F_1V_1 = e(K_1V_2 - V_1K_1). \tag{3}$$

By reference to Figure 39,

$$F_1V_2 - F_1V_1 = 2a,$$

$$K_1V_2 = a + K_1C,$$

and

$$V_1K_1 = a - K_1C.$$

Substituting these values in Equation (3), we have

$$2a = e[a + K_1C - (a - K_1C)],$$

or

$$K_1C = \frac{a}{e}. \tag{4}$$

After adding the corresponding members of Equations (1) and (2), we have

$$F_1V_1 + F_1V_2 = e(V_1K_1 + K_1V_2). \tag{5}$$

Again, by reference to Figure 39, we observe

$$F_1V_1 = F_1C - a,$$

$$F_1V_2 = F_1C + a,$$

and

$$V_1K_1 + K_1V_2 = 2a.$$

Substituting these values in Equation (5), we have

$$F_1C - a + F_1C + a = 2ae,$$

or

$$F_1C = ae. \tag{6}$$

Results (4) and (6) indicate, as in the case of the ellipse, that the focus F_1 is at $(-ae, 0)$ and the directrix AB has the equation $x = -a/e$. Of course, since $e > 1$, the relative positions of the focus and the directrix with respect to the origin have been changed. This was anticipated when Figure 39 was drawn.

From the definition of the hyperbola,

$$\frac{F_1P}{PB} = e.$$

This relation leads to the following succession of equations:

$$\frac{\sqrt{(x + ae)^2 + y^2}}{x + \frac{a}{e}} = e,$$

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2,$$

$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1),$$

and, finally,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1. \tag{7}$$

If $a^2(e^2 - 1)$ is denoted by b^2, Equation (7) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{8}$$

Equation (8) is symmetrical with respect to both axes and to the origin. If we solve the equation for y, we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}. \tag{9}$$

Equation (9) shows that there is no locus when $-a < x < a$, for within that range y becomes imaginary. Moreover, as x increases indefinitely in numerical value, so does y. The nature of the curve is de-

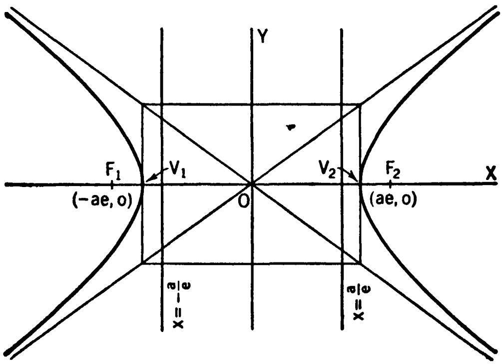


FIG. 40

picted in Figure 40. It may be observed that the graph consists of two distinct branches. From the symmetry we see, as in the case of the ellipse, that the curve has another focus at $(ae, 0)$ and another directrix $x = a/e$.

If we write Equation (9) in the form

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}, \tag{10}$$

we observe that the radical expression approaches the value 1 as x becomes numerically large. Thus, the straight lines

$$y = \frac{bx}{a} \quad \text{and} \quad y = -\frac{bx}{a}$$

become an excellent approximation to the form of the curve a long way from the origin. These two straight lines are called the *asymptotes of the curve* and serve as guide lines in its construction, as shown in Figure 40. It can be shown that the ordinates of $y = \pm(b/a)x$ are numerically larger than the ordinates given by (10), but they differ less and less as x continues to increase.

Definitions: The chord V_1V_2 through the foci is called the *axis of the hyperbola.* The lines F_1P and F_2P, when P is any point on the hyperbola, are called *focal radii.* The chord through a focus perpendicular to the axis is called a *latus rectum.*

If the axis is on the y axis, we shall represent our curve by the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

but the semiaxis is b, and in this case $a^2 = b^2(e^2 - 1)$. Also, in this case, the equations of the directrices are $x = \pm b/e$, and the coordinates of the foci are $(0, -be)$ and $(0, be)$.

Properties of the Hyperbola: (1) The length of the latus rectum of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $2(b^2/a)$, and the length of the latus

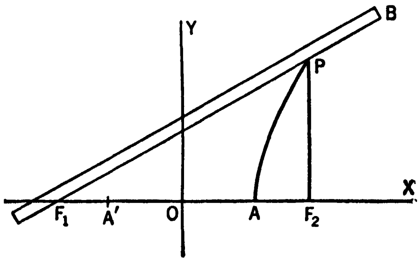


FIG. 41

rectum of $y^2/b^2 - x^2/a^2 = 1$ is $2(a^2/b)$. The student should prove this as an exercise.

(2) The difference between the two focal radii to any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $2a$, and the difference between the two

focal radii to any point on the hyperbola $y^2/b^2 - x^2/a^2 = 1$ is $2b$. It is left as an exercise for the student to establish this property.

Property (2) forms a basis for the mechanical construction of the hyperbola. The procedure follows: Take a straightedge F_1B, where $F_1B > 2a$ (note Figure 41). Fasten one end of a string of length $F_1B - 2a$ at B and the other end at the focus F_2. A pencil P held against the string and straightedge so as to keep the string taut will trace the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as the straightedge is revolved about F_1. If we reverse the position of the straightedge and string with respect to the foci, the other branch of the hyperbola may be drawn. The student should demonstrate that this result follows from Property (2).

EXERCISES 17

Find the eccentricity, the coordinates of the foci, and the equations of the asymptotes of the following hyperbolas, and sketch each curve.

1. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$
3. $x^2 - 2y^2 = 8$
4. $y^2 - x^2 = 1$
5. $2x^2 - 4y^2 = 16$
6. $24x^2 - y^2 = 144$

Find the equation of each of the following hyperbolas. The center is at the origin in each case.

7. One focus is at (5, 0), and the axis is 6.
8. One focus is at (0, 5), and the axis is 8.
9. One focus is at (5, 0), and the equation of the directrix is $x = 1$.
10. The latus rectum is $\frac{32}{3}$, and the equation of the directrix is $x = \frac{9}{5}$.
11. Determine the eccentricity of the hyperbola $x^2 - y^2 = k$, where k is any positive or negative constant.
12. The equations of the asymptotes to a hyperbola are $y = \pm\frac{5}{3}x$. Find the eccentricity.
13. A point (x, y) is joined by straight lines to the points $(-3, 0)$ and $(3, 0)$. The product of the slopes of the two lines is 2. Describe in detail the curve which the point (x, y) has for its locus.
14. Show that the curve representing the equation $x^2/a^2 - y^2/b^2 = 0$ is composed of two straight lines, which are the asymptotes of $x^2/a^2 - y^2/b^2 = 1$.
15. If a and b are given, determine the location of the asymptotes by a geometrical construction.
16. Use a straightedge and string to construct the hyperbola $x^2/25 - y^2/9 = 1$.
17. Use a straightedge and string to construct the hyperbola $y^2/9 - x^2/16 = 1$.

28. THE HYPERBOLA AND THE GENERAL QUADRATIC EQUATION

Let us consider the equation

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0 \qquad (A > 0,\ C < 0).$$

This equation may be written in the form

$$A\left(x^2 + \frac{2D}{A}x + \frac{D^2}{A^2}\right) + C\left(y^2 + \frac{2E}{C}y + \frac{E^2}{C^2}\right) = -F + \frac{D^2}{A} + \frac{E^2}{C},$$

or

$$A\left(x + \frac{D}{A}\right)^2 + C\left(y + \frac{E}{C}\right)^2 = \frac{D^2C + E^2A - FAC}{AC}.$$

If the member on the right is designated by G, and if $G \neq 0$, the equation may be written in the form

$$\frac{\left(x + \frac{D}{A}\right)^2}{\frac{G}{A}} + \frac{\left(y + \frac{E}{C}\right)^2}{\frac{G}{C}} = 1. \tag{1}$$

If the numerator of G, that is, $D^2C + E^2A - FAC = 0$, the equation becomes $A(x + D/A)^2 + C(y + E/C)^2 = 0$. Since C is negative, the left member may be regarded as the difference of two squares and factored accordingly. So the equation represents the real lines

$$\sqrt{A}\left(x + \frac{D}{A}\right) = \pm\sqrt{-C}\left(y + \frac{E}{C}\right).$$

Of course, $\sqrt{-C}$ is real, since C is negative.

If $D^2C + E^2A - FAC > 0$, then $G < 0$, and $G/A < 0$ and $G/C > 0$. If we make the transformations $x' = x + D/A$ and $y' = y + E/C$, the equation takes the form

$$\frac{y'^2}{b^2} - \frac{x'^2}{a^2} = 1,$$

where

$$b^2 = \frac{G}{C} \quad \text{and} \quad a^2 = -\frac{G}{A}.$$

This is the equation of a hyperbola with its center at the origin of the new x', y' axis system, which means that the center is located at $(-D/A, -E/C)$ with respect to the old axis system. Moreover, the axis of the hyperbola is located on the y' axis, which is parallel to the y axis.

If $D^2C + E^2A - FAC < 0$, the equation takes the form

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

which is a hyperbola with its axis on the x' axis, parallel to the x axis; in this equation,

$$a^2 = \frac{G}{A} \quad \text{and} \quad b^2 = -\frac{G}{C}.$$

We thus note that the equation

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0 \quad (A > 0,\ C < 0),$$

represents a hyperbola with its axis parallel to the x axis, a hyperbola with its axis parallel to the y axis, or two straight lines, according as $D^2C + E^2A - FAC$ is negative, positive, or zero, respectively.

If $D^2C + E^2A - FAC < 0$, Equation (1) may be written in the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

where a^2 and b^2 are defined as before, and

$$h = -\frac{D}{A} \quad \text{and} \quad k = -\frac{E}{C}.$$

If $D^2C + E^2A - FAC > 0$, Equation (1) may be written in the form

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

where a^2, b^2, h, and k have the meanings already described.

In practice, this algebraic manipulation may be carried out rather simply, as indicated by the following illustrations.

Illustration 1: Consider the curve that represents the equation

$$3x^2 - 5y^2 + 6x + 3 = 0.$$

This equation may be written

$$3(x^2 + 2x + 1) - 5y^2 = 0,$$

or

$$3(x+1)^2 - 5y^2 = 0.$$

Since the left member may be factored into

$$[\sqrt{3}(x+1) - \sqrt{5}y]\,[\sqrt{3}(x+1) + \sqrt{5}y],$$

the equation represents the two straight lines,

$$\sqrt{3}(x+1) = \pm\sqrt{5}y.$$

Illustration 2: Consider the graphical representation of the equation

$$3x^2 - 5y^2 + 6x + 18 = 0.$$

This equation may be written

$$3(x^2 + 2x + 1) - 5y^2 = -15,$$

or

$$3(x+1)^2 - 5y^2 = -15.$$

After dividing each member by -15, we obtain

$$\frac{y^2}{3} - \frac{(x+1)^2}{5} = 1.$$

This equation represents a hyperbola with its axis parallel to the y axis and its center at $(-1, 0)$.

Illustration 3: Consider the graphical representation of the equation

$$3x^2 - 5y^2 + 12x + 10y - 53 = 0.$$

The equation may be written

$$3(x^2 + 4x + 4) - 5(y^2 - 2y + 1) = 60,$$

$$3(x + 2)^2 - 5(y - 1)^2 = 60,$$

or

$$\frac{(x + 2)^2}{20} - \frac{(y - 1)^2}{12} = 1.$$

This equation represents a hyperbola with its axis parallel to the x axis and with its center at $(-2, 1)$. A sketch of the curve appears as Figure 42.

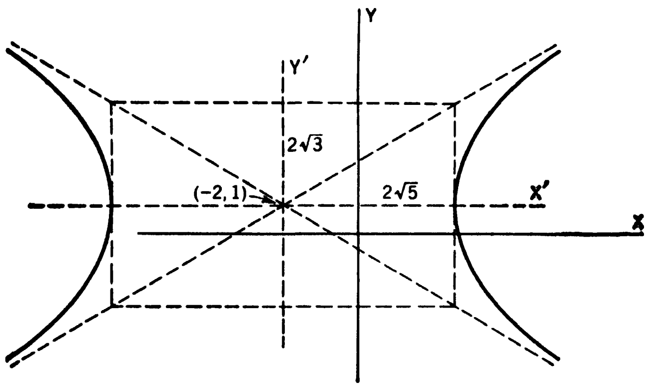


FIG. 42

In examining Figure 42, it is observed how easily the hyperbola may be sketched after constructing a rectangle of dimensions $2a$ and $2b$, with the center of the rectangle located at the center of the curve. When extended, the diagonals of the rectangle become the asymptotes of the desired hyperbola.

Of course, we could have compared the equations of the illustrations with the results previously obtained in our analysis of the general equation

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0.$$

Thus, in Illustration 1,

$$A = 3, \qquad C = -5, \qquad D = 3, \qquad E = 0, \qquad F = 3;$$

so

$$D^2C + E^2A - FAC = 0.$$

As a consequence, the curve degenerates into two intersecting straight lines.

In Illustration 2,

$$A = 3, \qquad C = -5, \qquad D = 3, \qquad E = 0, \qquad F = 18;$$

so
$$D^2C + E^2A - FAC = 24.$$

This positive result anticipates the fact that the axis of the curve is parallel to the y axis.

In Illustration 3,

$$A = 3, \qquad C = -5, \qquad D = 6, \qquad E = 5, \qquad F = -53;$$

so
$$D^2C + E^2A - FAC = -930.$$

Consequently, the axis of the curve is parallel to the x axis.

EXERCISES 18

1. Reduce each of the following equations to standard form, and find the coordinates of the center, the coordinates of the foci, and the equations of directrices and asymptotes, all with reference to the x and y axes. Sketch the curves.

(a) $25x^2 - 9y^2 - 100x - 54y = 206$
(b) $25x^2 - 9y^2 - 50x - 108y = 74$
(c) $25x^2 - 9y^2 - 50x + 108y = 299$
(d) $4x^2 - 24x - y^2 + 6y - 75 = 0$
(e) $2x^2 + 12y - 2y^2 + 4x - 29 = 0$

2. Find the equation of the hyperbola whose foci are at (2, 2) and (2, 12) and one of whose vertices is the point (2, 5).

3. The eccentricity of a hyperbola is 2, its center is at the point (2, 4), and the equation of one directrix is $x = \frac{7}{3}$. Find the equation of the hyperbola, and draw the curve.

4. A point moves so that in every position the ratio of its distance from the point (2, −1) to its distance from the line $y = 3$ is $\frac{5}{2}$. Find the equation of the locus, and sketch the curve.

5. Find the equation of a hyperbola, in polar coordinates, if its focus is at the pole and its directrix is perpendicular to the horizontal axis at the distance k to the left of the focus.

6. Find the equation of the locus of a point that moves so that the difference of its distances from (±12, 0) is 8.

7. A point moves so that its distance from (6, 0) is 5 units more than its distance from (−3, 0). Find the equation of its locus.

8. A point moves so that its distance from the origin is always twice its distance from the line $x = -10$. Determine the equation of its locus. What are the equations of its asymptotes?

9. Find the equation of the hyperbola that has vertices at (−3, −2) and (5, −2) and has one focus at (−5, −2).

10. The hyperbolas described by the equations $x^2/a^2 - y^2/b^2 = 1$ and $y^2/b^2 - x^2/a^2 = 1$ are said to be conjugate hyperbolas. Show that they have the same asymptotes.

11. Express the two asymptotes of the hyperbola $(x-h)^2/a^2 - (y-k)^2/b^2 = 1$ as a single quadratic equation in the variables x and y.

12. Determine the eccentricity of the curve $Ax^2 - Ay^2 + 2Dx + 2Ey + F = 0$, if $E^2 - D^2 + FA \neq 0$.

8 The Parabola

29. THE PARABOLA

If a point moves so that in every position the ratio of its distance from a fixed point, called the *focus*, to its distance from a fixed line, called the *directrix*, is equal to 1, the locus of the point is called a *parabola*. Thus, the eccentricity e of a parabola is always 1.

If, in Figure 43, the line AB is taken as the directrix and F as the focus, and if we insert the x axis through F perpendicular to AB at K, it is evident that there is a point V on KF, which is on the locus. From the definition of the parabola $VF/KV = 1$; that is, V is the mid-point of KF. If we let $KF = 2p$ and draw the y axis perpendicular to KF at V, and if we let $P(x, y)$ be any point on the locus, we have

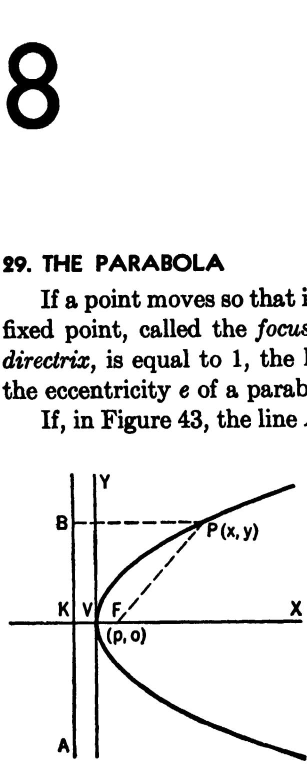


FIG. 43

$$\frac{\sqrt{(x-p)^2 + y^2}}{x+p} = 1,$$

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2,$$

or

$$y^2 = 4px. \tag{1}$$

From this equation we see that the curve of $y^2 = 4px$ is symmetrical with respect to the x axis.

It is evident that the curve of the equation

$$x^2 = 4py$$

is also a parabola, but with its focus at $(0, p)$. The directrix of this latter curve has the equation $y = -p$ (note Figure 44).

Definitions: The point V, where the line through the focus perpendicular to the directrix cuts the parabola, is called the *vertex* of the parabola. The line through the vertex and the focus of the parabola is known as the *axis* of the parabola. The chord through the focus perpendicular to the axis is called the *latus rectum*. The line joining any point of the parabola and the focus is called a *focal radius*.

30. CONSTRUCTION OF THE PARABOLA

The definition of the parabola leads to a simple method for its construction. Thus, to construct the parabola $y^2 = 4x$, draw the axes, and locate the focus at $(1, 0)$ and the directrix along $x = -1$. Then draw a collection of lines parallel to the directrix, cutting the x axis in points A, B, C, D, *etc.*, respectively. Now with F as a center and MA as a

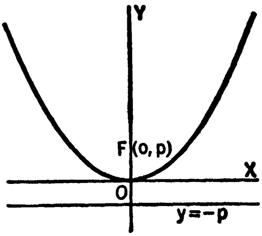


FIG. 44

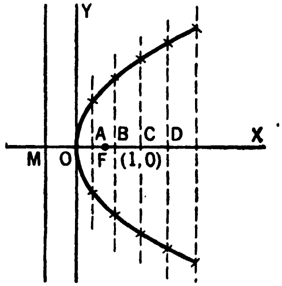


FIG. 45

radius, draw arcs cutting the line through A above and below the x axis; this gives two points of the parabola. Similarly, with F as a center and MB, MC, MD, *etc.*, as radii, draw arcs cutting the lines through B, C, D, *etc.*, respectively, above and below the axis, in each case obtaining two points of the parabola. The y axis is one of the set of parallel lines that is cut by the corresponding arc at only one point—the origin. We may thus locate as many points on the parabola as we wish and draw the parabola as shown in Figure 45. The above principle may also be used to trace the parabola mechanically by a continuously moving point, as follows: Place a straightedge along the directrix and a triangle against the straightedge as shown in Figure 46. Fasten one end of a string of length AB at B and the other end at the focus F. Now, if a pencil point is held against the string, keeping it taut and against AB, while the triangle is moved along the directrix, the pencil will trace a parabola. Why?

31. THE PARABOLA AND THE QUADRATIC EQUATION

Let us consider the equation $Cy^2 + 2Dx + 2Ey + F = 0$, $C \neq 0$. This may be rewritten in the form

$$C\left(y^2 + \frac{2E}{C}y + \frac{E^2}{C^2}\right) = -2Dx - F + \frac{E^2}{C},$$

or

$$\left(y + \frac{E}{C}\right)^2 = -\frac{2D}{C}\left(x + \frac{FC - E^2}{2DC}\right). \tag{1}$$

If $D \neq 0$ and we make the transformations

$$y' = y + \frac{E}{C} \quad \text{and} \quad x' = x + \frac{FC - E^2}{2DC},$$

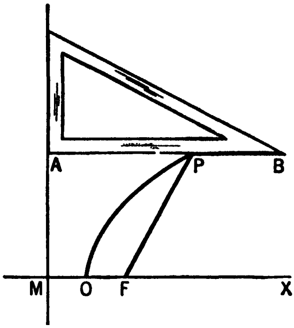


Fig. 46

the equation takes the form

$$y'^2 = 4px'.$$

This equation represents a parabola with its axis on the x' axis, which is parallel to the x axis. The vertex of the parabola is at the origin of the x', y' axis system, which corresponds to the point $[-E/C, -(FC-E^2)/2DC]$ relative to the original axes. These coordinates of the vertex are readily detected if the original equation is reduced to the form

$$(y - k)^2 = 4p(x - h),$$

where, of course,

$$k = -\frac{E}{C}, \qquad h = -\frac{FC - E^2}{2DC}, \qquad \text{and} \qquad p = -\frac{D}{2C}.$$

If $D = 0$, the original equation becomes

$$Cy^2 + 2Ey + F = 0.$$

The solution of this quadratic equation in y yields

$$y = \frac{-2E \pm \sqrt{4E^2 - 4CF}}{2C},$$

or

$$y = \frac{-E \pm \sqrt{E^2 - FC}}{C}.$$

If $E^2 - FC = 0$, this result represents the line

$$y = -\frac{E}{C},$$

usually referred to as *two coincident lines.*

If $E^2 - FC < 0$, the locus is imaginary.

If $E^2 - FC > 0$, we have two lines parallel to the x axis.

Following a similar analysis, the equation

$$Ax^2 + 2Dx + 2Ey + F = 0 \qquad (A \neq 0),$$

may be written

$$A\left(x^2 + \frac{2D}{A}x + \frac{D^2}{A^2}\right) = -2Ey - F + \frac{D^2}{A},$$

or

$$\left(x + \frac{D}{A}\right)^2 = -\frac{2E}{A}\left(y + \frac{FA - D^2}{2EA}\right). \tag{2}$$

If $E \neq 0$, it is apparent that the equation has been expressed in the

form

$$(x - h)^2 = 4p(y - k),$$

where $h = -\dfrac{D}{A}$, $\quad k = -\dfrac{FA - D^2}{2EA}$, $\quad$ and $\quad p = -\dfrac{E}{2A}$.

The point (h, k) is the vertex of the parabola, and the axis of the parabola is parallel to the y axis.

If $E = 0$ and $D^2 - FA = 0$, the original equation represents two coincident lines.

If $E = 0$ and $D^2 - FA > 0$, the original equation represents two lines parallel to the y axis.

If $E = 0$ and $D^2 - FA < 0$, the original equation represents an imaginary locus.

We thus note that the quadratic equation

$$Cy^2 + 2Dx + 2Ey + F = 0,$$

or

$$Ax^2 + 2Dx + 2Ey + F = 0,$$

represents a parabola, two parallel lines, two coincident lines, or an imaginary locus, depending on the conditions above. In practice, the actual examination of the curve representing the given equation can be carried out through the application of simple algebraic procedures. The following illustration typifies the method.

Illustration: Consider the equation

$$y^2 + 4x + 4y - 8 = 0.$$

This equation may be written

$$y^2 + 4y + 4 = -4x + 12,$$

or

$$(y + 2)^2 = -4(x - 3).$$

Hence, the equation represents a parabola with its vertex at $(3, -2)$. The axis is the line passing through this vertex, parallel to the x axis; so it is the line $y = -2$. Since $4p = -4$, it follows that $p = -1$, which means that the focus is 1 to the left of the vertex; this gives the point $(2, -2)$. The directrix is perpendicular to the axis of the parabola and, in this case, must be 1 to the right of the vertex, so it is the line $x = 4$.

EXERCISES 19

1. Construct the parabola $y^2 = 8x$ by the method of Section 30.

2. Construct the parabola $x^2 = 6y$ by the method of Section 30.

3. Find the length of the latus rectum of the parabola $y^2 = 4px$.

4. Find the equation of a parabola of vertical axis with its vertex at $(2, 3)$ whose latus rectum is 4 units. Note the result of Exercise 3.

5. Write the equation of a parabola whose focus is at the point $(6, 0)$ and whose directrix is the line $x = -6$.

6. Determine the coordinates of the focus and the equation of the directrix for each of the following parabolas:

(a) $y^2 = 8x$ (b) $y^2 = 12x$ (c) $x^2 = 10y$
(d) $y^2 = 9x$ (e) $x^2 = 12y$ (f) $y^2 = 5x$

7. Determine the coordinates of the vertex, the coordinates of the focus, and the equation of the directrix for each of the following parabolas:

(a) $y^2 = 8(x - 3)$ (b) $(y + 2)^2 = 12(x - 5)$ (c) $(x - 1)^2 = 6y$

8. Write the equation of a parabola whose focus is at (6, 0) and whose directrix is the line $x = 3$.

NOTE: The vertex is midway between the directrix and focus.

9. Write the equation of a parabola whose focus is at the point (0, 5) and whose directrix is the line $y = -5$.

10. Find the equation of a parabola whose focus is the point (2, 3) and whose directrix is the line $x = 9$.

11. Find the equation of a parabola whose axis is parallel to the x axis, whose vertex is at the point (2, 3), and which passes through the point (6, −3).

12. Find the equation of a parabola whose axis is parallel to the x axis and which passes through the three points (1, 1), (2, 3) and (3, −5).

13. Find the equation of the parabola whose focus is the point (7, 0) and whose directrix is the line $x = 2$.

14. Find the equation of the parabola whose focus is the point (7, 0) and whose directrix is the line $x = 10$.

15. Construct the parabola whose equation is $x^2 = 9y$.

16. It is desired to construct a parabolic searchlight which will be 36 in. across the front and 18 in. deep. Determine the equation of a cross section that contains the axis of the parabolic surface if this axis is made to coincide with the x axis and the vertex is located at the origin.

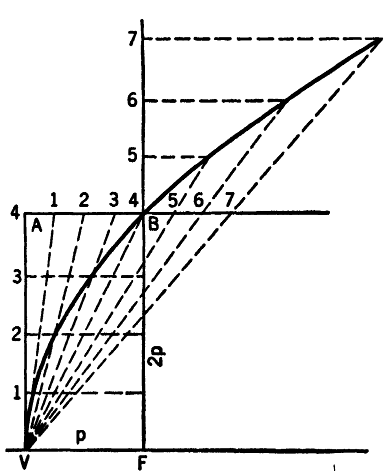


FIG. 47

17. A concrete arch for a bridge to span a distance of 40 ft is to be constructed in the form of a parabola. The highest point of the arch is 15 ft above the piers. Construct a form for the arch, using a scale 1 in. = 4 ft. Find the height, above the level of the piers, of a point on the arch which is horizontally 5 ft from one pier.

18. (a) As in Figure 47, draw a rectangle with base p and altitude $2p$, and divide the upper base and altitude into the same number of equal parts. Then through the points of division of the altitude draw lines parallel to the base. Also draw radial lines connecting V with points of division of the upper base, as shown in Figure 47. Prove that the intersections of the horizontal lines and radial lines through corresponding points of divisions lie on the parabola $y^2 = 4px$.

(*b*) Extend FB through B, and lay off any number of units each equal to $V1$; also extend AB and lay off the same number of units each equal to $A1$. Show that the intersections of horizontal lines through the new division points on FB with the corresponding radial lines through the new division points on AB lie on the parabola $y^2 = 4px$.

19. Use the method of Exercise 18 to draw the parabola $y^2 = 4x$ from $x = 0$ to $x = 6$.

20. Simplify each of the following equations by translating the axes, and determine the locus of each:

(*a*) $2y^2 - 3x + 14y + 44 = 0$

(*b*) $3x^2 - 6x + 5y - 7 = 0$

(*c*) $5y^2 + 10y - 14x = 0$

(*d*) $4x^2 - 16x + 5y = 2$

(*e*) $3y^2 + 2x - 7y = 13$

(*f*) $2x^2 - 3x + 4y - 9 = 0$

(*g*) $3x^2 - 6x - 7 = 0$

(*h*) $4y^2 - 6y + 9 = 0$

9 The General Equation of the Second Degree

32. ROTATION OF AXES

If a curve is given relative to a set of rectangular axes OX and OY, it is sometimes desirable to know the equation of the curve with respect to a set of rectangular axes OX' and OY', where the angle from OX to OX' is θ (see Figure 48).

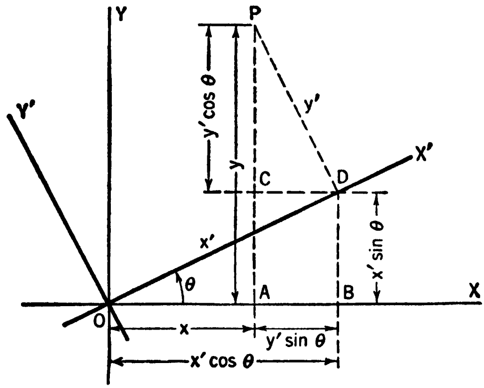


Fig. 48

Let P be any point whose coordinates are (x, y) relative to the original axes and (x', y') relative to the new axes. We draw AP perpendicular to OX, DP perpendicular to OX', and CD parallel to OX. Then, as indicated in the figure, $OA = x$, $AP = y$, $OD = x'$, and $DP = y'$. It is easily shown by elementary geometry that $\angle CPD = \theta$.

It follows that $OB = x' \cos\theta$, $CP = y' \cos\theta$, $CD = y' \sin\theta$, $BD = x' \sin\theta$. Hence, we observe that

$$x = x' \cos\theta - y' \sin\theta \tag{1}$$

and

$$y = x' \sin\theta + y' \cos\theta. \tag{2}$$

Equations (1) and (2) are known as the transformations for rotating the axis.

We note that if we rotate the OX' and OY' axes back to the OX and OY positions, we should have

$$x' = x \cos(-\theta) - y \sin(-\theta) = x \cos\theta + y \sin\theta \tag{3}$$

and $$y' = x \sin(-\theta) + y \cos(-\theta) = -x \sin\theta + y \cos\theta. \tag{4}$$

Equations (3) and (4) may also be obtained by solving (1) and (2) for x and y. Thus, if we multiply the members of Equation (1) by $\cos\theta$ and those of (2) by $\sin\theta$, we have

$$x \cos\theta = x' \cos^2\theta - y' \sin\theta \cos\theta$$

and $$y \sin\theta = x' \sin^2\theta + y' \sin\theta \cos\theta.$$

After adding the corresponding members, we have

$$x \cos\theta + y \sin\theta = x'(\cos^2\theta + \sin^2\theta) = x'.$$

This is Equation (3).

If we multiply the members of Equation (1) by $\sin\theta$ and those of (2) by $\cos\theta$, we have

$$x \sin\theta = x' \sin\theta \cos\theta - y' \sin^2\theta$$

and $$y \cos\theta = x' \sin\theta \cos\theta + y' \cos^2\theta.$$

After combining these equations by subtraction, we obtain

$$y \cos\theta - x \sin\theta = y'(\cos^2\theta + \sin^2\theta) = y'.$$

This is Equation (4).

The utility of these transformation relations is immediately apparent. If the equation of a curve is $f(x, y) = 0$, relative to the axes OX and OY, then the equation of the curve relative to OX' and OY' is obtained by substituting $x' \cos\theta - y' \sin\theta$ for x, and $x' \sin\theta + y' \cos\theta$ for y in the equation $f(x, y) = 0$. It is frequently desirable to "rotate the axes" through 45°. If $\theta = 45°$, the transformation relations become

$$x = x' \cos 45° - y' \sin 45° = \frac{x' - y'}{\sqrt{2}},$$

$$y = x' \sin 45° + y' \cos 45° = \frac{x' + y'}{\sqrt{2}}.$$

Illustration: If the equation of a curve is

$$3x^2 + 2xy + 3y^2 = 1,$$

the equation of the curve relative to OX' and OY', where the angle from OX to OX' is 45°, is

$$3\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 3\left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 1,$$

or $$3\left(\frac{x'^2 - 2x'y' + y'^2}{2}\right) + 2\left(\frac{x'^2 - y'^2}{2}\right) + 3\left(\frac{x'^2 + 2x'y' + y'^2}{2}\right) = 1,$$

which may be simplified to

$$4x'^2 + 2y'^2 = 1,$$

or $$\frac{x'^2}{\frac{1}{4}} + \frac{y'^2}{\frac{1}{2}} = 1.$$

We now see that the curve is an ellipse with its major axis on OY' and its minor axis on OX', as displayed in Figure 49.

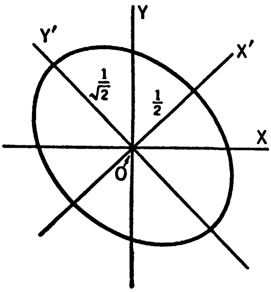


FIG. 49

It is now a simple matter to determine all the characteristics of the ellipse in terms of x' and y' and, by (1) and (2), or (3) and (4), express them in terms of x and y.

Thus, $a^2 = b^2(1 - e^2)$, where $a^2 = \frac{1}{4}$ and $b^2 = \frac{1}{2}$. Hence,

$$e = \frac{1}{\sqrt{2}}.$$

The foci are at $(0, \pm be)$, that is, $(0, \pm\frac{1}{2})$, relative to the x', y' axes. From Relations (1) and (2), the corresponding x and y coordinates are readily obtained. Thus, if $x' = 0$ and $y' = \frac{1}{2}$, it follows that

$$x = (0)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4},$$

and $$y = (0)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}. \tag{2}$$

A similar computation may be made when $x' = 0$ and $y' = \frac{1}{2}$. Hence, the foci are at $(-\sqrt{2}/4, \sqrt{2}/4)$ and at $(\sqrt{2}/4, -\sqrt{2}/4)$, relative to the old axes.

The equations of the directrices relative to OX' and OY' are

$$y' = \pm\frac{b}{e} = \pm\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \pm 1.$$

Hence, if $y' = 1$, we have from relation (4):

$$1 = \frac{-x + y}{\sqrt{2}} \quad \text{or} \quad y = x + \sqrt{2}.$$

If $y' = -1$, we have

$$-1 = \frac{-x + y}{\sqrt{2}} \quad \text{or} \quad y = x - \sqrt{2}.$$

These are the equations of the directrices relative to the OX and OY axes.

EXERCISES 20

Determine the equation of each of the following curves when the axis system is rotated through the angle specified:

1. $2x - 5y + 6 = 0;\ \theta = 45°$
2. $7x + 2y - 3 = 0;\ \theta = 30°$
3. $3x - 5y = 7;\ \theta = 60°$
4. $y^2 = 4x;\ \theta = 90°$
5. $x^2 + y^2 = 36;\ \theta = 30°$ (Explain your result)
6. $x^2 - y^2 = 5;\ \theta = 45°$
7. $xy = 6;\ \theta = 45°$ (Determine the eccentricity of the curve)
8. $(3x + 4y)^2 + 7x = 0;\ \theta = \tan^{-1} \frac{4}{3}$. What is the eccentricity of the curve?
9. $4x^2 + 2\sqrt{3}xy + 2y^2 = 9;\ \theta = 30°$. Determine the eccentricity of the curve.
10. $2x^2 + 2xy + 2y^2 = 5;\ \theta = 45°$. Determine the length of the major axis of the curve.
11. $x^2 - 2xy + y^2 + 6x = 0;\ \theta = 45°$
12. $x^2 - y^2 = 0;\ \theta = 45°$

33. THE EFFECT OF ROTATION OF AXES ON DEGREE OF EQUATION

The degree of an equation is not altered by the rotation of the axes. This may be seen from the fact that the transformation equations (1) and (2) are of first degree, and, hence, the equation of the curve in terms of x' and y' will not be higher than the degree of the original equation. The degree of the equation of the curve cannot be lowered, for if it were lowered, then by returning to the original equation through the transformation equations (3) and (4) the degree of the equation would have to be raised, which is impossible.

34. THE GENERAL EQUATION OF THE SECOND DEGREE

The general equation of the second degree may be written in the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

If $B = 0$, the equation reduces to some form already considered in the previous chapters, and the nature of the locus may be determined from previous considerations.

We therefore assume in the following discussion that $B \neq 0$. If we

rotate the axes through an angle θ, the equation relative to OX' and OY' is

$$A(x' \cos \theta - y' \sin \theta)^2 + 2B(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + C(x' \sin \theta + y' \cos \theta)^2 + 2D(x' \cos \theta - y' \sin \theta) + 2E(x' \sin \theta + y' \cos \theta) + F = 0. \quad (1)$$

If (1) is expanded, the coefficient of $x'y'$ is

$$2B(\cos^2 \theta - \sin^2 \theta) + 2(C - A) \sin \theta \cos \theta.$$

As a consequence of well-known trigonometric relations, this coefficient may be written

$$2B \cos 2\theta - (A - C) \sin 2\theta.$$

Hence, if we choose θ so that

$$2B \cos 2\theta - (A - C) \sin 2\theta = 0, \quad (2)$$

which means that

$$\tan 2\theta = \frac{2B}{A - C}, \qquad A \neq C, \quad (2a)$$

Equation (1) will result in an equation of the second degree without an $x'y'$ term. In other words, the rotation of the axes through θ determined by Equation (2) transforms the general equation of the second degree involving an xy term, and where $A \neq C$, to a new second-degree equation in x' and y' which does not involve an $x'y'$ term. Under this new form, the equation may be classified under the cases already considered in previous chapters, and the nature of the locus may be determined from previous considerations.

If $A = C$, Equation (2) yields the result

$$2B \cos 2\theta = 0,$$

or

$$2\theta = 90°, \qquad \theta = 45°.$$

In the illustration of Section 32 we considered the equation,

$$3x^2 + 2xy + 3y^2 = 1.$$

Here $A = C$; so by the result just obtained, $\theta = 45°$, and the rotation of the axes through 45° will result in a new equation without the $x'y'$ term. This is precisely the angle that we chose for the illustration.

To consider a case when $A \neq C$, let us analyze the equation

$$9x^2 - 24xy + 16y^2 + 10x = 0.$$

Here, $A = 9$, $B = -12$, and $C = 16$. Thus, the angle through which the axes may be rotated to eliminate the $x'y'$ term is given by Relation (2a). In fact,

$$\tan 2\theta = \frac{-24}{9 - 16} = \frac{24}{7}.$$

We have from trigonometry

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \tag{3}$$

Consequently,

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{24}{7},$$

$$12 \tan^2 \theta + 7 \tan \theta - 12 = 0,$$

and $\tan \theta = \frac{3}{4}$ or $-\frac{4}{3}$.

If the first value is chosen, $\sin \theta = \frac{3}{5}$ and $\cos \theta = \frac{4}{5}$. For these values of $\sin \theta$ and $\cos \theta$, by transformation equations (1) and (2) of Section 32,

$$x = \frac{4x' - 3y'}{5},$$

$$y = \frac{3x' + 4y'}{5}.$$

After substituting for x and y in the given equation, we obtain

$$25y'^2 - 6y' + 8x' = 0.$$

This is the equation of a parabola, which may be studied by methods already outlined.

EXERCISES 21

Simplify each of the following equations by rotating the axes so as to eliminate the $x'y'$ term. Draw the various axes and the curve corresponding to each equation.

1. $xy = 7$
2. $x^2 - 2xy + y^2 + 3x = 0$
3. $5x^2 - 2xy + 5y^2 = 12$
4. $5x^2 - 26xy + 5y^2 + 72 = 0$
5. $9x^2 + 24xy + 16y^2 - 80x + 60y = 0$
6. $7x^2 + 48xy - 7y^2 - 6x + 138y + 137 = 0$
7. $9x^2 - 12xy + 4y^2 - 18x + 12y + 34 = 0$
8. $15x^2 + 24xy + 8y^2 + 30x + 20y = 915/2$
9. $15x^2 - 24xy + 8y^2 + 30x - 20y = 35.5$
10. $5x^2 + 4xy + 2y^2 + 6\sqrt{5}x = 22.2$
11. $3x^2 - 4xy + 6y^2 + 20x + 10y = 7.5$

35. DEGENERATE LOCI

By referring to previous chapters and the considerations of this chapter, we see that the general equation of the second degree represents either an ellipse (the circle may be regarded as a particular case of an ellipse), a hyperbola, a parabola, or the possible degenerate cases of two intersecting lines, two parallel or two identical lines, only a single point, or an imaginary locus. It is possible to find criteria that may be used to determine the

nature of the locus without removing the xy term from the equation by the rotation of the axes through the required angle; these criteria are not treated in this book. In general, it is preferable in practice to remove the xy term by rotation, if such a term is present, and then determine the nature of the locus and its characteristics. However, the degenerate cases mentioned above may be detected in advance by the use of special considerations, and such detection may save time in making the desired analysis.

Suppose, in the general quadratic equation, that

$$A = C = 0 \qquad \text{and} \qquad B \neq 0;$$

then we have

$$2Bxy + 2Dx + 2Ey + F = 0,$$

or

$$xy + \frac{D}{B}x + \frac{E}{B}y + \frac{F}{2B} = 0.$$

This may be rewritten in the form

$$\left(x + \frac{E}{B}\right)\left(y + \frac{D}{B}\right) = \frac{DE}{B^2} - \frac{F}{2B} = \frac{2DE - BF}{2B^2}.$$

If $2DE - BF \neq 0$, the equation represents a hyperbola with the lines $x = -E/B$ and $y = -D/B$ as asymptotes. In fact, the curve may be subjected to the type of study already outlined.

However, if $2DE - BF = 0$, the locus consists of two straight lines, namely, $x = -E/B$ and $y = -D/B$. For example, the equation $xy - 2y - 3x + 6 = 0$ can be written $(x - 2)(y - 3) = 0$; so the desired locus is merely the pair of intersecting lines $x = 2$ and $y = 3$.

Let us now, as a more common case, consider the factorability of the general quadratic equation where $C \neq 0$, $B \neq 0$. We shall write the general equation, namely,

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{1}$$

in the form

$$Cy^2 + (2Bx + 2E)y + Ax^2 + 2Dx + F = 0 \tag{2}$$

and solve for y, thereby obtaining

$$y = -\frac{2Bx + 2E \pm 2\sqrt{(Bx + E)^2 - C(Ax^2 + 2Dx + F)}}{2C}. \tag{3}$$

The expression under the radical may be written

$$(B^2 - AC)x^2 + 2(EB - CD)x + (E^2 - CF). \tag{4}$$

This quadratic in x is a perfect square if

$$4(EB - CD)^2 - 4(B^2 - AC)(E^2 - CF) = 0.$$

This may be simplified and rewritten as

$$AE^2 + CD^2 + FB^2 - 2EBD - ACF = 0 \tag{5}$$

or, in determinant form,

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0. \tag{6}$$

If the coefficients of the given equation satisfy condition (6), it is possible for the expression under the radical in (3) to be in the form $(Lx + M)^2$ or $-(Lx + M)^2$; the second possibility follows from the fact that a change in sign of all the coefficients of quadratic (4) would still yield the same equality (5).

If the radicand in (3) is of the form $(Lx + M)^2$, then (3) is of the form

$$y = \frac{Bx + E \pm (Lx + M)}{C}; \tag{7}$$

and then (7) may represent two intersecting lines, two distinct parallel lines, or two identical lines, depending on the values of L and M.

If, however, the radicand in (3) is of the form $-(Lx + M)^2$, then (3) may be written

$$y = \frac{Bx + E \pm (Lx + M)\,i}{C}; \tag{8}$$

and then (8) may represent two identical lines, a single point, or an imaginary locus, depending on the values of L and M.

In each of the following equations condition (6) is satisfied:

$$x^2 - 2xy + y^2 + 2x - 2y + 5 = 0. \tag{a}$$

$$6x^2 - 2xy + y^2 + 2x - 2y + 1 = 0. \tag{b}$$

$$x^2 - 2xy + y^2 - 3x + 3y + 2 = 0. \tag{c}$$

$$2x^2 - 3xy + y^2 - 3x + 2y + 1 = 0. \tag{d}$$

$$x^2 - 2xy + y^2 - 2x + 2y + 1 = 0. \tag{e}$$

But, Equation (a) represents an imaginary locus; (b) represents the point $(0, 1)$; (c) represents the parallel lines $x - y - 1 = 0$ and $x - y - 2 = 0$; (d) represents the two intersecting lines $x - y - 1 = 0$ and $2x - y - 1 = 0$; and (e) represents the identical lines $x - y - 1 = 0$.

If $C = 0$ and $A \neq 0$, $B \neq 0$, we may solve Equation (1) for x instead of y. The condition that the expression under the radical shall be a perfect square in this case is exactly the same as (6).

In summary, if we have no information relative to the locus of an equation of the second degree which contains the xy term, and if A and C

are not both zero, it is desirable to apply condition (6) to determine if perchance the locus is one of the degenerate forms considered above. If condition (6) is not fulfilled, then rotation of the axes for the elimination of the xy term is desirable.

EXERCISES 22

Examine each of the following equations to discover which represent degenerate conics, and discuss the nature of those that are degenerate.

1. $2x^2 - 3xy + y^2 + 7x - 5y + 6 = 0$
2. $x^2 - 6xy + 9y^2 + 10x - 30y + 25 = 0$
3. $5x^2 - 4xy + 3y^2 + 2x - y = 0$
4. $6x^2 + 7xy - 3y^2 - 3x + y = 0$
5. $x^2 - xy + 8x - 7y + 7 = 0$
6. $xy + 2y^2 + 5x + 7y - 15 = 0$
7. $x^2 - 2xy + y^2 + 4x - 4y + 4 = 0$
8. $xy - 2y^2 - 10x + 5y - 12 = 0$
9. $2x^2 + xy - 14x - 7y = 0$
10. $3x^2 - 5xy + 9y^2 = 0$

36. TANGENT TO A CURVE

Let $f(x, y) = 0$ be the equation of a curve, as shown in Figure 50. Moreover, let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the curve, and the secant line P_1S through these points is drawn. As P_2 moves along the curve to P_1, the secant line rotates in the plane about P_1, approaching, under common circumstances, the position designated in the figure by P_1T. The line P_1T is defined to be the tangent line to the curve at P_1.

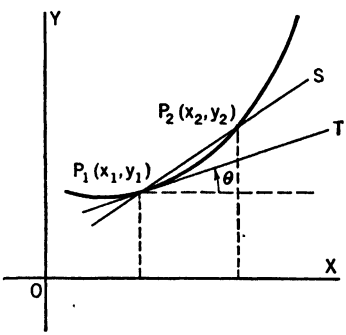


FIG. 50

The equation of the secant line P_1P_2 is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The right member is the slope of the line. As P_2 approaches P_1 along the curve, both numerator and denominator of the right member approach zero; yet the fraction can, and usually does, approach a definite limit. This limit is the slope of the tangent line P_1T, that is, $\tan \theta$, and is defined as the slope of the curve at (x_1, y_1).

Thus, in Figure 50, the slope, $\tan \theta$, of the tangent line P_1T is determined by

$$\lim_{P_2 \to P_1} \frac{y_2 - y_1}{x_2 - x_1}.$$

This important symbolic expression is read, "The limit of the fraction $(y_2 - y_1)/(x_2 - x_1)$ as P_2 approaches P_1." It is one of the fundamental problems of the calculus to determine this limit, if it has a value.

We shall illustrate the method of determining this limiting value for a few particular equations and then for the general equation of the second degree.

Illustration 1: Determine the slope and the equation of the tangent line at some point $P_1(x_1, y_1)$ for the curve of $y^2 = 4x$.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the given curve. Hence, both points must satisfy the equation of the curve; that is,

$$y_2^2 = 4x_2, \tag{2}$$

$$y_1^2 = 4x_1, \tag{3}$$

and

$$y_2^2 - y_1^2 = 4(x_2 - x_1).$$

From this relation we desire to obtain an expression for

$$\frac{y_2 - y_1}{x_2 - x_1};$$

this may be done by dividing the two members by $(y_2 + y_1)(x_2 - x_1)$, thereby obtaining

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4}{y_2 + y_1}.$$

As P_2 approaches P_1, y_2 must approach y_1. So,

$$\tan\theta = \lim_{P_2 \to P_1} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{y_2 \to y_1} \frac{4}{y_2 + y_1} = \frac{2}{y_1}. \tag{4}$$

Consequently, the equation of the tangent line at $P_1(x_1, y_1)$, given by the point-slope form of the straight line, is

$$\frac{y - y_1}{x - x_1} = \frac{2}{y_1}. \tag{5}$$

This equation completes the solution of the exercise, but some additional algebraic manipulation yields interesting results.

Equation (5) may be transformed to

$$yy_1 - y_1^2 = 2x - 2x_1. \tag{6}$$

From Equations (3) and (6), we have

$$yy_1 - 4x_1 = 2x - 2x_1,$$

$$yy_1 = 2(x + x_1). \tag{7}$$

This form, Equation (7), may be obtained in a mechanical way from $y^2 = 4x$ by writing the variable of the second degree as yy and the variable

of the first degree as $(x + x)/2$. Then we write $y^2 = 4x$ as

$$yy = 4\left(\frac{x + x}{2}\right).$$

If we now apply the subscript to one of the y's and to one of the x's, we have

$$yy_1 = 4\left(\frac{x + x_1}{2}\right),$$

or

$$yy_1 = 2(x + x_1).$$

This interesting mechanical device receives justification in the next section.

Of course, to obtain the equation of the tangent line to the curve at some particular point, x_1 and y_1 should be given their appropriate values.

Illustration 2: Determine the slope and equation of the tangent line at the point $P_1(x_1, y_1)$ for the curve,

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the curve. Since their coordinates must satisfy the equation, we have

$$\frac{x_2^2}{25} + \frac{y_2^2}{16} = 1$$

and

$$\frac{x_1^2}{25} + \frac{y_1^2}{16} = 1.$$

Consequently,

$$\frac{x_2^2 - x_1^2}{25} + \frac{y_2^2 - y_1^2}{16} = 0,$$

or

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{16(x_2 + x_1)}{25(y_2 + y_1)}.$$

The limit of this ratio, as P_2 approaches P_1, is the desired slope of the tangent at the point (x_1, y_1). Therefore,

$$\tan\theta = \lim_{P_2 \to P_1} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{P_2 \to P_1} -\frac{16}{25}\,\frac{(x_2 + x_1)}{(y_2 + y_1)} = -\frac{16}{25}\,\frac{(2x_1)}{(2y_1)} = -\frac{16x_1}{25y_1},$$

and the equation of the tangent line at $P_1(x_1, y_1)$ is

$$\frac{y - y_1}{x - x_1} = -\frac{16x_1}{25y_1}.$$

The last equation may be transformed to

$$25yy_1 - 25y_1^2 = -16xx_1 + 16x_1^2,$$

or
$$16xx_1 + 25yy_1 = 16x_1^2 + 25y_1^2,$$

or
$$\frac{xx_1}{25} + \frac{yy_1}{16} = \frac{x_1^2}{25} + \frac{y_1^2}{16}.$$

The right member of this equation equals 1, since the point $P_1(x_1, y_1)$ is on the curve $x^2/25 + y^2/16 = 1$. Hence, the equation of the tangent line is

$$\frac{xx_1}{25} + \frac{yy_1}{16} = 1.$$

We note that the mechanical device for obtaining this equation from the given equation, as explained in Illustration 1, may also be applied this time; that is, we write the given equation in the form

$$\frac{xx}{25} + \frac{yy}{16} = 1,$$

and attach the subscript to one of the x's and to one of the y's; then we have

$$\frac{xx_1}{25} + \frac{yy_1}{16} = 1.$$

37. EQUATION OF THE TANGENT LINE TO ANY SECOND-DEGREE CURVE

Take the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the curve of

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \tag{1}$$

Since the coordinates of the points satisfy the equation, we have

$$Ax_2^2 + 2Bx_2y_2 + Cy_2^2 + 2Dx_2 + 2Ey_2 + F = 0 \tag{2}$$

and
$$Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dx_1 + 2Ey_1 + F = 0. \tag{3}$$

After subtracting the members of Equation (3) from the corresponding members of (2), we have

$$\begin{aligned} A(x_2^2 - x_1^2) + 2B(x_2y_2 - x_1y_1) + C(y_2^2 - y_1^2) \\ + 2D(x_2 - x_1) + 2E(y_2 - y_1) = 0. \end{aligned} \tag{4}$$

The term $2B(x_2y_2 - x_1y_1)$ may be written in the form

$$2B(x_2y_2 - x_1y_2 + x_1y_2 - x_1y_1) = 2By_2(x_2 - x_1) + 2Bx_1(y_2 - y_1).$$

Hence, (4) may be written

$$\begin{aligned} A(x_2 - x_1)(x_2 + x_1) + 2By_2(x_2 - x_1) + 2Bx_1(y_2 - y_1) \\ + C(y_2 - y_1)(y_2 + y_1) + 2D(x_2 - x_1) + 2E(y_2 - y_1) = 0. \end{aligned} \tag{5}$$

After dividing the two members by $x_2 - x_1$, we have

$$A(x_2 + x_1) + 2By_2 + 2Bx_1 \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$+ C\frac{(y_2 - y_1)}{(x_2 - x_1)}(y_2 + y_1) + 2D + 2E\frac{(y_2 - y_1)}{(x_2 - x_1)} = 0. \quad (6)$$

The solution of this equation for $(y_2 - y_1)/(x_2 - x_1)$ yields

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{A(x_2 + x_1) + 2By_2 + 2D}{2Bx_1 + C(y_2 + y_1) + 2E}. \quad (7)$$

If the limit of the right member exists as P_2 approaches P_1, the limit is the desired value of $\tan\theta$.

When x_2 approaches x_1, y_2 approaches y_1, and the right member of (7) has the limiting value

$$-\frac{2Ax_1 + 2By_1 + 2D}{2Bx_1 + 2Cy_1 + 2E};$$

that is,

$$\tan\theta = -\frac{Ax_1 + By_1 + D}{Bx_1 + Cy_1 + E}. \quad (8)$$

Hence, the equation of the tangent to the curve at $P_1(x_1, y_1)$ is

$$\frac{y - y_1}{x - x_1} = -\frac{Ax_1 + By_1 + D}{Bx_1 + Cy_1 + E}. \quad (9)$$

After clearing of fractions, we have

$$Byx_1 - Bx_1y_1 + Cyy_1 - Cy_1^2 + Ey - Ey_1 = -Axx_1 + Ax_1^2$$
$$-Bxy_1 + Bx_1y_1 - Dx + Dx_1$$

or

$$Axx_1 + Byx_1 + Bxy_1 + Cyy_1 + Ey + Ey_1 + Dx + Dx_1 + F$$
$$= Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dx_1 + 2Ey_1 + F.$$

Since, from Equation (3), the right member is zero, we obtain as the equation of the tangent line

$$Axx_1 + B(x_1y + xy_1) + Cyy_1 + D(x + x_1) + E(y + y_1) + F = 0. \quad (10)$$

A comparison of the original Equation (1) with Equation (10) shows how (10) may be obtained from (1); that is, we write the given Equation (1) as

$$Axx + B(xy + xy) + Cyy + D(x + x) + E(y + y) + F = 0,$$

and then replace one of the x's in each term by x_1 and one of the y's in each term by y_1. It should be especially noted that in the term $B(xy + xy)$ we replace the x by x_1 in one of the xy's and the y by y_1 in the other xy. Of

course, this mechanical procedure is the one already employed in connection with the two previous illustrations.

Illustration 1: The equation of the tangent line at the point (5, 6) on the circle $(x-1)^2 + (y-3)^2 = 25$ is found by writing the equation in the form

$$x^2 - 2x + y^2 - 6y = 15$$

or
$$xx - (x + x) + yy - 3(y + y) = 15.$$

After replacing one of the x's in each term by 5 and one of the y's in each term by 6, we have

$$5x - (x + 5) + 6y - 3(y + 6) = 15$$

or
$$4x + 3y - 38 = 0.$$

Illustration 2: The equation of the tangent line at the point (1, 2) on the hyperbola $xy + 2x + y = 6$ is found by writing the equation in the form

$$\frac{1}{2}(xy + xy) + (x + x) + \left(\frac{y + y}{2}\right) = 6.$$

After replacing one of the x's by 1 and one of the y's by 2, noting that in $\frac{1}{2}(xy + xy)$ we replace x by 1 in one of the xy's and y by 2 in the other xy, we have

$$\frac{1}{2}(y + 2x) + (x + 1) + \left(\frac{y + 2}{2}\right) = 6$$

or
$$y + 2x - 4 = 0.$$

38. NORMAL TO A CURVE

The line perpendicular to the tangent to a curve at the point of tangency is called a *normal to the curve*. Since the normal is perpendicular to the tangent line, the slope of the normal is the negative reciprocal of the slope of the tangent line. Hence, in the special case of a curve of second degree, the slope m of the normal is the negative reciprocal of the value of $\tan\theta$ obtained in Equation (8) of the previous section; that is,

$$m = \frac{Bx_1 + Cy_1 + E}{Ax_1 + By_1 + D}.$$

Consequently, the equation of the normal to a curve of second degree at the point (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = \frac{Bx_1 + Cy_1 + E}{Ax_1 + By_1 + D}.$$

As an illustration, the equation of the normal to the curve

$$x^2 + xy + 2x + 2y + 12 = 0$$

at the point $(2, -5)$ is

$$\frac{y+5}{x-2} = \frac{\frac{1}{2}(2)+1}{1(2)+\frac{1}{2}(-5)+1} = \frac{2}{\frac{1}{2}} = 4$$

or

$$4x - y - 13 = 0.$$

EXERCISES 23

1. Determine the slope of the tangent to each of the following curves at the point specified:

(a) $y^2 = 5x$; $(5, 5)$ (b) $x^2 + y^2 = 25$; $(3, 4)$
(c) $x^2 = 4y$; $(2, 1)$ (d) $xy + x^2 = 1$; $(1, 0)$

2. (a) Find the equation of the tangent to the parabola $y^2 = 8x$ at the point $(2, 4)$.
(b) Find the equation of the normal to $y^2 = 8x$ at $(2, 4)$.

3. (a) Find the equation of the tangent to $x^2 = 4y$ at the point $(2, 1)$.
(b) Find the equation of the normal to $x^2 = 4y$ at the point $(2, 1)$.

4. Show that the x intercept of the tangent to the parabola $y^2 = 4px$ at (x_1, y_1) is $-x_1$.

5. From the result of Exercise 4 explain how a tangent may be accurately drawn to a parabola at any point.

6. Prove that a line from the focus of a parabola to any point P on the curve and a line through P, parallel to the axis, make equal angles with the tangent at P.

7. (a) Write the equation of the tangent to the ellipse $x^2/25 + y^2/16 = 1$ at the point $(3, 3\frac{1}{5})$.
(b) Find the equation of the normal to the ellipse $x^2/25 + y^2/16 = 1$ at the point $(3, 3\frac{1}{5})$.

8. Find the equations of the tangents to the ellipse $9x^2 + 36x + 16y^2 - 48y = 72$ at the points on the curve whose abscissa is zero.

9. The tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at a point P meets the tangent at the vertex $(a, 0)$ in the point Q. Show that the line joining Q to the center is parallel to the line joining P to the other vertex.

10. Prove: The lines from the foci of an ellipse to any point on the curve make equal angles with the line that is tangent to the ellipse at that point.

11. Find the equations of the tangents to the hyperbola $x^2/36 - y^2/16 = 1$ at the point where $x = 7.5$.

12. Write the equation of the tangent to $xy = 10$ at the point $(2, 5)$.

13. Find the equations of the tangents to the curve $y^2 = 16x - 32$ at the extremities of the latus rectum. Show that they are perpendicular and meet on the directrix.

14. Find the equation of the tangent and of the normal to the curve $x^2 + y^2 - 6x + 4y = 0$ at $(1, 1)$.

15. Prove that the tangents at the extremities of a latus rectum of the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect on the corresponding directrix.

16. Prove that the tangent at one end of the latus rectum of the parabola $y^2 = 4px$ is parallel to the normal at the other end.

39. EQUATIONS OF THE TANGENTS WITH A GIVEN SLOPE TO A CURVE OF THE SECOND DEGREE

If we assume the equation of the tangent line to be $y = mx + k$, where m is known, we may consider the system of equations composed of the equation of the curve and $y = mx + k$. If we substitute $mx + k$ for y in the equation of the curve, we obtain a quadratic equation in x involving m and k. Since the line is to be tangent to the curve, the two solutions for x must be identical. Hence, the discriminant of the quadratic in x must be zero. This fact enables us to determine k in terms of the known m, and the desired equation of the tangent is completely determined.

As an illustration, suppose that the line given by $y = mx + k$, where m is regarded as known, is to be tangent to $y^2 = 4px$. We consider the system

$$y = mx + k$$

$$y^2 = 4px.$$

If, in the second equation, y is replaced by its value from the first equation, we have

$$(mx + k)^2 = 4px$$

or

$$m^2x^2 + (2mk - 4p)x + k^2 = 0.$$

Now, if the discriminant, that is, the part under the radical in the quadratic equation, is equated to zero, we have

$$(2mk - 4p)^2 - 4m^2k^2 = 0,$$

$$4m^2k^2 - 16mkp + 16p^2 - 4m^2k^2 = 0,$$

or

$$k = \frac{p}{m}.$$

Hence, the equation of the tangent of given slope m to the curve $y^2 = 4px$ is

$$y = mx + \frac{p}{m}. \tag{1}$$

Similarly, the tangents, with a given slope m, to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be found to be, respectively,

$$y = mx \pm \sqrt{a^2m^2 + b^2} \tag{2}$$

and

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{3}$$

The determination of these latter equations is left as an exercise for the student.

EXERCISES 24

1. Find the equation of the line whose slope is 2 and which is tangent to the parabola $y^2 = 8x$.

2. Derive a formula for the tangent to the parabola $x^2 = 4py$ in terms of its slope m.

3. Use the formula derived in Exercise 2 to find the equation of the line tangent to $x^2 = 6y$ and having the slope 2.

4. Find the equations of the lines through the point (2, 6) and tangent to the parabola $y^2 = 8x$.

HINT: Use the equation of the tangent to the parabola in terms of slope, and determine the slope so that the line will pass through the point (2, 6).

5. Find the coordinates of the points of tangency for the tangents determined in Exercise 3.

6. Write the equations of the lines tangent to the ellipse $16x^2 + 25y^2 = 400$, and which have the slope $\frac{1}{2}$.

7. Find the equations of the lines tangent to the ellipse $x^2/25 + y^2/9 = 1$ and which pass through the point (4, 5).

8. Find the equations of the lines tangent to the hyperbola $x^2/36 - y^2/16 = 1$ which have the slope 2.

9. Find the equation of a line tangent to $xy = k$ and having the slope m.

10. Find the equations of the lines through the point $(-1, 5)$ and tangent to the curve $xy = 10$.

11. Find the equations of the lines that are tangent to the hyperbola $x^2 - 4y^2 = 36$ and parallel to the line $4x + 6y = 15$.

12. Find the equations of the lines that are tangent to the circle $x^2 + y^2 - 6x = 0$ and perpendicular to the line $2x + 3y = 5$.

13. Show that the circle tangent to the x axis and to each of the circles $x^2 - 2x + y^2 - 2y + 1 = 0$ and $x^2 + 2x + y^2 + 2y + 1 = 0$ has the radius $\frac{1}{4}$.

10 Curve Fitting

40. THE PROBLEM OF CURVE FITTING

The experimenter collects data indicating how certain variables appear to be related. Thus, in the case of two variables, if one variable is taken as x and another as y, a set of experiments merely provides a table of pairs of related values of x and y. Of course, these pairs of corresponding values may be displayed relative to a coordinate system, thereby portraying to better advantage any trends which may be present.

Collections of data in tabular form are usually inconvenient to handle, especially if a mathematical analysis of the data is desired. So, it is common "to fit a formula to the data," or, to state it in equivalent fashion, "to fit a curve to the points representing the data." The human element is very strong in this latter process, for it is first necessary for the mathematical scientist to select a general type of curve that possesses the same general behavior as the trend indicated by the points. There is no unique curve to be found, and the ultimate choice will involve, to a certain extent, the scientist's prejudices.

After the type of curve has been selected, it is necessary to determine the arbitrary constants in the equation of the curve so that the curve follows the points in an acceptable manner. For large collections of data it is usually impossible to determine values for the constants that will permit the curve to pass through all the points. Thus, we speak of obtaining the curve that best fits the points or data; of course, a definition must be given of the word "best."

41. TYPES OF EQUATIONS COMMONLY USED IN CURVE FITTING

The following types of equations are frequently considered in fitting a curve to experimental data:

(a) $y = A + Bx.$

(b) $y = A + Bx + Cx^2.$

(c) $y = A + \dfrac{B}{x}.$

(d) $y = \dfrac{A + Bx}{C + Dx}.$

(e) $y = AB^x, B > 0.$

(f) $y = Ax^n.$

Equation (a) is the equation of a straight line with slope B and y intercept A.

Equation (b) is the equation of a parabola with its axis parallel to the y axis.

Equation (c) is the equation of a hyperbola with asymptotes $y = A$ and $x = 0$.

Equation (d) is the equation of a hyperbola with asymptotes $x = -C/D$ and $y = B/D$.

Equation (e) provides an exponential curve. If $B = e^r$, where e is the base of the natural logarithm system, then the equation may be written $y = Ae^{rx}$. Written this way, the formula is known as the compound-interest law. The graph of $y = AB^x$ takes one of the forms represented in Figure 51.

Equation (f) is called the *power law*. If $n = 1$, then we have $y = Ax$, and the graph of the function is a straight line. If $n = -1$, we have a

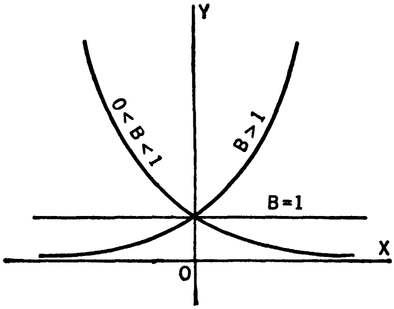


FIG. 51

special case of the type (c). If $n = 2$, we have a special case of the curve of type (b). If $n > 2$, and if n is an even integer, we have curves somewhat

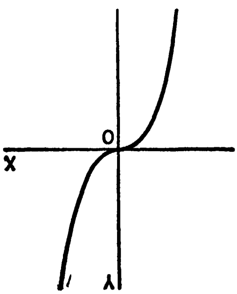


FIG. 52

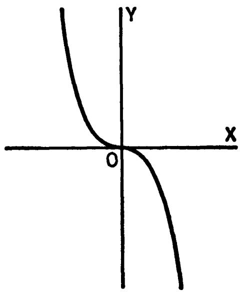


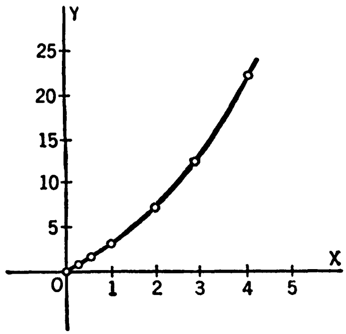
FIG. 53

similar to the parabolic type obtained when $n = 2$, but rising more and more rapidly with larger values of n. If $n = 3$, we have the cubical

parabola displayed in Figure 52 if $A > 0$, and that of Figure 53 if $A < 0$. If $n > 3$, and if n is an odd integer, we obtain curves somewhat similar to those in Figures 52 and 53.

If in type (f) n is a rational fraction, that is, in the form p/q, we may have q odd or even. If q is odd, there is no ambiguity since $x^{p/q}$ will have only one real value for each value of x. But, if q is even, $x^{p/q}$ equals $\pm\sqrt[q]{x^p}$. Frequently, however, we restrict our consideration to the function $+\sqrt[q]{x^p}$. If n is irrational, we shall, by definition, restrict our consideration to the function $+\sqrt[q]{x^p}$, where p/q is a rational approximation to n, and limit ourselves to positive values of x.

Thus, the graph $y = 3x^{\sqrt{2}}$ is approximately that of $y = 3x^{1.4}$, or perhaps $y = 3x^{707/500}$, and the approximate graph is given in Figure 54. In obtaining points on such a curve, the values of y are calculated by the use of logarithms.



x	y
0	0
0.2	0.32
0.5	1.13
1.0	3
2.0	7.99
3.0	14.19
4.0	21.30

FIG. 54

EXERCISES 25

1. Draw the graph of $y = A + Bx + Cx^2$, for each set of values of A, B, and C given below:

(a) $A = 1, B = 2, C = 3$
(b) $A = 1, B = 2, C = -3$
(c) $A = 0, B = 2, C = 3$
(d) $A = 0, B = 0, C = -3$
(e) $A = 5, B = 2, C = 0.3$
(f) $A = 5, B = 2, C = 0.03$
(g) $A = 5, B = 2, C = 0.003$

2. Draw the graph of $y = A + B/x$ for the following values of A and B:

(a) $A = 2, B = 3$
(b) $A = -2, B = 3$
(c) $A = -2, B = -3$
(d) $A = 2, B = -3$
(e) $A = 0, B = 3$

3. Draw the graph of $y = (A + Bx)/(C + Dx)$ for the following values of A, B, C, and D. Also draw the asymptotes in each case.

(a) $A = 2, B = -3, C = 3, D = 5$
(b) $A = -7, B = 5, C = 2, D = 7$
(c) $A = 3, B = -15, C = -4, D = 6$

4. Draw the graph of $y = AB^x$ for the following values of A and B:
(a) $A = 2, B = 10$ (b) $A = 2, B = \frac{1}{10}$
(c) $A = 2, B = 1$
(d) What is the graphical significance of the constant A in the equation $y = AB^x$?
(e) If A is negative, what effect will it have on the graph?
(f) Draw the graph for $A = -3$ and $B = 5$.

5. Let $A = 1$ in the equation $y = Ax^n$, and draw the graph of this function for each of the following values of n: 0, 1, 2, 3, 4, -1, -2, -3, -4. Draw all these curves on the same set of axes, and note the graphical significance as n increases or decreases. Draw a similar set of curves for $A = 2$. What is the graphical significance of the constant A?

6. Draw the graph of $y = Ax^n$ for each of the following values of A and n:
(a) $A = 2, n = \frac{1}{2}$ (b) $A = 2, n = \frac{1}{3}$
(c) $A = 3, n = \frac{2}{3}$ (d) $A = 3, n = 3$.

7. (a) Rewrite the function $y = (2.3)^x$ in the form $y = 10^{rx}$.
HINT: By the use of a table of common logarithms, write 2.3 in the form 10^r.
(b) Draw the curve representing the function of part (a). Do you see any advantage in using the second form of the function?

8. Rewrite the function $y = (3.6)^x$ in the form $y = e^{rx}$. Do you see any advantage in using the second form of the function?

9. Compare the graphs of the curves, $y = 10^x$ and $y = \log x$, where it is understood that the logarithm is in the common system.

10. Write the function, $x = \log y - 2$, in the form $y = A \cdot 10^x$.

42. EQUATIONS OF GRAPHS THROUGH GIVEN POINTS

In Chapter IV we derived a formula for finding the equation of a straight line through two points. It is often more convenient to use the method illustrated by the following example.

Illustration 1: Let us find the equation of a straight line through the points (2, 3) and (5, -1).

Assume that the straight line given by the equation $y = A + Bx$ passes through these two points. Then the coordinates of each of the points must satisfy the equation, and we have, by substituting the coordinates,

$$3 = A + 2B \tag{1}$$

and

$$-1 = A + 5B. \tag{2}$$

After solving these equations for A and B, we obtain

$$A = \frac{17}{3} \quad \text{and} \quad B = -\frac{4}{3}.$$

When these values are substituted in the equation $y = A + Bx$ and the result simplified, we have $4x + 3y = 17$, which is the required equation.

Illustration 2: The same method may be used to find the equations of other types of curves through two points. For example, find the equation of a curve of type $y = A + B/x$ through the two points (2, 3) and (5, -1).

After substituting, we have

$$3 = A + \frac{B}{2}. \tag{1}$$

and

$$-1 = A + \frac{B}{5}. \tag{2}$$

The solution of this system for A and B yields $B = \frac{40}{3}$ and $A = -\frac{11}{3}$. Hence, the required equation is

$$y = -\frac{11}{3} + \frac{40}{3x}.$$

In general, this method may be used to find the equation of any curve through two or more points if the number of given points is equal to the number of arbitrary constants in the standard equation of the curve. Thus, to find the equation of a curve of the type,

$$y = A + Bx + Cx^2,$$

we must have three points given. If the number of points is less than the number of arbitrary constants in the formula assumed, an unlimited number of such functions may be obtained.

Illustration 3: Find a curve of the form $y = A + Bx + Cx^2$ passing through the points $(1, 3)$ and $(2, 7)$.

We have, then,

$$3 = A + B + C \tag{1}$$

and

$$7 = A + 2B + 4C. \tag{2}$$

These two equations are not sufficient to determine A, B, and C. However, we may eliminate C from these two equations as follows:

$$12 = 4A + 4B + 4C, \tag{1}$$
$$7 = A + 2B + 4C. \tag{2}$$

Therefore,

$$5 = 3A + 2B,$$

or

$$B = \frac{5 - 3A}{2}.$$

In a similar manner we may eliminate B from (1) and (2) as follows:

$$6 = 2A + 2B + 2C \tag{1}$$

and

$$7 = A + 2B + 4C. \tag{2}$$

Therefore,

$$-1 = A - 2C,$$

or
$$C = \frac{A + 1}{2}.$$

Hence, the original equation may be written
$$y = A + \left(\frac{5 - 3A}{2}\right)x + \left(\frac{A + 1}{2}\right)x^2.$$

We may now assign any value to A except $A = -1$, and thus, through the given points, an unlimited number of such parabolas may be determined.

Illustration 4: On the other hand, let the given points be (0, 0), (1, 1), and (1, 7), apparently a sufficient number of points to determine A, B, and C of the formula
$$y = A + Bx + Cx^2.$$

We now have
$$0 = A, \tag{1}$$
$$1 = B + C, \tag{2}$$
$$7 = B + C. \tag{3}$$

Equations (2) and (3) are inconsistent. Hence, we cannot determine a curve of the required form through the given points. Thus, we see that it is not always possible to find the equation of a given type through points chosen at random. Further illustrations will be found in some of the problems of the following exercises.

EXERCISES 26

By use of the method explained in the previous section, find the equations of the following curves:

1. Find the equation of a straight line through the points $(-1, 3)$ and $(2, -7)$.

2. Find the equation of a curve of the type $y = A + B/x$ through the two points in Exercise 1. Draw the graph.

3. Find the equation of a curve of the type $y = AB^x$ through the points $(-1, 10)$ and $(3, 160)$.

4. Find the equation of a curve of the type $y = Ax^n$ through the points $(1, 5)$ and $(2, 20)$. Draw the graph.

5. Find the equation of a curve of the type $y = A + Bx + Cx^2$ through the points $(1, 0)$, $(0, 0)$, and $(3, 5)$. Draw the graph.

6. Attempt to find the equation of a curve of type $y = A + \frac{B}{x} + \frac{C}{x^2}$ through the points $(1, 2)$, $(1, 0)$, and $(1, 5)$.

7. Given the points (1, 3) and (3, 0). Determine which of the following types of curves may be made to pass through these points, and find the equation of each type that is possible.

(a) $y = A + Bx$

(b) $y = A + \dfrac{B}{x}$

(c) $y = AB^x$

(d) $y = Ax^n$

(e) $y = A + Bx + Cx^2$

8. Given the points (0, 3) and (1, 2). Can you find an equation of the type $y = A + B/x$ through them? Explain. Can you find an equation of the type $y = Ax^n$ through them? Explain.

9. Find the equation of a curve of the type $y = AB^x$ through the points $(-2, 10)$ and $(\frac{1}{2}, 1)$. Draw the curve.

10. Find the equation of a curve of the type $y = A + Bx + Cx^2$ through the points $(-1, -7)$, $(1, 3)$, and $(6, 0)$. Draw the curve.

43. EXPERIMENTAL DATA ON A LINE

We shall now consider problems in which the points representing the experimental data are exactly on a straight line or approximately on a straight line.

If we graph the following data obtained in measuring the amount of heat h of a pound of steam at various temperatures θ°C, the graph of the data indicates that the points lie exactly on a straight line (note Figure 55). The fact that these points are all on the same straight line is quickly confirmed by noting that if (θ_1, h_1), (θ_2, h_2), (θ_3, h_3) denote any three points on the curve; it is always true that

$$\frac{h_3 - h_2}{\theta_3 - \theta_2} = \frac{h_2 - h_1}{\theta_2 - \theta_1}.$$

h	624.8	626.3	627.8	629.3	630.8	632.3	633.8	636.8	639.8
θ	60	65	70	75	80	85	90	100	110

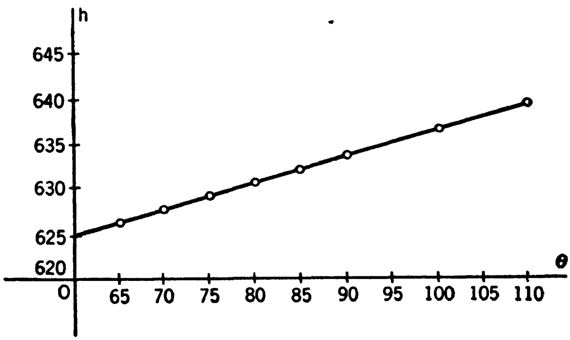


FIG. 55

In order to find the proper linear relation between h and θ, we employ the formula (1) $h = A + B\theta$ and determine A and B from any two equations obtained by substituting corresponding values of h and θ from the

given table. Thus, we might take

$$629.3 = A + 75B \tag{2}$$

and

$$636.8 = A + 100B. \tag{3}$$

After solving these equations for A and B, we have $A = 606.8$ and $B = 0.3$. Hence, the required linear relation between h and θ is $h = 606.8 + 0.3\theta$.

Obviously this process is merely one of the methods of finding the equation of a straight line through two given points, a familiar problem.

In the above data all the points were exactly on a straight line. However, this may not always be the case, although the graphed data may indicate that the trend is essentially that of a straight line. In such a case it is evidently desirable to determine the equation of a straight line that will be approximately representative of the observed data.

If we plot the points corresponding to the pairs of values of x and y given in the following table, we find that they lie approximately on a straight line. To obtain an equation of a straight line that they will satisfy approximately, we may use a transparent straightedge and by trial draw a straight line that will appear to divide the points into two groups, so that half of them will be on each side of the line, as shown in Figure 56. Then, if we observe the coordinates of two points on the line, we can find the desired equation.

x	.7	.7	1.0	1.6	2.0	2.5	3.0	3.2	3.8	4.5	5.5
y	.4	.8	1.0	1.5	2.5	2.7	3.3	3.0	4.2	4.5	6.0

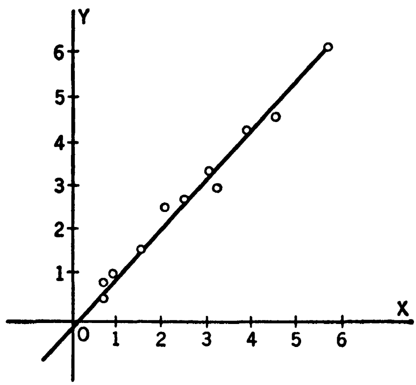


FIG. 56

From the figure we note that the points (0.2, 0) and (5.5, 6) appear to be on the line. After substituting these values in the equation $y = A + Bx$, we have

$$0 = A + 0.2B \tag{1}$$

and

$$6 = A + 5.5B. \tag{2}$$

The solution of this system yields $A = -0.23$, $B = 1.13$; so the required equation is

$$y = -0.23 + 1.13x. \tag{3}$$

It is of interest to note how the observed data compares with the calculated data obtained from the resulting equation. These values are obtained by substituting the original values of x in Equation (3) and solving for y. The values are called y_c in the accompanying table; $y_c - y$ has also been calculated. If we add the negative values of $y_c - y$, we get -1.30, and the sum of the positive values gives 0.95, showing a numerical difference of only 0.35. This indicates that for many purposes the line is "sufficiently" representative of the given data.

x	y_c	$y_c - y$
0.7	0.56	0.16
0.7	0.56	−0.24
1.0	0.90	−0.10
1.6	1.58	0.08
2.0	2.03	−0.47
2.5	2.60	−0.10
3.0	3.16	−0.14
3.2	3.39	0.39
3.8	4.06	−0.14
4.5	4.82	0.32
5.5	5.89	−0.11

The method just given of "fitting a straight line" to the given data depends too much on whim to be satisfying to most scientists. Of course,

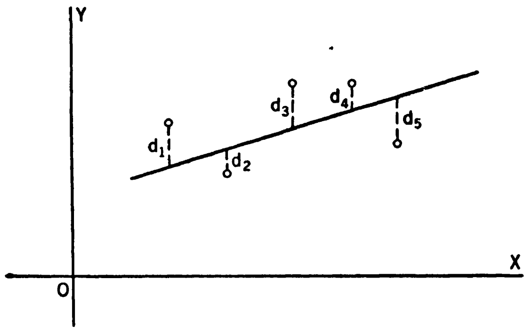


FIG. 57

any interpretation of "the line that best fits the data" must be arbitrary, but the line determined by the method of least squares is commonly accepted as furnishing quite a satisfactory solution to the problem. The theory underlying the method of least squares is too advanced to be treated

here. Suffice it to say, however, that a line so determined is such that the sum of the squares of the vertical discrepancies d_1, d_2, d_3, and so on, between the given points and the proposed line, as indicated in Figure 57, must be a minimum.

The actual mechanical method of applying the principle of least squares in the case of a straight line will be explained by considering the particular data already treated above.

If corresponding values of x and y are substituted into the equation $y = A + Bx$, we obtain the following set of eleven equations:

$$\begin{aligned}
A + 0.7B &= 0.4,\\
A + 0.7B &= 0.8,\\
A + 1.0B &= 1.0,\\
A + 1.6B &= 1.5,\\
A + 2.0B &= 2.5,\\
A + 2.5B &= 2.7,\\
A + 3.0B &= 3.3,\\
A + 3.2B &= 3.0,\\
A + 3.8B &= 4.2,\\
A + 4.5B &= 4.5,\\
A + 5.5B &= 6.0.
\end{aligned}$$

The *first normal equation* is obtained by multiplying each equation by its respective coefficient of A and then adding all the left members and all the right members. Since the coefficient of A is 1 in each case, the first normal equation is merely the sum of the eleven equations; the result is

$$11A + 28.5B = 29.9. \tag{1}$$

The second normal equation is obtained in the same manner after first multiplying each member of each equation by its respective coefficient of B. We obtain

$$\begin{array}{rcl}
0.7A + 0.49B &=& 0.28\\
0.7A + 0.49B &=& 0.56\\
1.0A + 1.0B &=& 1.0\\
1.6A + 2.56B &=& 2.4\\
2.0A + 4.0B &=& 5.0\\
2.5A + 6.25B &=& 6.75\\
3.0A + 9.0B &=& 9.9\\
3.2A + 10.24B &=& 9.6\\
3.8A + 14.44B &=& 15.96\\
4.5A + 20.25B &=& 20.25\\
5.5A + 30.25B &=& 33.0\\
\hline
28.5A + 98.97B &=& 104.70.
\end{array} \tag{2}$$

The solution of the system of Equations (1) and (2) yields the desired values of A and B that are to be substituted into the equation $y = A + Bx$. The equation finally determined is

$$y = -0.8 + 1.08x.$$

It is observed that this line approximates quite closely the one obtained by the rough method.

EXERCISES 27

Obtain a linear relationship satisfying the data, at least approximately, in each of the following exercises by both the first and the second method:

1. S is the weight of sodium nitrate dissolved in 100 gm of water at temperature t°C. Find a law for S as a linear function of t.

S	69.3	72.9	80.2	87.5	94.7
t	−5	0	10	20	30

2. S is the specific heat of mercury at temperature t°C. Find a law for S as a linear function of t.

t	75	88	100	120	130
S	0.03258	0.03246	0.03235	0.03216	0.03207

3. P is the pull required to lift a weight W by means of a differential pulley block. Find a law for P as a linear function of W.

W	145	230	273	315	358	400
P	20	30	35	40	45	50

4. The theoretical horsepower-hours per acre-foot of storage area of water for different heads is given by the following table. H is the head in feet and E is the energy in horsepower-hours. Find a law for E as a linear function of H.

H	5	10	20	35	50	75	100	150	200
E	6.88	13.75	27.50	48.12	68.75	103.12	137.50	206.25	275.00

5. The horsepower required by standard boring mills using one cutting tool of water-hardened steel at a cutting speed of about 20 fpm is found by experiment to be about as given in the following table. P is the horsepower required and R is the swing of the mill in inches. Find a law for P as a linear function of R.

P	5	7.4	9.7	12	14.4	18.7
R	30	40	50	60	70	80

6. In measuring the elongation E of a spring due to different forces F applied to it, the following observations were made in the laboratory. Find a law for E as a linear function of F.

F	100	200	300	400	500
E	0.7	1.3	2.0	2.6	3.2

7. The following data show the relation between torque T and the armature current I in a shunt motor. Find a formula for T as a linear function of I.

I	5.5	12.5	33.5	42.5	48.7	61.0
T	3.15	36.8	126	167	194	246

44. FITTING A PARABOLA TO EMPIRICAL DATA

Much of the time a collection of points, obtained empirically, does not even suggest a straight line. Note the points in Figure 58, for example. Of course, even in such a case, it is possible to "fit a straight line to the data," but there would not be much correspondence between the linear function thus obtained and the data under consideration; in fact, the law would not have much value. It is much better to attempt to fit another type of curve to the data. The curves commonly employed were discussed in the first part of this chapter.

The parabola is used frequently in practice. In fact, the points of Figure 58 immediately suggest a parabola. The data for these points are given immediately above the figure.

x	0	1	2	3	4	5
y	8	3	2	5	12	23

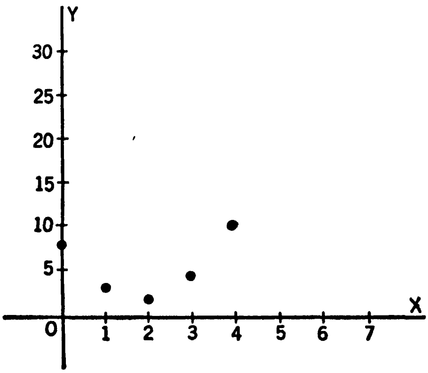


FIG. 58

Three points are sufficient to determine the constants in the equation

of the parabola when it is in the form $y = A + Bx + Cx^2$. If the six given points determine a perfect parabola, the remaining three points will satisfy the equation thus determined. Usually a given set of points does not determine a perfect parabola, and such a method as that of least squares must be employed. This time, however, let us see what parabola in the form $y = A + Bx + Cx^2$ is determined by three of the points.

The coordinates of any three points given in the table may be substituted in the equation $y = A + Bx + Cx^2$, and A, B, and C may then be determined.

Thus,

$$\text{for } x = 0, \quad y = 8, \quad \text{we have} \quad 8 = A; \tag{1}$$

$$\text{for } x = 3, \quad y = 5, \quad \text{we have} \quad 5 = A + 3B + 9C; \tag{2}$$

$$\text{for } x = 5, \quad y = 23, \quad \text{we have} \quad 23 = A + 5B + 25C. \tag{3}$$

Since $A = 8$, Equations (2) and (3) may be written

$$3B + 9C = -3 \tag{4}$$

and

$$5B + 25C = 15. \tag{5}$$

The solution of this system yields $C = 2$ and $B = -7$.

Hence, the equation of the parabola through these three points is

$$y = 8 - 7x + 2x^2.$$

When we test the other three values in the table, we find that they also satisfy the equation. Thus, all six points were exactly on a parabola of the desired form.

When we change the data only slightly as in the following table, the trend still suggests a parabola, but no equation of the form $y = A + Bx + Cx^2$ can be found which the coordinates of all the points will satisfy. Thus, we shall apply the method of least squares to obtain the parabola of "best fit."

x	0.5	1	2	3	4	5
y	6	3	2.5	4.5	12	23

First, as in our previous study of least squares, we substitute each pair of coordinates in the assumed equation of the form $A + Bx + Cx^2 = y$. We obtain the following set of six equations:

$$\begin{aligned}
A + 0.5B + 0.25C &= 6,\\
A + B + C &= 3,\\
A + 2B + 4C &= 2.5,\\
A + 3B + 9C &= 4.5,\\
A + 4B + 16C &= 12,\\
A + 5B + 25C &= 23.
\end{aligned}$$

As before, the first normal equation is found by multiplying the members of each equation by the coefficient of A and adding the resulting equations; the second normal equation is found by multiplying the members of each equation by the coefficient of B and adding the resulting equations; and the third normal equation is found by multiplying the members of each equation by the coefficient of C and adding the resulting equations. The three normal equations, thus obtained, form a system in A, B, and C. After solving for these three unknowns, the desired equation $y = A + Bx + Cx^2$ is determined. It is suggested that the student complete the illustration as an exercise.

EXERCISES 28

1. Fit a parabola of the form $y = A + Bx + Cx^2$ to the data (0, 3), (2, −1), (4, 2).

2. Determine the constants in the equation $y = A + Bx + Cx^2$ so that the coordinates of the following points will satisfy it: (1, 2.2), (3, 6.7), (5, 4.3).

3. Fit a curve of the form $y = A + Bx + Cx^2$ to the following data: (−1, 0), (0, 1), (1, 6), (2, 15), (3, 28).

4. Find the parabola $y = A + Bx + Cx^2$ that fits the following data the best: (1, 0), (2, 2), (3, 2), (4, −1).

5. By the method of least squares, fit a curve of the form $y = A + Bx + Cx^2$ to the coordinates: (−1, 2), (1, 1), (3, 1), (5, 3), (6, 5).

6. The following data were taken from a test on a 500-kw Curtis steam turbine:

Average load, kw	Approx. load	Steam used per hour, lb per kw
125.87	0.252	25.9
250.06	0.500	22.4
393.8	0.785	20.9
511.7	1.023	20.5
613.5	1.227	20.9

The values in the second column of the above table were found by dividing each of the values in the first column by 500. Find the equation of a parabola that will be satisfied approximately by these data, using values in the second column as values of x and number of pounds of steam used per kilowatt as values of y.

7. The following data were taken from a test made on a 500-hp Rateau turbine:

Electrical hp at brushes	Approx. load	Steam consumption, lb per electrical hp-hr at brushes
135	0.27	21.3
259	0.52	18.0
525	1.05	15.8
627	1.25	15.39

Using the approximate loads as values of x and the steam consumption as values of y, find the equation of a parabolic curve that will express approximately the relation between x and y.

45. FITTING A CURVE OF THE FORM $y = A + \frac{B}{x}$.

For a set of values $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ to satisfy an equation of the form

$$y = A + \frac{B}{x},$$

the points determined by the set of coordinates $(u_1, y_1), \cdots, (u_n, y_n)$, where $u_i = 1/x_i$, $i = 1, 2, \cdots, n$, should be on a straight line, for the substitution of u for $1/x$ results in the linear equation

$$y = A + Bu.$$

Hence, if the graph of $(u_1, y_1), \cdots, (u_n, y_n)$, where $u_i = 1/x_i$, $i = 1, 2, \cdots, n$, is practically linear, we may find the equation of the straight line that best fits the data involving u and y by either of the methods previously discussed and thus have an equation of the form $y = A + Bu$, where A and B are now determined. After substituting $1/x$ for u, we have the desired equation

$$y = A + \frac{B}{x}$$

in terms of x and y.

Illustration: Fit some convenient curve to the following data:

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5
y	5	4	3	2.5	2.33	2.25	2.20

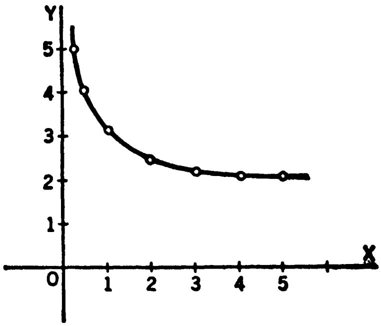


FIG. 59

When these coordinates are graphed, they have the trend indicated by the curve of Figure 59. The curve of Figure 59 resembles a hyperbola.

Let us, then, adjust the given data as shown in the following table:

$u = 1/x$	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
y	5	4	3	2.5	2.33	2.25	2.20

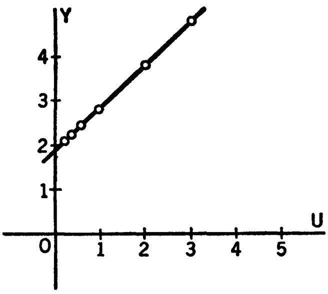


Fig. 60

Evidently these points lie exactly on a straight line, as shown in Figure 60; this fact is readily confirmed. Hence, we have by substituting, for example, $u = 2$, $y = 4$, and $u = \frac{1}{4}$, $y = 2.25$ in the equation $y = A + Bu$, the system of equations

$$4 = A + 2B$$

$$2.25 = A + \frac{B}{4}.$$

The solution of this system yields $A = 2$, $B = 1$, and we have the linear relation

$$y = 2 + u.$$

Hence, the required function $y = 2 + \dfrac{1}{x}$ is satisfied by all the given data.

If the (u, y) coordinates had been located only approximately upon a straight line, but if a linear relation between them seems to provide a generally satisfactory law, the equation $y = A + Bu$ may be obtained by the method of selected points or that of least squares.

EXERCISES 29

1. Determine a curve of the form $y = A + B/x$ that passes through the points (2, 3) and (5, 8).

2. What hyperbola of the form $y = A + B/x$ passes through the points $(-1, -1)$, (1, 5), and (3, 1)?

3. Fit some convenient curve to the points (1, 1), $(2, -1)$, $(4, -2)$, $(8, -2.5)$.

4. Fit a curve to the points (1, 8.8), (3, 5.1), (6. 3.9), (8, 3.8), (10, 3.6).

5. Determine y as a function of x that appears to be compatible with the following data:

x	1	3	5	10	15	20	30
y	7.7	2.7	1.7	1.0	0.8	0.6	0.5

6. A relation between x and y is indicated by the following pairs of values. Find an appropriate formula.

x	1	5	10	20	30
y	73.7	62.7	61.4	60.7	60.4

7. The following data were obtained experimentally by measuring the relation between the voltage v and current i in a circuit containing a copper carbon arc of 1 mm length in an illuminating-gas atmosphere with a magnetic field. Find the formula for v as a function of i.

v, volts	49.0	39.8	30.0	22.0	20.8	18.0	16.3
i, amperes	1.3	1.6	2.0	2.65	3.0	4.0	5.0

46. FITTING A CURVE OF THE FORM $y = A \cdot B^x, B > 0$

For a set of values $(x_1, y_1), \cdots, (x_n, y_n)$ to satisfy an equation of the form

$$y = A \cdot B^x, \qquad B > 0,$$

the graph determined by $(x_1, v_1), \cdots, (x_n, v_n)$, where $v_i = \log y_i$, $i = 1, 2, 3, \cdots, n$, should be linear. This follows from the fact that

$$\log y = \log A + (\log B)x.$$

So, if $\log y$ is denoted by v, and if the constant $\log A$ is designated by A_1 and $\log B$ by B_1, we have the linear function $v = A_1 + B_1x$.

As a result, if the graph of $(x_1, v_1), \cdots, (x_n, v_n)$, where $v_i = \log y_i$, $i = 1, 2, \cdots, n$, is practically linear, we may find the best line for the relation between x and v, by either of the methods already discussed and then have an equation of the form

$$v = A_1 + B_1x.$$

Since $A_1 = \log A$ and $B_1 = \log B$, A and B may now be determined, and we have a satisfactory equation of the form

$$y = A \cdot B^x, \qquad B > 0,$$

for the relation between x and y.

Illustration: Find the functional relation between x and y for the following data:

x	0	1	2	3	4	5	6
y	3	6	12	24	48	96	192

The graph sketched through the points determined by the data is displayed in Figure 61, and it shows some resemblance to the exponential function $y = A \cdot B^x$, where $B > 1$.

To examine the situation more carefully, let us study the behavior of the points determined by the data that follows:

x	0	1	2	3	4	5	6
$v = \log y$	0.477	0.778	1.079	1.380	1.681	1.982	2.283

It is quickly confirmed that these points lie on the straight line that appears in Figure 62. Hence, we have confirmed the fact that the function for the given data must be of the form $y = A \cdot B^x$.

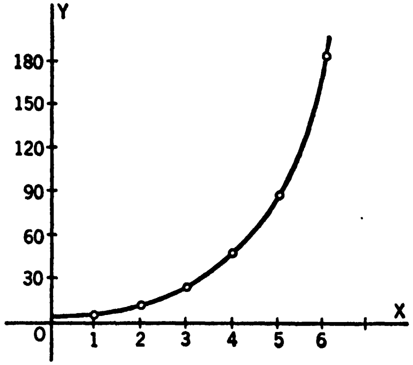


FIG. 61

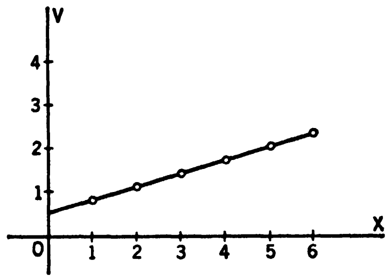


FIG. 62

The equation $v = A_1 + B_1x$ of the straight line in Figure 62 is found to be $v = 0.477 + 0.301x$; that is, $A_1 = \log A = 0.477$ and $B_1 = \log B = 0.301$.

Hence, $B = 2$ and $A = 3$; so the required function is $y = 3(2^x)$.

47. SEMILOGARITHMIC PAPER

In testing a function of the type

$$y = A \cdot B^x$$

to determine whether it is satisfied by the given data, the substitution $v = \log y$ requires the looking up of the logarithms corresponding to the values of y. Note the illustration of the previous section. In order to avoid the necessity of looking up logarithms and preparing the data of the second table, special paper for graphing has been devised (see Figure 63). This special paper, called *semilogarithmic paper*, is constructed by laying off on the horizontal scale the actual values of x and on the vertical scale the logarithms of the values of y.

The paper of Figure 63 is divided into four cycles, all marked from 1 to 10. We may choose any one of these 1's as any multiple of 10, and mark the other points accordingly. Thus, the first 1 may be called 0.1; and the next 1 will then be 1; the next 1 will be 10; and so on. In each cycle the number designations, such as 1, 2, 3, 4, really indicate the corresponding logarithms of these numbers on the vertical scale. In Figure 63 we have graphed the original data of the illustration in Section 46, and we have obtained a straight line, as expected. Thus, we have avoided the looking up of logarithms.

From Figure 63 we may note that the y intercept equals the A of the

function $A \cdot B^x$. The value of B may then be found by substituting from the table a pair of corresponding values of x and y in the equation $y = A \cdot B^x$, where A is now known.

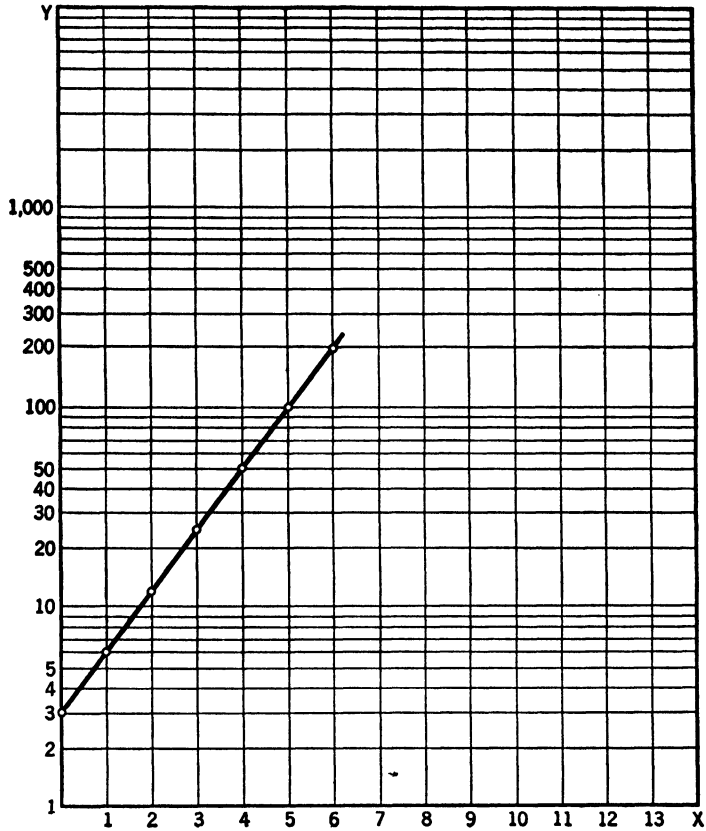


FIG. 63

Of course, once it is known that the given data satisfy a function of the exponential form, A and B may be found by several methods.

By substituting two pairs of corresponding values of x and y, such as (x_1, y_1) and (x_2, y_2), in $y = A \cdot B^x$, we obtain the system of equations

$$y_1 = A \cdot B^{x_1}, \tag{1}$$

and

$$y_2 = A \cdot B^{x_2}. \tag{2}$$

The division of equals by equals yields

$$\frac{y_1}{y_2} = B^{x_1 - x_2}$$

Hence,

$$\log B = \frac{\log y_1 - \log y_2}{x_1 - x_2},$$

from which B is found. The value of A may then be found from either Equation (1) or (2) above.

Quite often the points corresponding to (x, v) may not lie exactly on a straight line but sufficiently near a straight line for practical purposes. By the method of least squares or some other method, we may determine a line for this set of points and then find A and B as above.

EXERCISES 30

1. Determine a curve of the form $y = A \cdot B^x$ that passes through the points $(0, 3)$, $(0.5, 6)$, $(1.5, 24)$.

2. If x denotes the number of a term of a certain geometric progression, and if y is the value of that term, show that the pairs of values thus obtained satisfy a function of the type $y = A \cdot B^x$.

3. The number N of bacteria in a culture t hr after they were first counted is shown in the following table. Find a formula that will express approximately the value of N as a function of t.

t	0	1	2	3	4	5	6	7
N	100	165	272	450	742	1222	2010	3305

4. The area A of a healing wound decreased in size, after t days, as shown in the following table. Find a formula that will express approximately the value of A as a function of t.

t	0	2	4	6	8	10	12
A	6.2	4.8	3.4	2.6	1.8	1.3	1.0

5. The temperature T possessed by a cooling body after t min is shown in the following table. Find the formula for T in terms of t.

t	0	5	10	15	20	30	40	60
T	18	16.5	15.3	14.1	13.2	12.0	10.5	8

6. To obtain core loss in an induction motor, the input (in watts) and the voltage are measured, and the core loss is computed from the results. The following table gives the results of such a test on a 1-hp 550-v three-phase motor. Determine the equation of the curve which gives approximately the relation between voltage v and watts w.

v (volts)	280	360	395	495	545	595	640	710	740
w (watts)	42	60	64	97	104	125	140	164	190

7. The production of petroleum in Argentina from 1919 to 1926 in thousands of barrels was as shown in the table. Find a formula that expresses the relation of number of barrels to years.

Year	1919	1920	1921	1922	1923	1924	1925	1926
Number of barrels, in thousands	1331	1651	2036	2866	3400	4639	5997	6500

8. The following table gives the production of rayon products in the United States. Plot these data, and note that there was an increase each year from 1912 to 1916 and also from 1918 to 1926. Find an approximate formula for the amount P, in terms of t (years), for each of these periods.

Year	Pounds	Year	Pounds
1912	1,100,000	1920	10,250,000
1913	1,560,000	1921	15,000,000
1914	2,400,000	1922	23,500,000
1915	4,100,000	1923	35,400,000
1916	5,750,000	1924	37,719,600
1917	6,700,000	1925	51,792,000
1918	5,800,000	1926	65,750,000
1919	8,180,000		

48. FITTING A CURVE OF THE FORM $y = Ax^n$

For a set of values $(x_1, y_1), \cdots, (x_k, y_k)$ to satisfy an equation of the form

$$y = Ax^n,$$

the graph determined by $(u_1, v_1), \cdots, (u_k, v_k)$, where $u_i = \log x_i$, $i = 1, 2, \cdots, k$, and $v_i = \log y_i$, $i = 1, 2, \cdots, k$, should be linear. This follows from the fact that

$$\log y = \log A + n \log x.$$

So, if the constant $\log A$ is designated by A_1 and $\log y$ is replaced by v and $\log x$ by u, we have the linear equation

$$v = A_1 + nu.$$

If the graph of $(u_1, v_1), \cdots, (u_k, v_k)$, where $u_i = \log x_i$, $i = 1, 2, \cdots, k$, and $v_i = \log y_i$, $i = 1, 2, \cdots, k$, is practically linear, we may find the best line for the relation between u and v, by either of the methods previously discussed, thereby obtaining an equation of the form

$$v = A_1 + nu.$$

It is then a simple step to obtain the desired function

$$y = Ax^n.$$

Illustration: Examine the possibility of fitting an equation of the form $y = Ax^n$, to the following data:

x	20	40	60	80	100
y	28.69	87	166.6	263.7	376.6

Recalling the relations

$$\log y = v, \qquad \text{and} \qquad \log x = u,$$

we obtain the following table of values:

u	1.301	1.602	1.778	1.903	2
v	1.458	1.939	2.222	2.421	2.576

The points corresponding to this table of values are essentially on a line, as shown in Figure 64.

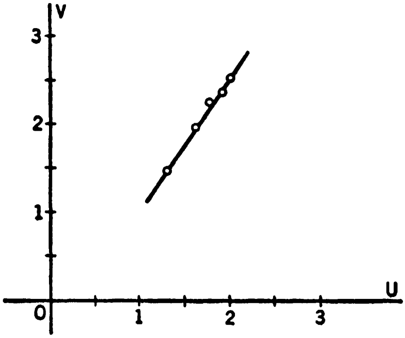


FIG. 64

An equation for the straight line in Figure 64 is determined to be

$$v = -0.623 + 1.6u.$$

So, in the equation $y = Ax^n$, n, which is the coefficient of u, is 1.6. Moreover,

$$A_1 = \log A = -0.623, \qquad \text{or} \qquad 0.377 - 1.$$

Hence, $$A = 0.238.$$

Consequently, the required function is

$$y = 0.238x^{1.6}.$$

49. LOGARITHMIC PAPER

In attempting to fit the function $y = Ax^n$ to the given data, the substitution $u = \log x$ and $v = \log y$ required the looking up of two sets of

logarithms corresponding to the values of x and y. In order to avoid the necessity of looking up these logarithms and preparing the data of the second table, another kind of special paper has been devised, called *logarithmic paper*. In this paper, on both the horizontal and vertical scales, the logarithms of x and y are measured and designated as on the vertical scale of the semilogarithmic paper. The name "semilogarithmic"

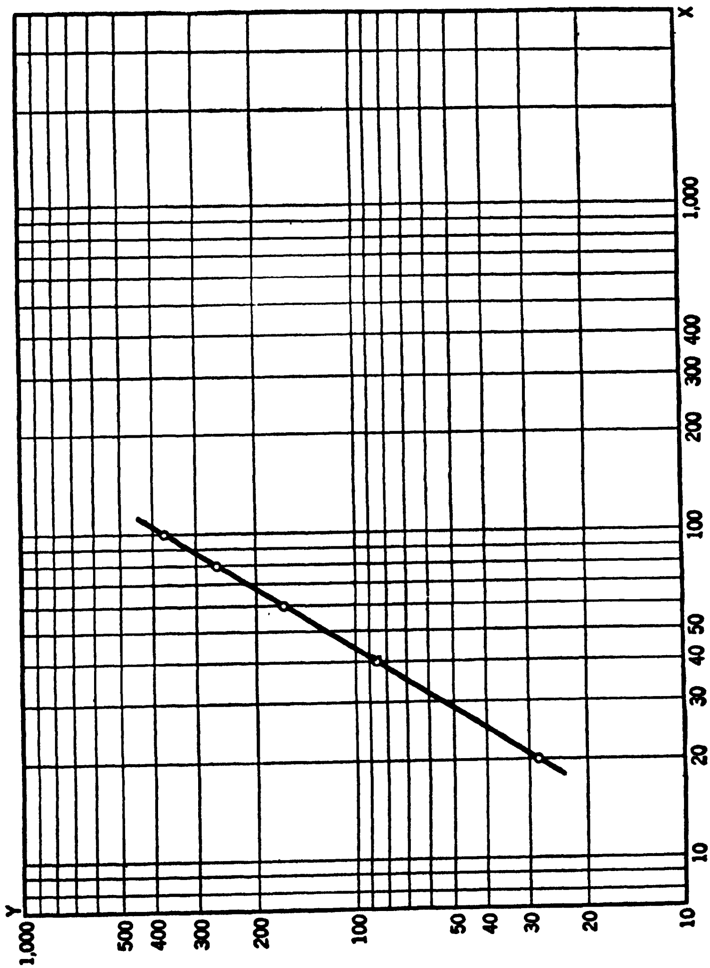


Fig. 65

refers to the fact that only one scale is logarithmic, whereas in this second type of paper both scales are logarithmic. The graph of the original data of the illustration in 48 is shown on logarithmic paper in Figure 65. It is a straight line, as one would anticipate.

By substituting from the first table in Section 48, two pairs of corresponding values of x and y in $y = Ax^n$, A and n may be found.

Thus, after substituting (20, 28.69) and (60, 166.6), we have

$$28.69 = A(20)^n \tag{1}$$

and

$$166.6 = A(60)^n. \tag{2}$$

By dividing the members of Equation (2) by the corresponding members of Equation (1), we have

$$\frac{166.6}{28.69} = \frac{(60)^n}{(20)^n} = (3)^n.$$

Consequently, after taking the logarithm of each member, there results

$$\log 166.6 - \log 28.69 = n \log 3,$$

$$n = \frac{\log 166.6 - \log 28.69}{\log 3}.$$

The completion of the computation is left as an exercise.

The value of A may be found by substituting this value of n in either Equation (1) or (2).

EXERCISES 31

1. The distances of the planets from the sun and their periods of revolution are given below. Determine a formula for T as a function of D. (NOTE: the distance from the earth to the sun is taken as the unit of distance.)

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
D (distance)	0.387	0.723	1.00	1.52	5.20	9.54	19.2	30.1
T (years)	0.24	0.615	1.00	1.88	11.9	29.5	84	165

2. The following data show the relation between the voltage v and the alternating current i flowing across a copper-carbon arc in air. Find the formula for v as a function of i.

v	36	32.5	30	28	25	23.2	20.1	19.2
i	1.2	1.4	2.0	2.5	3.5	4.6	7.0	9.5

3. The following data were obtained under the same conditions as in Exercise 2, except that a shorter arc was used. Find the formula for v as a function of i.

v	46	40.8	37	32.8	30	26.8	23.2
i	1.1	1.55	2.05	3.0	4.05	6.0	9.0

4. In a test to determine the impedance of a 1-hp 550-v three-phase motor, the following values of watts input and volts between terminals were observed. Find a formula expressing w as a function of v.

w	50	120	300	550	1150	1360	1860
v	60	98	150	200	290	320	370

50. FITTING A CURVE OF THE TYPE $y = A + Bx + Cx^2$

For a set of values (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) to satisfy an equation of the form

$$y = A + Bx + Cx^2,$$

the graph satisfied by (u_1, v_1), $\cdots$, (u_n, v_n), where

$$u_i = x_i + x', \qquad v_i = \frac{y_i - y'}{x_i - x'}, \qquad i = 1, 2, 3, \cdots, n,$$

and (x', y') is a point on the curve $y = A + Bx + Cx^2$, should be linear. This follows from the fact that the substitution of u for $x + x'$ and of v for $\dfrac{y - y'}{x - x'}$ results in a linear equation in u and v. This is easily demonstrated. If we take

$$y = A + Bx + Cx^2,$$

then

$$y' = A + Bx' + C(x')^2,$$

since (x', y') is on the curve.

Hence,

$$y - y' = B(x - x') + C[x^2 - (x')^2],$$

or

$$\frac{y - y'}{x - x'} = B + C(x + x').$$

Consequently, after making the substitutions already indicated for u and v, we have

$$v = B + Cu.$$

If the graph of (u_1, v_1), $\cdots$, (u_n, v_n) is approximately linear, we may find the equation of the best line in u and v. Then we know the desired values of B and C. Since (x', y') is on the curve, we may obtain A from the equation

$$y' = A + Bx' + C(x')^2,$$

and hence all the constants are determined. There is, however, a lack of mathematical precision in this method, inasmuch as it requires an element of guesswork to determine (x', y'). To determine a choice for (x', y'), it is advisable to graph the given values, and draw a freehand curve that

fits the values more or less closely. Any point that lies on this curve and is within the range of the given values will be sufficiently accurate for the test.

Illustration: Fit a curve to the following data:

x	0	0.5	1.0	1.5	2.0	3.0	3.5	4
y	1	1.7	2.0	1.4	−1.0	−8.0	−15.0	−19.0

If the points of the tabulated data are graphed, they seem to lie on a curve of the form

$$y = A + Bx + Cx^2.$$

A sketch appears as Figure 66.

We first make our selection of the point (x', y') and tabulate the data for u and v. Let us choose $x' = 1$ and $y' = 2$. We then construct the following table:

x	y	$u = x + x'$	$y - y'$	$x - x'$	$v = \dfrac{y - y'}{x - x'}$
0	1.0	1.0	−1.0	−1.0	1.0
0.5	1.7	1.5	−0.3	−0.5	0.6
1.0	2.0	2.0	0	0	
1.5	1.4	2.5	−0.6	0.5	−1.2
2.0	−1.0	3.0	−3.0	1.0	−3.0
3.0	−8.0	4.0	−10.0	2.0	−5.0
3.5	−15.0	4.5	−17.0	2.5	−6.8
4.0	−19.0	5.0	−21.0	3.0	−7.0

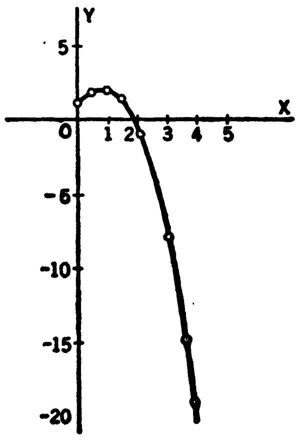


FIG. 66

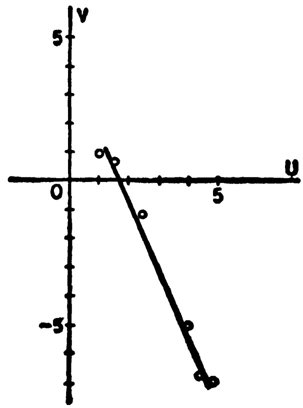


FIG. 67

The graph in u and v is practically linear, as shown in Figure 67, so the

graph in x and y may be expressed to a close degree of approximation by $y = A + Bx + Cx^2$. From Figure 67 we note that the points $(3, -3)$ and $(1.5, 0.4)$ are approximately on the line. Hence, after substituting these values in the equation $v = B + Cu$, we obtain

$$-3 = B + 3C$$

and

$$0.4 = B + 1.5C,$$

from which $C = -2.3$ and $B = 3.9$.

To find A we substitute $x' = 1$ and $y' = 2$ in the equation,

$$y = A + Bx + Cx^2,$$

thereby giving

$$2 = A + 3.9(1) - 2.3(1)^2,$$

or

$$A = 0.4.$$

Hence, the desired relation is

$$y = 0.4 + 3.9x - 2.3x^2.$$

51. FITTING A CURVE OF THE TYPE $y = \dfrac{A + Bx}{C + Dx}$

For a set of values $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ to satisfy an equation of the form

$$y = \frac{A + Bx}{C + Dx},$$

the graph satisfied by $(v_1, y_1), \cdots, (v_n, y_n)$, where

$$v_i = \frac{y_i - y'}{v_i - x'}, \qquad i = 1, 2, 3, \cdots, n,$$

and where (x', y') is a point on the curve $y = \dfrac{A + Bx}{C + Dx}$, should be linear. The derivation of this fact is as follows:

The original equation

$$y = \frac{A + Bx}{C + Dx}$$

may be written

$$y = \frac{\frac{A}{D} + \frac{B}{D}x}{\frac{C}{D} + x}$$

or

$$y = \frac{A_1 + B_1x}{C_1 + x},$$

which becomes

$$C_1y + xy = A_1 + B_1x. \tag{1}$$

Then, if (x', y') is a point on the curve, it follows that

$$C_1y' + x'y' = A_1 + B_1x'. \tag{2}$$

After subtracting the members of (2) from the corresponding members of (1), we obtain the following equalities:

$$C_1(y - y') + xy - x'y' = B_1(x - x'),$$

$$C_1(y - y') + xy - yx' + yx' - x'y' = B_1(x - x'),$$

$$C_1(y - y') + y(x - x') + x'(y - y') = B_1(x - x'),$$

$$\frac{y - y'}{x - x'} = \frac{B_1 - y}{C_1 + x'}. \tag{3}$$

If we let

$$v = \frac{y - y'}{x - x'} \quad \text{and} \quad K = C_1 + x', \tag{4}$$

then

$$v = \frac{B_1 - y}{K} \quad \text{or} \quad y = B_1 - Kv, \tag{5}$$

which is linear in v and y.

After determining the linear relation in v and y, the coefficients B_1 and K are known. Then C_1 may be found from (4) and A_1 from (2). These values substituted in (1) give the desired relation between x and y.

Illustration: Let us find the functional relation between x and y for the following data:

x	0	1	2	3	4	5	6
y	4.00	2.43	1.56	1.00	0.62	0.33	0.12

The points representing these values have been located in Figure 68, and a curve indicating the trend of the data has been sketched.

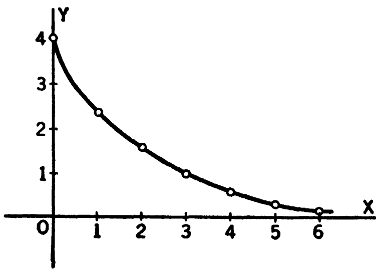


FIG. 68

From the graph one might suppose the points to lie on a curve either of type $y = AB^x$ or of type $y = \dfrac{A + Bx}{C + Dx}$. Plotting the data on semilogarithmic paper shows that the curve is not of type $y = AB^x$. Hence, we shall try the test for $y = \dfrac{A + Bx}{C + Dx}$.

Let $v = \dfrac{y - y'}{x - x'}$. Assume the point (0, 4) as the point (x', y'). Then calculate the values of the following table:

x	y	$x - x'$	$y - y'$	v
0	4.00	0	0	
1	2.43	1	−1.57	−1.57
2	1.56	2	−2.44	−1.22
3	1.00	3	−3	−1
4	0.62	4	−3.38	−0.84
5	0.33	5	−3.67	−0.73
6	0.12	6	−3.88	−0.65

Since the points of coordinates (v, y) are essentially on a straight line, as shown in Figure 69, the original points must lie on a curve of type $y = \dfrac{A + Bx}{C + Dx}$.

After substituting points (−1.57, 2.43) and (−0.73, 0.33) in Equation (5), we have

$$2.43 = B_1 + 1.57K$$

and

$$0.33 = B_1 + 0.73K.$$

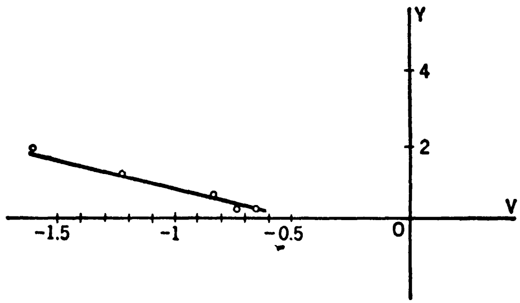


FIG. 69

The solution of this system yields $K = 2.5$ and $B_1 = -1.5$. From Equation (4),

$$C_1 = 2.5 - 0 = 2.5,$$

and from Equation (2),

$$2.5(4) + (0)(4) = A_1 + (-1.5)(0).$$

Therefore, $A_1 = 10$. Hence, by employing Equation (1), the required equation is

$$y = \frac{10 - 1.5x}{2.5 + x}.$$

EXERCISES 32

1. The temperature T of water as it cooled in a calorimeter was observed at frequent time intervals t, and the results were recorded in the following table:

t	0	5	10	15	20	25	30	35	40	45
T	79	68.4	61.5	56.1	52.1	48.7	45.9	43.5	41.2	39.6

Show that these data satisfy approximately an equation of type $T = \dfrac{A + Bt}{C + Dt}$.

Find the equation that is consistent with the given data, obtaining the values of A, B, C, and D correct to three significant figures.

2. The densities d of cerous chloride solution at 25°, in terms of molality m, are given in the following table:

m	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
d	0.99707	1.1079	1.2124	1.3107	1.4029	1.4902	1.5726	1.6506

Find a formula for d in terms of m, obtaining values of the constants accurate to five significant figures.

MISCELLANEOUS EXERCISES 33

1. The following data show the relation of length L and the corresponding weight W of the trout in Bearcamp River:

L, cm.....	10	12.6	16	18.6	20.5	21.6	22.7
W, gr.....	10	20	40	60	80	100	120

Find a formula expressing the weight as a function of the length for the trout in this river.

2. In an experiment to study the deflection d of the needle of a ballistic galvanometer caused by varying charges of electricity C through a capacitor, the following data were obtained:

C, μf...........	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
d, cm..........	10.50	9.51	8.50	7.50	6.34	5.34	4.31	3.22	2.14	1.06

Find a formula for d in terms of C.

3. The following table shows the horsepower H transmitted per inch of width of a double leather belt running at various speeds S:

H..................	0.6	1.2	1.75	2.6	2.8	3.2	3.4	3.38	3.15	2.7	2.4
S, fpm...............	500	1000	1500	2000	2500	3000	3500	4000	4500	5000	5500

Determine a formula for H as a function of S.

4. The following table shows the relation between the unit cost of production of a certain manufactured article and the number of pieces per lot:

C, cost per piece in dollars	5	3	1.80	1.40	1.20	1.13	1.08
n, number of pieces manufactured	1	2	5	10	20	30	50

Find a formula for C in terms of n.

5. Gas was allowed to expand adiabatically in a cylinder and the following corresponding values of pressure p and volume v were measured:

p, lb	1.10	0.70	0.46	0.31	0.21
v, cu ft	0.47	0.71	1.10	1.65	2.29

Find p as a function of v.

11 Parametric Equations

52. PARAMETRIC REPRESENTATION

The coordinates x and y of a point on a curve are often related through the medium of a third variable, or parameter, by expressing both x and y as functions of the parameter. If t is the parameter, and

$$\left.\begin{aligned} x &= f(t) \\ y &= \phi(t) \end{aligned}\right\}, \tag{1}$$

each choice of t within its permissible range gives a value of x and a value of y, it being assumed that the functions exist for some range of t. The points (x, y) determined in this manner constitute a curve, and Equations (1) are called the parametric equations of the curve.

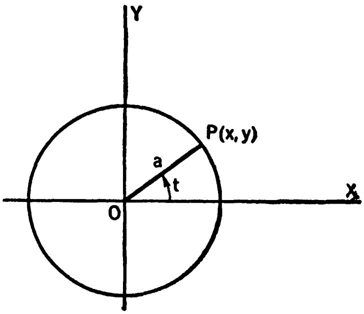


FIG. 70

Such parametric equations may arise in many ways. For example, the parametric equations for the circle of radius a and center at (0, 0) may be found by reference to Figure 70 as follows:

If we let the angle XOP be t, and choose t as our parameter, we have the two equations

$$x = a\cos t \tag{2}$$

and

$$y = a\sin t. \tag{3}$$

By giving t various values from 0 to 2π, we may find the corresponding values for x and y and thus graph the curve.

In the case of the circle, it is not necessary, nor particularly desirable for most purposes, to use parametric representation. On the other hand, let us now derive the parametric equations of the curve traced by a point on the circumference of a circle that rolls upon a straight line. This curve is called a *cycloid* and is shown in Figure 71.

Let the circle C roll upon the straight line OX. We wish to find the

parametric equations of the curve traced by the point P. Let a be the radius of the rolling circle, and assume that the circle has rolled from its position tangent to OX at O to the position tangent to OX at A, and that

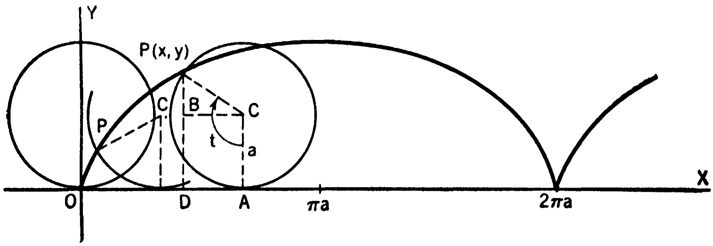


FIG. 71

in so doing the point P has traced the arc of the cycloid OP. Then, arc $AP = at$, where t is measured in radians, but arc $AP = OA = at$. It follows that

$$x = OD = OA - DA = at - BC = at - a \cos (t - \pi/2).$$

But, since $\cos(-\alpha) = \cos \alpha$, we have

$$x = at - a \cos (\pi/2 - t),$$

or $$x = at - a \sin t = a(t - \sin t). \tag{4}$$

Also $$y = DP = DB + BP = a + a \sin (t - \pi/2).$$

But, since $\sin(-\alpha) = -\sin \alpha$, we have

$$y = a - a \sin (\pi/2 - t)$$

or $$y = a - a \cos t = a(1 - \cos t). \tag{5}$$

Equations (4) and (5) are the required parametric equations for this curve, the cycloid.

In this case it would have been quite inconvenient to derive the Cartesian equation directly. Moreover, the properties of the curve are much easier to study by reference to its parametric representation.

The actual mechanics of graphing a pair of parametric equations is illustrated by the following example.

Illustration: Graph the parametric equations

$$\left.\begin{aligned} x &= 2t \\ y &= 3t - 1 \end{aligned}\right\}.$$

In the following table the values of t are chosen arbitrarily, after which the corresponding values of x and y are determined. Of course, only the

x and y values are used in locating points on the curve, which is a straight line (note Figure 72).

t	-2	-1	0	1	2	3	4
x	-4	-2	0	2	4	6	8
y	-7	-4	-1	2	5	8	11

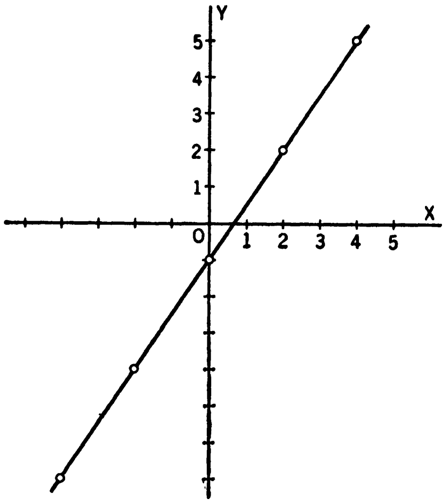


Fig. 72

53. ELIMINATION OF THE PARAMETER

Frequently it is possible to eliminate the parameter from two parametric equations, thereby obtaining a single equation in x and y. For example, if we square each member of Equations (2) and (3), we have

$$x^2 = a^2 \cos^2 t \qquad \text{and} \qquad y^2 = a^2 \sin^2 t.$$

After adding the right and left members, we obtain

$$x^2 + y^2 = a^2(\cos^2 t + \sin^2 t)$$

or

$$x^2 + y^2 = a^2.$$

This is the equation of a circle, as one would anticipate.

We may eliminate the parameter t from Equations (4) and (5) as follows: From Equation (5) we have

$$\cos t = \frac{a - y}{a} \qquad \text{or} \qquad t = \cos^{-1}\frac{a - y}{a},$$

whence

$$\sin t = \sqrt{1 - \left(\frac{a - y}{a}\right)^2} = \frac{\sqrt{2ay - y^2}}{a}.$$

After substituting the values for t and $\sin t$ in Equation (4), we have the Cartesian equation.

$$x = a\left(\cos^{-1}\frac{a - y}{a} - \frac{\sqrt{2ay - y^2}}{a}\right).$$

The student must not obtain the impression that the graph of a pair of parametric equations is necessarily identical with the graph of the equation resulting after the elimination of the parameter. The graph of the equation in x and y, after the parameter has been eliminated, will contain the graph of the parametric equations, but it may also contain additional points. This fact may be illustrated by considering the parametric equations

$$\left.\begin{aligned} x &= \sin t \\ y &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2t \end{aligned}\right\}.$$

Since $\cos 2t = 1 - 2\sin^2 t$, the parameter is readily eliminated as follows:

$$\begin{aligned} y &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2t = \tfrac{1}{2} - \tfrac{1}{2}(1 - 2\sin^2 t) \\ &= \sin^2 t = x^2. \end{aligned}$$

The graph of $y = x^2$ is a parabola with its vertex at the origin; obviously, such points as $(2, 4)$, $(-2, 4)$, $(3, 9)$, $(-3, 9)$, and so on, are on the curve. In examining the original parametric equation, however, it is observed that x and y are narrowly restricted in magnitude; for instance, $x = \sin t$ is restricted to the range $-1 \leqq x \leqq 1$. In fact, the graph of the parametric equations is merely the portion of the parabola in the neighborhood of the origin from $(-1, 1)$ to $(1, 1)$. The student should actually construct these graphs as an exercise.

EXERCISES 34

1. Draw the graphs of the following pairs of parametric equations:

(a) $x = 2t, \quad y = 4t - 1$ (b) $x = t^3, \quad y = 2t + 1$

(c) $x = \cos t, \quad y = \sin t$ (d) $x = \sin t, \quad y = \cos t$

(e) $x = \cos^3 t$, $y = \sin^3 t$. (This curve is known as a hypocycloid of four cusps.)

(f) $x = 4\cos t, \quad y = 2\sin t$.

2. Eliminate t from the parametric equations in Exercise 1(f), and show that the resulting equation is that of an ellipse.

3. Graph the curve whose parametric equations are

$$x = \frac{3t}{1 + t^3}, \qquad y = \frac{3t^2}{1 + t^3}.$$

This curve is known as the *folium of Descartes*.

4. Graph the curve $x = 10(t^3 - t)$; $y = 10t(t^3 - t)$. Find the Cartesian equation.

5. Show that the graph of $x = 2e^t$, $y = 4e^t - 1$ is only part of a straight line. What is the equation of the entire line in Cartesian coordinates?

6. Graph the curve $x = a \sin \theta$; $y = b \cos^3 \theta$. Find the Cartesian equation.

7. Graph the curve $x = a \tan \theta$; $y = a \cos^2 \theta$.

8. If a projectile starts with an initial velocity v_o in a direction which makes an angle α with the x axis, its position at any time t is given by the equations $x = v_o t \cos \alpha$, $y = v_o t \sin \alpha - \frac{1}{2}gt^2$. Find the Cartesian equation of the path of the projectile. What kind of a curve is it?

9. A gun stands on a cliff 1000 ft above the water. From the equations of Exercise 8, what elevation must be given to the gun so that a projectile may strike a point in the water 2 miles away from the base of the cliff? Given $v_o = 2000$ fps, $g = 32.3$

10. From the equations of Exercise 8, what elevation must be given to a gun to obtain a maximum range on a horizontal line passing through the muzzle?

11. Draw the graph of each of the following pairs of parametric equations, and find the corresponding Cartesian equation in (x, y) for each pair of parametric equations. In each case call attention to any difference between the graph of the parametric equations and that of its corresponding Cartesian equation.

(a) $x = t^2$; $y = \dfrac{12}{t^2}$ (b) $x = t - \dfrac{1}{t}$; $y = t + \dfrac{1}{t}$

(c) $x = t^2$; $y = 4t - t^3$ (d) $x = \sin t$; $y = \cos 2t$

(e) $x = \tan^2 t$; $y = \sec t$ (f) $x = 2t^2$; $y = 6t^2 + 7$

(g) $x = 2(1 + \cos t)$; $y = 5 \cos t$ (h) $x = \sin t$; $y = 1 - \cos 2t$

54. INVOLUTE OF A CIRCLE

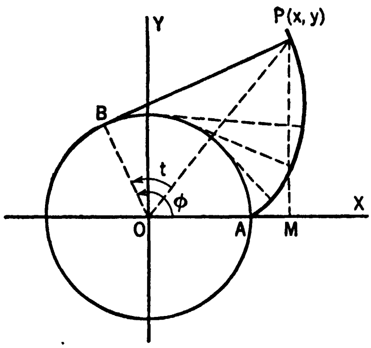


FIG. 73

If a string kept taut is unwound from the circumference of a circle, the end describes a curve called the *involute of a circle* (note Figure 73).

To find the parametric equations of the involute of a circle, let O be the center of the circle and AP a portion of the involute traced by the point P. Let $\angle AOB = \phi$, $\angle POB = t$, and the radius of the circle $= a$. Then arc $AB = BP$.

But arc $AB = a\phi$, if ϕ is measured in radians, and $BP = a \tan t$. Therefore,

$$a\phi = a \tan t \quad \text{or} \quad \phi = \tan t.$$

Also,

$$\begin{aligned} x &= OP \cos (\phi - t) = a \sec t \cos (\phi - t) \\ &= a \sec t(\cos \phi \cos t + \sin \phi \sin t) \\ &= a(\cos \phi + \sin \phi \tan t), \end{aligned}$$

which means that

$$x = a(\cos\phi + \phi\sin\phi).$$

Moreover,

$$\begin{aligned} y &= OP\sin(\phi - t) = a\sec t\sin(\phi - t) \\ &= a\sec t(\sin\phi\cos t - \cos\phi\sin t) \\ &= a(\sin\phi - \cos\phi\tan t); \end{aligned}$$

so

$$y = a(\sin\phi - \phi\cos\phi).$$

The two equations for x and y are the parametric equations for the involute of a circle. The involute is useful in gear design.

12 Solid Analytic Geometry

55. RECTANGULAR COORDINATES

In the rectangular coordinate system for solid analytic geometry a point is determined by its three distances from three intersecting planes, mutually perpendicular to one another. These planes divide space into eight portions called *octants*. The portion $O - XYZ$ is sometimes called the *first octant*. The other octants are not usually numbered.

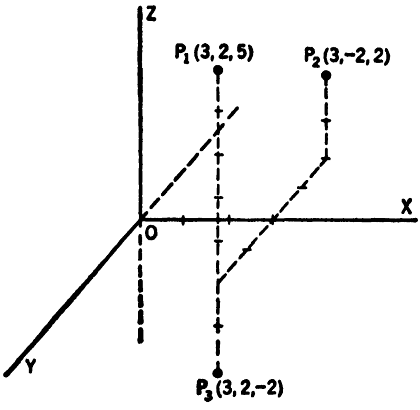


FIG. 74

Thus, in Figure 74 the three reference planes are XOY, XOZ, and YOZ. For short these planes are designated respectively as the xy, xz, and the yz planes. The point P_1 is determined by $x = 3$, $y = 2$, $z = 5$. The point P_2 is determined by $x = 3$, $y = -2$, $z = 2$. The point P_3 is determined by $x = 3$, $y = 2$, $z = -2$. It is evident how a point is determined in any octant of Figure 74.

56. EQUATIONS OF CERTAIN PLANES

In Figure 75, $x = 3$ is the equation of the plane parallel to the yz plane and 3 units to the right of it. The equation states that *any point* in this plane has 3 for its x coordinate. The plane parallel to the yz plane and 3 units to the left of it has for its equation $x = -3$. Similarly, the plane

parallel to the xz plane and 3 units forward on the y axis has for its equation $y = 3$.

In general, a plane $x = a$ is parallel to the yz plane, a plane $y = b$ is parallel to the xz plane, and a plane $z = c$ is parallel to the xy plane. In particular, the reference planes xy, xz, and yz have the equations $z = 0$, $y = 0$, and $x = 0$, respectively.

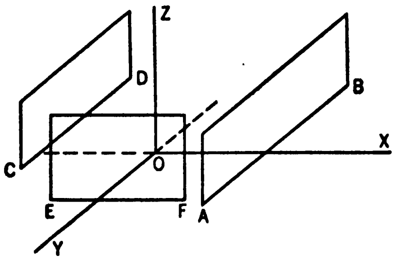


FIG. 75

We may also note that when a point is determined by the coordinates $x = a$, $y = b$, $z = c$, this point may be considered as the intersection of the *planes* $x = a$, $y = b$, $z = c$.

57. EQUATIONS OF CERTAIN LINES

In Figure 75, the line AB is determined by the intersection of the plane $x = 3$ with the plane $z = 0$. Hence, we say that the line AB has for its equations $x = 3$, $z = 0$; similarly, the line CD has for its equations $x = -3$, $z = 0$; and the line EF has for its equations $y = 3$, $z = 0$. In particular, the x axis is given by $y = 0$, $z = 0$; the y axis is given by $x = 0$, $z = 0$; and the z axis is given by $x = 0$, $y = 0$.

We have noted so far that the *planes* considered are given by *one* equation of the first degree; that the *lines* considered are given by *two* equations of the first degree, and that a *point* is given by *three* equations of the first degree. We shall see later that this conclusion applies to any plane and any line.

58. EQUATION OF A SURFACE

We have noted that the equation of a plane, which of course is a special case of a surface, is given by *one* equation. In general, the equation of a surface is given by one equation involving x, y, and z. The form of the surface is determined from the form of the function in x, y, and z.

As in plane analytic geometry, we have the problems of determining the surface, having given the equation, and of finding the equation, having given the surface.

Thus, having the equation $x^2 + y^2 + z^2 = 25$, we may readily determine

the surface. It is seen from the equation that

$$-5 \leqq x \leqq 5,$$

$$-5 \leqq y \leqq 5,$$

$$-5 \leqq z \leqq 5.$$

If we consider the intersection of the xy plane, that is, the plane of $z = 0$, and the surface of $x^2 + y^2 + z^2 = 25$, we are considering $z = 0$ and $x^2+y^2+z^2 = 25$ as a system of equations. Hence, we obtain $x^2+y^2 = 25$,

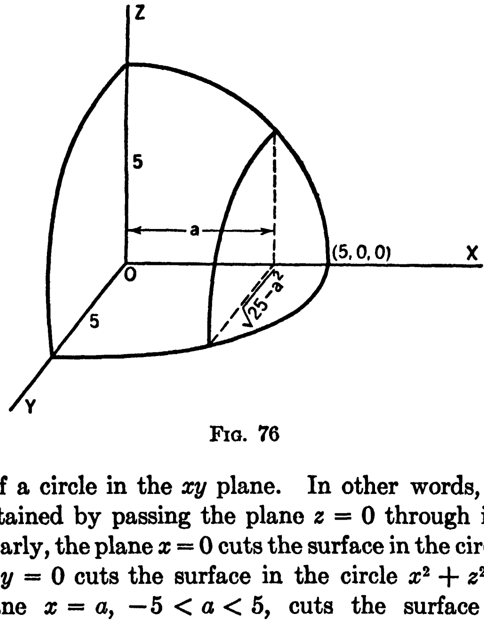


FIG. 76

the equation of a circle in the xy plane. In other words, the section of the surface obtained by passing the plane $z = 0$ through it is a circle of radius 5. Similarly, the plane $x = 0$ cuts the surface in the circle $y^2+z^2 = 25$, and the plane $y = 0$ cuts the surface in the circle $x^2 + z^2 = 25$. Moreover, any plane $x = a$, $-5 < a < 5$, cuts the surface in the circle $a^2 + y^2 + z^2 = 25$ or $y^2 + z^2 = 25 - a^2$. From the symmetry of the figure it is seen that the surface $x^2 + y^2 + z^2 = 25$ is a sphere and that one eighth of the surface is represented in Figure 76.

Conversely, we may now find the equation of the sphere whose center is $(0, 0, 0)$ and whose radius is 5.

In Figure 77 let P be any point on the sphere of center $(0, 0, 0)$ and radius 5. Hence, we have from right triangle OPP'

$$u^2 + z^2 = 25 \tag{1}$$

and from right triangle OAP'

$$u^2 = x^2 + y^2. \tag{2}$$

Hence, from (1) and (2) we have the equation of the sphere, namely, $x^2 + y^2 + z^2 = 25$.

In Section 59 and some of the following sections we shall present a more systematic study of points, lines, planes, and certain surfaces.

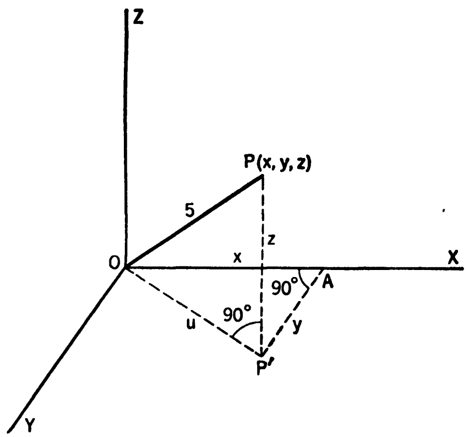


FIG. 77

EXERCISES 35

1. Plot the following points: $A(2, 3, -5)$, $B(7, -1, 3)$, $C(-3, -2, -5)$, and $D(-3, 3, 3)$.

2. Draw the following planes: (a) $x = 5$, (b) $y = -3$, (c) $x = 4$.

3. Draw the straight lines represented by each of the following systems of equations: (a) $x = 3$, $y = 2$; (b) $y = 5$, $z = 4$; (c) $x = 4$, $z = -3$.

4. Write the equation of a sphere with the center at the origin and radius 10.

5. What is the locus of all points 7 units below the xy plane? Write its equation.

6. Write the equations of the locus of all points 5 units above the xy plane and 3 units to the right of the yz plane.

7. Write the equation of the locus of all points 13 units from the origin.

59. DISTANCE BETWEEN TWO POINTS

We shall now derive a formula for the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

In Figure 78,

$$OB = x_1, \qquad BD = y_1, \qquad DP_1 = z_1,$$
$$OA = x_2, \qquad AC = y_2, \qquad CP_2 = z_2.$$

P_1F is drawn parallel to OX, and EF is drawn parallel to YO. The angle P_1EP_2 is a right angle, and the angle P_1FE is a right angle. Hence,

$$\overline{P_1P_2}^2 = d^2 = \overline{P_1E}^2 + \overline{EP_2}^2.$$

But $$\overline{P_1E}^2 = \overline{P_1F}^2 + \overline{EF}^2;$$

hence, $$d^2 = \overline{P_1F}^2 + \overline{EF}^2 + \overline{EP_2}^2.$$

Now $P_1F = x_2 - x_1$, $EF = y_2 - y_1$, and $EP_2 = z_2 - z_1$,

so $$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

or $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

which is the formula for the distance between P_1 and P_2.

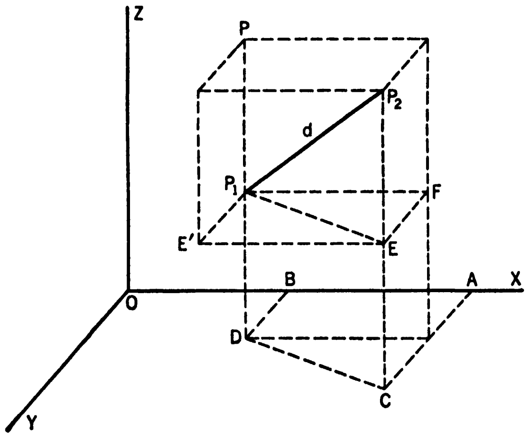


FIG. 78

60. DIRECTION COSINES OF A LINE

In Figure 79 we reproduce a portion of Figure 78 and draw P_2R parallel to EP_1. If angle $FP_1P_2 = \alpha$, where P_1F is parallel to OX, angle $MP_1P_2 = \beta$,

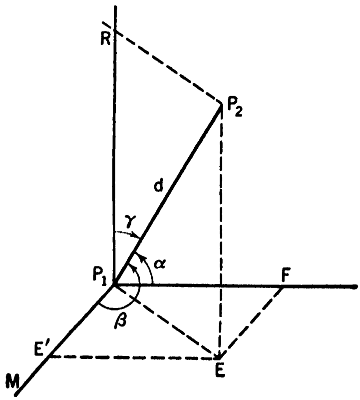


FIG. 79

where P_1M is parallel to OY, and angle $P_2P_1R = \gamma$, where P_1R is parallel to OZ, we refer to cos α, cos β, and cos γ as the direction cosines of the line

determined by P_1 and P_2. Since

$$P_1F = x_2 - x_1, \qquad \text{we have} \qquad \cos\alpha = \frac{x_2 - x_1}{d};$$

$$\text{and from} \quad P_1E' = FE = y_2 - y_1, \quad \text{we obtain} \quad \cos\beta = \frac{y_2 - y_1}{d};$$

$$\text{and from} \quad P_1R = EP_2 = z_2 - z_1, \quad \text{we have} \quad \cos\gamma = \frac{z_2 - z_1}{d}.$$

From these relations we have the following equation:

$$\frac{(x_2 - x_1)^2}{d^2} + \frac{(y_2 - y_1)^2}{d^2} + \frac{(z_2 - z_1)^2}{d^2} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

But from Section 59 we have seen that

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Hence, we have the important result

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

61. ANGLE BETWEEN TWO LINES

We shall define the angle between any two directed straight lines in space, whether or not they lie in the same plane, as the angle θ between the lines that pass through the origin parallel to the given lines.

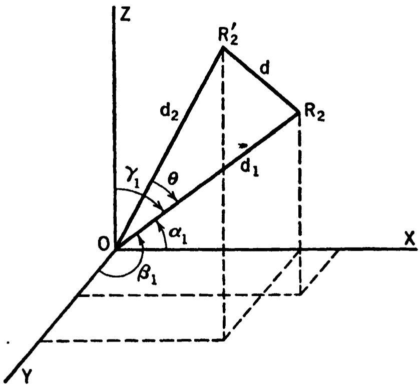


FIG. 80

We shall now develop a formula for $\cos\theta$ which serves as a means to determine θ, since between two directed lines there is but one angle θ such that $0 < \theta < 180°$.

Let P_1P_2 be the line whose direction angles are α_1, β_1, γ_1 and let $P_1'P_2'$ be the line whose direction angles are α_2, β_2, γ_2. We shall assume that the

lines are not parallel and shall draw through O lines respectively parallel to P_1P_2 and $P_1'P_2'$.

Let $R_2(x_1, y_1, z_1)$ and $R_2'(x_2, y_2, z_2)$ be any two points, except the origin, on the lines parallel to P_1P_2 and $P_1'P_2'$, respectively, and passing through the origin, and let $OR_2 = d_1$, $OR_2' = d_2$, and $R_2R_2' = d$ (note Figure 80). From the law of cosines in trigonometry we have

$$d^2 = d_1^2 + d_2^2 - 2d_1d_2 \cos\theta$$

or

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2d_1d_2\cos\theta,$$

from which we obtain $\cos\theta = \dfrac{(x_1x_2 + y_1y_2 + z_1z_2)}{d_1d_2}$.

Now $\quad x_1 = d_1 \cos\alpha_1,\ y_1 = d_1\cos\beta_1,\ z_1 = d_1\cos\gamma_1,$

and $\quad x_2 = d_2\cos\alpha_2,\ y_2 = d_2\cos\beta_2,\ z_2 = d_2\cos\gamma_2.$

Hence,

$$\cos\theta = \frac{d_1d_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2)}{d_1d_2},$$

or

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

Since α_1, β_1, γ_1, and α_2, β_2, γ_2 are supposed to be known, the formula determines $\cos\theta$, and hence θ.

EXERCISES 36

1. Find the length of each side of the triangle $A(3, 2, 5)$, $B(-1, 5, 2)$, and $C(7, 3, -1)$.

2. Find the direction cosines of each side of the triangle in Exercise 1.

3. Find the angle ABC; the angle BCA; the angle CAB, Exercise 1. Check by showing that the sum of these three angles equals 180°.

4. Show that the numbers $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ cannot be the direction cosines of a line.

5. If $\cos\alpha = \frac{1}{2}$, $\cos\beta = \frac{1}{4}$, find $\cos\gamma$, where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of a line.

6. Find the direction cosines of a line that makes equal angles with the positive end of the x, y, and z axes. Find the angles.

7. Find the equation of a sphere with center at the point $(2, 5, -1)$ and whose radius is 5.

8. Find the equation of the locus of all points equidistant from the points $(2, 1, 7)$ and $(-3, -5, 1)$.

9. If $\cos\alpha = \dfrac{1}{\sqrt{2}}$, $\cos\beta = \dfrac{1}{\sqrt{2}}$, and $\cos\gamma = 0$ are the direction cosines of a line through the point $(2, 3, 4)$, draw the line.

10. Find the equation of a sphere whose center is the point $(2, 9, 6)$ and which is tangent to the xz plane.

62. THE PLANE

We know from solid geometry that three noncollinear points determine a plane. Let us consider a portion of a plane BCD, whose perpendicular distance from the origin is p.

Let P_1 be the foot of the perpendicular from O to the plane, $P(x, y, z)$ any point in the plane other than P_1, α, β, γ the direction angles of OP, α_1, β_1, γ_1 the direction angles of OP_1, and the angle $POP_1 = \theta$. (Refer to Figure 81.) Then from Section 61,

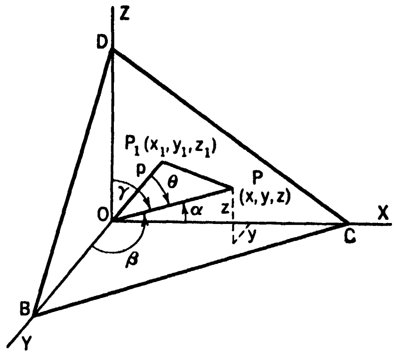


FIG. 81

$$\cos\theta = \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1.$$

But $$\cos\alpha = \frac{x}{OP}, \cos\beta = \frac{y}{OP}, \cos\gamma = \frac{z}{OP}, \text{ and } \cos\theta = \frac{p}{OP}.$$

Therefore, $$\frac{p}{OP} = \frac{x}{OP}\cos\alpha_1 + \frac{y}{OP}\cos\beta_1 + \frac{z}{OP}\cos\gamma_1,$$

or $$x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 = p, \tag{1}$$

which is the required equation of the first degree.

Conversely, the equation $Ax + By + Cz + D = 0$, $A \neq 0$, may be written in the form $x + \frac{By}{A} + \frac{Cz}{A} + \frac{D}{A} = 0$

or $$x + k_1y + k_2z + k_3 = 0. \tag{2}$$

This equation involves three undetermined constants k_1, k_2, and k_3. If we have three noncollinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), we may substitute them for x, y, and z of Equation (2) and thus obtain three linear equations, from which k_1, k_2, and k_3 may be determined.

If $Ax + By + Cz + D = 0$ is rewritten as

$$KAx + KBy + KCz + KD = 0$$

and compared with Equation (1), we note that $KA = \cos\alpha_1$, $KB = \cos\beta_1$, and $KC = \cos\gamma_1$,

or $$K^2A^2 + K^2B^2 + K^2C^2 = \cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1,$$

from which we obtain $K^2(A^2 + B^2 + C^2) = 1$.

Therefore, $$K = \frac{1}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

Hence, the general equation of first degree

$$Ax + By + Cz + D = 0$$

may be written in the form

$$\frac{Ax}{\pm\sqrt{A^2+B^2+C^2}}+\frac{By}{\pm\sqrt{A^2+B^2+C^2}}+\frac{Cz}{\pm\sqrt{A^2+B^2+C^2}}=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}},$$

which represents a plane whose perpendicular distance from the origin is $\dfrac{D}{\pm\sqrt{A^2+B^2+C^2}}\cdot$ The direction cosines of a perpendicular to this plane are

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}},\ \frac{B}{\pm\sqrt{A^2+B^2+C^2}},\ \frac{C}{\pm\sqrt{A^2+B^2+C^2}},$$

where the sign of the radical is taken opposite to that of D, if $D \neq 0$, to render the perpendicular distance always positive. If $D = 0$, then $p = 0$, and either sign may be used before the radical to determine the direction cosine of a perpendicular to this plane.

63. DISTANCE FROM A PLANE TO A POINT

Let $P_1(x_1, y_1, z_1)$ be a given point and $x \cos\alpha + y \cos\beta + z \cos\gamma = p$ be a given plane. Then a plane through P_1 parallel to the given plane has for its equation

$$x \cos\alpha + y \cos\beta + z \cos\gamma = q.$$

If d is the perpendicular distance from the given plane to the given point, then $q = p + d$. Hence, the equation of the plane through P_1 is

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p + d.$$

But since (x_1, y_1, z_1) is a point on this plane,

$$d = x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma - p.$$

It should be noted that d may be positive or negative.

64. THE ANGLE BETWEEN TWO PLANES

If

$$A_1x + B_1y + C_1z + D_1 = 0$$

and

$$A_2x + B_2y + C_2z + D_2 = 0$$

are two given planes, these two planes may be written as

$$\frac{A_1x + B_1y + C_1z + D_1}{\pm\sqrt{A_1^2+B_1^2+C_1^2}} = 0 \text{ and } \frac{A_2x + B_2y + C_2z + D_2}{\pm\sqrt{A_2^2+B_2^2+C_2^2}} = 0.$$

By definition the angle between two planes is the angle between the

normals to these planes. Let θ be this angle; then

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2)}}. \tag{1}$$

If the two planes are parallel, their perpendiculars are parallel. Hence, the two planes are parallel if

$$\frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{A_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{B_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

and

$$\frac{C_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{C_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

In other words, the two planes are parallel if

$$A_1 = A_2k,$$

$$B_1 = B_2k,$$

$$C_1 = C_2k,$$

where

$$k = \frac{\sqrt{A_1^2 + B_1^2 + C_1^2}}{\sqrt{A_2^2 + B_2^2 + C_2^2}} \neq 0.$$

If the two planes are perpendicular, $\cos\theta = 0$. Hence, by reference to Relation (1), the two planes are perpendicular if $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

65. THE EQUATION OF A PLANE IN TERMS OF ITS INTERCEPTS

The equation of the plane $Ax + By + Cz + D = 0$, where $D \neq 0$, may be written

$$\frac{x}{-\frac{D}{A}} + \frac{y}{-\frac{D}{B}} + \frac{z}{-\frac{D}{C}} = 1.$$

From this equation we see that if $y = 0$, and $z = 0$, then $x = -\frac{D}{A}$; if $x = 0$, and $y = 0$, then $z = -\frac{D}{C}$; and if $x = 0$, and $z = 0$, then $y = -\frac{D}{B}$. If we let $-\frac{D}{A} = a$, $-\frac{D}{B} = b$, and $-\frac{D}{C} = c$, we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

which is known as the intercept form of the equation of the plane, and a, b, and c are, respectively, the x, y, and z intercepts.

EXERCISES 37

1. Find the equation of the plane if the length of the perpendicular upon it from the origin is 7 units, and the direction cosines of this perpendicular with the x, y, and z axes, respectively, are $\frac{1}{2}$, $\frac{2}{3}$, and $\sqrt{11}/6$.

2. (a) Find the equation of the plane through the three points $A(3, 2, 5)$, $B(-1, 5, 2)$, and $C(7, 3, -1)$.
(b) Find the distance from the origin to this plane.
(c) Find the direction cosines of the perpendicular from the origin to the plane.

3. Draw a portion of the plane represented by the equation $2x + 3y - z = 6$.

HINT: Find the intercept on each axis and connect the points.

Definition: The intersection of a plane with a coordinate plane is called the *trace* of the plane on the coordinate plane. Thus, the trace of the plane in Exercise 3 on the xy plane is $2x + 3y = 6$, since $z = 0$ for all points in the xy plane.

4. Given the plane $3x - 5y + 10z = 20$. Find its trace on each coordinate plane.

5. Change the equation of the plane in Exercise 4 to the intercept form.

6. Find the perpendicular distance from the plane $2x - 3y + 6z = 12$ to the point $(10, 3, -1)$.

7. Find the angle between the planes $2x - 3y + 6z = 12$ and $x + y - z = 4$.

8. Find the equation of a plane parallel to $2x - 6y + 3z = 14$ and 10 units from the origin.

9. Find the equation of the plane tangent at $P(4, 12, 6)$ to the sphere whose center is the origin.

10. Find the equation of the plane passing through the points $(5, 5, 6)$ and $(-1, 3, 0)$, and parallel to the z axis.

11. Find the equation of a plane through the point $(6, 1, 5)$ parallel to the plane $3x - 2y - 6z = 5$.

12. Find the equation of the plane perpendicular to the line joining $(2, 1, 3)$ and $(4, -3, -1)$ at its mid-point.

13. The equation $2x - y + kz = 10$ represents a system of planes. Draw three or four planes of this system. How are the planes of this system related? Find the equation of the particular plane of this system that passes through the point $(1, 1, 1)$.

14. Describe the system of planes represented by the equation $3x + 6y - 2z = k$. Find the value of k so that one of these planes will pass through the point $(2, 1, 3)$.

15. (a) Draw the plane $2x - 3y = 6$.
(b) Draw the plane $x - z = 3$.

66. THE STRAIGHT LINE

In Section 57 we showed how certain lines are determined by two equations of the first degree. In general, every equation of the first degree represents a plane, and two nonparallel planes intersect in a straight line. Hence, in general, a straight line is determined by two equations of the

first degree. Thus, the system of equations

$$A_1x + B_1y + C_1z + D_1 = 0$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

are the equations of a straight line if the planes are not parallel.

If the straight line is to be determined by its direction cosines, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and a point (x_1, y_1, z_1), its equation may be found as follows: Let (x, y, z) be *any* point on the line other than (x_1, y_1, z_1); then from Section 60,

$$\cos\alpha = \frac{x - x_1}{d}, \quad \cos\beta = \frac{y - y_1}{d}, \quad \cos\gamma = \frac{z - z_1}{d},$$

where $$d = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}.$$

Hence, $$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{1}$$

If l, m, n are any three numbers proportional to the direction cosines of the line, they are called *direction numbers*.

Since $l = k\cos\alpha$, $m = k\cos\beta$, and $n = k\cos\gamma$, Equation (1) may be written

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \tag{2}$$

If the straight line is to be determined by two points (x_1, y_1, z_1) and (x_2, y_2, z_2), its equations may be found as follows: Let (x, y, z) be *any* point on the line other than (x_1, y_1, z_1) and (x_2, y_2, z_2); then,

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma},$$

and $$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}.$$

Thus, $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{3}$$

The direction cosines of a line may be determined by finding the equations of the line in Form (2). Thus, if the equations of the line are

$$x + 2y - 3z = 6, \quad \text{and} \quad 3x + 4y + z = 5,$$

we may eliminate any variable, such as z, and obtain

$$10x + 14y = 21 \quad \text{or} \quad y = \frac{-10x + 21}{14}.$$

If we now eliminate any other variable, such as x, we obtain

$$2y - 10z = 13 \qquad \text{or} \qquad y = \frac{10z + 13}{2}.$$

Hence,
$$\frac{-10x + 21}{14} = y = \frac{10z + 13}{2},$$

or
$$\frac{x - \frac{21}{10}}{-\frac{7}{5}} = \frac{y}{1} = \frac{z + \frac{13}{10}}{\frac{1}{5}}.$$

Comparing these equations with (2), we have

$$l = -\tfrac{7}{5}, \qquad m = 1, \qquad \text{and} \qquad n = \tfrac{1}{5}.$$

Since
$$\cos\alpha = \frac{l}{k}, \qquad \cos\beta = \frac{m}{k}, \qquad \text{and} \qquad \cos\gamma = \frac{n}{k},$$

and
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

we have
$$\frac{49}{25k^2} + \frac{1}{k^2} + \frac{1}{25k^2} = 1.$$

Restricting k to be positive, we find that $k = \sqrt{3}$.

Hence,
$$\cos\alpha = -\frac{7}{5\sqrt{3}}, \qquad \cos\beta = \frac{1}{\sqrt{3}}, \qquad \cos\gamma = \frac{1}{5\sqrt{3}}.$$

EXERCISES 38

1. Sketch the lines represented by each of the following pairs of equations:

(a) $z = 1,$ $x = 2.$

(b) $y + z = 3,$ $x + y = 3.$

(c) $x + 2y + z = 5,$ $z = 3.$

(d) $x + 2y + z = 5,$ $x + 2y = 6.$

(e) $2x - 3y + z = 6,$ $x + y + 2z = 4.$

2. Find the coordinates of the points in which the line $x + y - 2z = -10$, $3x - y + 3z = 6$, cuts each of the coordinate planes. Draw the line.

3. Find the coordinates of the point in which the line whose equations are $x - 3y + 5z = 18$, $2x + y - 3z = 3$, meets the plane $5x + 4y + z = 23$.

4. Find the equations of the straight lines through each of the following pairs of points:

(a) (0, 1, 0), (2, 5, 6)

(b) (3, −1, 4), (−2, 3, 5)

(c) (5, 0, −3), (0, −2, 5)

(d) (2, 0, −1), (0, 0, 5)

5. Find the equations of a line which passes through the point (1, 3, −5) and whose direction cosines are $\cos\alpha = \frac{1}{2}$, $\cos\beta = -\frac{2}{3}$, and $\cos\gamma = \sqrt{11}/6$.

6. Find the equations of a line through the point (2, −3, 4) and perpendicular to the plane $x - 2y + 2z = 7$. Draw a portion of the plane and the perpendicular. Find the coordinates of the point of intersection of the perpendicular and the plane.

7. Find the direction cosines of the lines represented by each of the following pairs of equations:

(a) $x - 2y + 3z = 12$, $2x + y - 2z = 8$ (b) $x + 2y = 10$, $5x - 7y + z = 22$

8. Show that the following lines are parallel:

(a) $4x - y + z + 4 = 0$, $2x + y + 2z + 5 = 0$ and $2x - 2y - z - 9 = 0$, $2x + y + 2z + 3 = 0$

(b) $x - 3y + 5z = 8$, $5x + 4y - 7z = 20$ and $19x + 19y - 33z = 52$, $16x + 9y - 16z = 11$

9. Find the angle between the two lines:

$x - 2y + 2z = 10$, $2x + y - 2z = 15$ and $2x - 3y + 6z = 12$, $6x + 2y - 3z = 30$

10. Show that the following lines are perpendicular:

(a) $\dfrac{x-1}{2} = \dfrac{y-2}{-5} = \dfrac{z+1}{2}$; $\dfrac{x+3}{1} = \dfrac{y}{2} = \dfrac{z-5}{4}$

(b) $x + z = 1$, $2x - 3y - 3z + 16 = 0$; $5x - y - z = 14$, $3x + 5y - 3z + 6 = 0$

67. SURFACES OF REVOLUTION

Let $z = f(x)$ be some curve in the xz plane. If we rotate a portion of the curve $z = f(x)$ about the x axis, we generate a surface of revolution

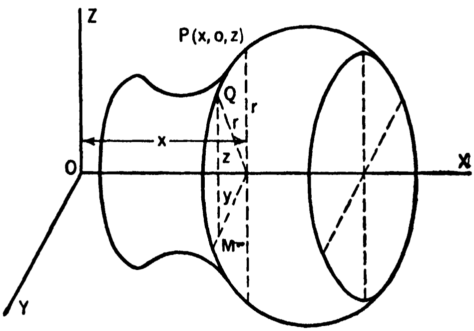


FIG. 82

whose equation may be found as follows: Let P be any point on the curve $z = f(x)$, as shown in Figure 82. Then, the rotation of $f(x)$ causes P to generate the circle $y^2 + z^2 = r^2$, where $r = z$ for the point P. Hence, $r = f(x)$. Therefore, we have $y^2 + z^2 = [f(x)]^2$ as the required equation of the surface.

Thus, as an illustration, if we rotate a line $z = a$ about the x axis, we obtain the cylinder of revolution displayed in Figure 83, whose equation is $y^2 + z^2 = a^2$.

Similarly, the equation of the cone generated by the line $z = 2x$ rotated about the x axis is $y^2 + z^2 = 4x^2$ (Figure 84). The equation of the cone

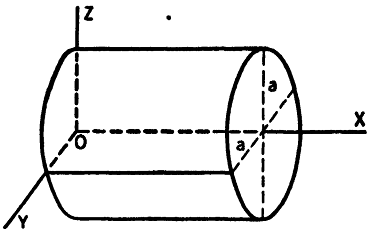


FIG. 83

generated by rotating the line $z = 2x$ about the z axis is $x^2 + y^2 = z^2/4$ (Figure 85).

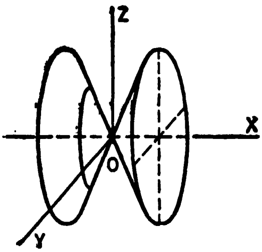


FIG. 84

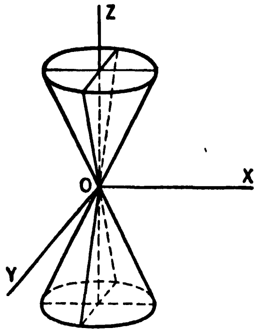


FIG. 85

The paraboloid of revolution generated by rotating $z^2 = 4x$ about the x axis has for its equation $y^2 + z^2 = 4x$ (Figure 86). If it is rotated about the z axis, the equation of the surface is $x^2 + y^2 = z^4/16$ (Figure 87). This surface is not designated a paraboloid.

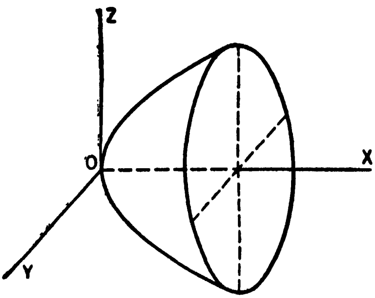


FIG. 86

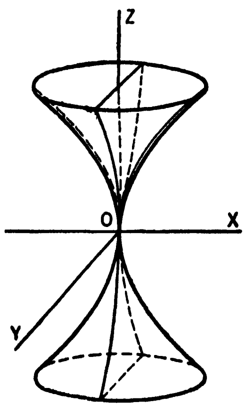


FIG. 87

If we rotate the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$ about the x axis, we obtain a surface whose equation is

$$y^2 + z^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

or

$$a^2y^2 + a^2z^2 + b^2x^2 = a^2b^2.$$

This may be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1.$$

This surface is called an *ellipsoid of revolution*.

If the ellipse is rotated about the z axis, we obtain the ellipsoid of revolution whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

EXERCISES 39

1. Write the equation of the surface generated by revolving the curve $y^2 = 6x$ about the x axis; about the y axis. Draw a figure representing the surface in each case.

2. Write the equation of the surface generated by revolving the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$ about the x axis. Draw a figure representing the surface.

3. Write the equation of the surface generated by revolving the curve $\frac{x^2}{9} - \frac{y^2}{4} = 1$ about the x axis; about the y axis. Draw the diagrams in each case.

4. Write the equation of the surface generated by revolving the curve $(x - 5)^2 + (y - 6)^2 = 16$ about the x axis. This surface is called a *torus*.

68. CERTAIN CONICOIDS

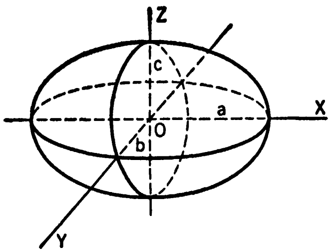
Any surface whose equation is of the second degree in three variables is called a *conicoid*, or a *quadric surface*. The general equation of a quadric surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + M = 0.$$

This equation, by translation and rotation of axes, is reducible to various standard forms. We list below the equations of a few important quadric surfaces. Compare each with the corresponding surface of revolution.

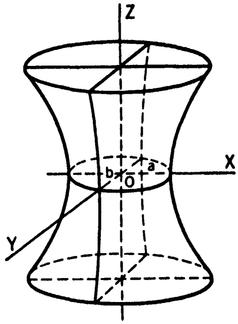
69. DRAWING SURFACES AND INTERSECTIONS OF SURFACES

In practice it is often necessary to find the area of certain surfaces, or certain portions of surfaces, as well as the volume bounded by a surface or surfaces. In general, these problems are solved by means of calculus. It is of great importance that the student be able to picture surfaces and the intersections of surfaces.



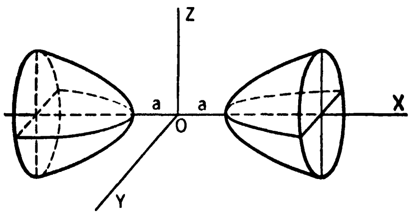
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

FIG. 88. ELLIPSOID



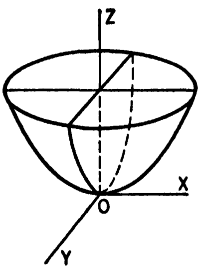
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

FIG. 89. HYPERBOLOID OF ONE SHEET



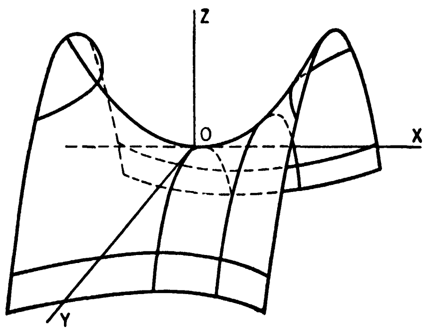
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

FIG. 90. HYPERBOLOID OF TWO SHEETS



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$

FIG. 91. ELLIPTIC PARABOLOID



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$$

FIG. 92. HYPERBOLIC PARABOLOID

In drawing a surface, what we really do is to draw various sections of the surface or lines on the surface, which show its character. In drawing the intersections of surfaces, it is a good practice to cut the various surfaces by some plane or planes whose intersections with them can be recognized. These planes cut curves out of the surfaces. A curve drawn through common points of the surfaces is a curve in which the surfaces intersect.

The quadric surfaces of Section 68 were drawn by picturing the sections of the surfaces with the coordinate planes and planes parallel to the co-

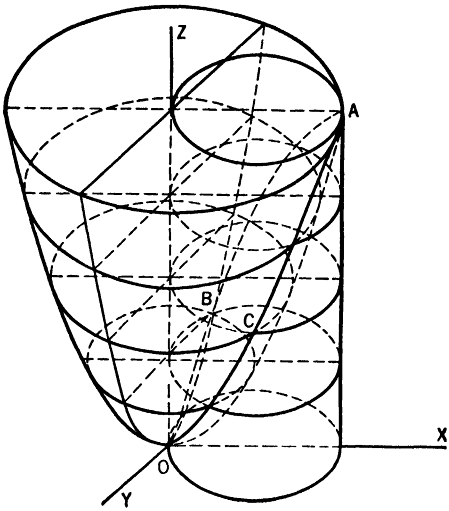


FIG. 93

ordinate planes. As an example in the drawing of surfaces and their intersections, we consider the surfaces $x^2 + y^2 = az$ and $x^2 + y^2 = 2ax$ and their intersections. If the surface $x^2 + y^2 = az$ is cut by any plane parallel to the xy plane, such as the plane $z = k$, we have a circle $x^2 + y^2 = ak$. In particular, if $k = 0$, we have the point (0, 0, 0). If the surface is cut by the plane $y = 0$, we have the parabola $x^2 = az$. Consequently, the surface may be pictured as a circle parallel to the xy plane sliding along the parabola $x^2 = az$ (note Figure 93). The surface $x^2 + y^2 = 2ax$ is a cylinder. Since the equation of the surface $x^2 + y^2 = 2ax$ is independent of z, the intersection of this surface by any plane parallel to the xy plane, such as $z = k$, will always give the circle $x^2 + y^2 = 2ax$.

The intersection of the circle $x^2 + y^2 = ak$ in the plane $z = k$ with the circle $x^2 + y^2 = 2ax$ in the plane $z = k$ will give points on the intersection of the surfaces. Solving these equations, we have $x = k/2$, and $y = \pm\frac{1}{2}\sqrt{4ak - k^2}$. Thus we have two points on the intersection of the two surfaces, namely, $(k/2, \frac{1}{2}\sqrt{4ak - k^2}, k)$ and $(k/2, -\frac{1}{2}\sqrt{4ak - k^2}, k)$.

For every value of k, two points on the intersection are determined. If these points are joined, we obtain the curve of intersection of the surfaces. This is the curve $OCABO$ in Figure 93.

As another example of drawing surfaces and their intersections, we consider the spherical surface $x^2 + y^2 + z^2 = a^2$ and the cylindrical surface $x^2 + y^2 = ax$.

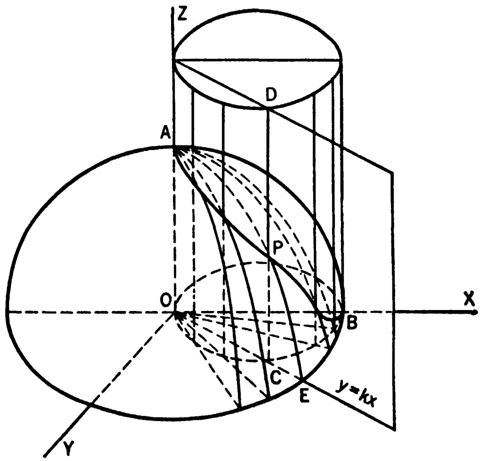


FIG. 94

The plane $y = kx$ will cut out of the cylindrical surface the line OA and the line CD, and out of the spherical surface a circle of which the arc AE is a quadrant. If we solve the system

$$y = kx, x^2 + y^2 = ax,$$

and
$$x^2 + y^2 + z^2 = a^2$$

for x, y, and z, we will get points of intersection of the two given surfaces. Solving simultaneously $x^2 + y^2 = ax$ and $y = kx$, we have $x = 0$ and $x = \dfrac{a}{1 + k^2}$. Substituting these values in $y = kx$, we have $y = 0$ and $y = \dfrac{ak}{1 + k^2}$. Substituting $x = 0$, $y = 0$ and $x = \dfrac{a}{1 + k^2}$, $y = \dfrac{ak}{1 + k^2}$ in $x^2 + y^2 + z^2 = a^2$, we have $z = \pm a$ and $z = \pm \dfrac{ak}{\sqrt{1 + k^2}}$. Hence, we have obtained the points of intersection of the surfaces as $(0, 0, \pm a)$ and

$$\left(\frac{a}{1 + k^2}, \frac{ak}{1 + k^2}, \pm \frac{ak}{\sqrt{1 + k^2}}\right).$$

For every value of k, four points on the intersection are determined. If

these points are joined, we obtain the curve of intersection of the surfaces. Figure 94 pictures the curve of intersection for the first octant only.

EXERCISES 40

1. Draw figures to represent each of the following surfaces:

(a) $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{25} = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} + \frac{z^2}{25} = 1$

(c) $y^2 + x^2 = 4z^2$ (d) $y^2 + z^2 = 25$

(e) $\frac{y^2}{16} + \frac{z^2}{9} = 4x$

2. (a) Draw a diagram representing the surface $z = x^2 + 4y^2$, and its intersection with the plane $z = 10$.
(b) Draw the intersection of this surface with the plane $x + y + z = 5$.

3. Draw a diagram showing the volume bounded by the surfaces $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.

4. Draw a diagram showing the intersection of the surfaces $x^2 + y^2 = 10x$ and $4x^2 + 4y^2 = z^2$.

5. Draw a figure showing the intersection of the surfaces $x^2 + y^2 = 100$ and $y^2 + z^2 = 100$, and the volume bounded by these surfaces.

6. Draw a diagram showing the intersection of the surface of the paraboloid $y^2 + z^2 = 2ax$ and the cylindrical surface $y^2 = ax$. Show also the volume bounded by these two surfaces and the planes $x = a$ and $z = 0$.

70. SPHERICAL COORDINATES AND CYLINDRICAL COORDINATES

It is often convenient to use polar coordinates in space; these are usually referred to as spherical coordinates. The spherical coordinates of a point P are ρ, its distance from the origin; α, the angle made by the projection of OP on the xy plane with the x axis; and β, the angle made by OP with the z axis. Hence, if the Cartesian coordinates of P are x, y, z, we have from Figure 95 the equations

$$x = \rho \sin \beta \cos \alpha,$$

$$y = \rho \sin \beta \sin \alpha,$$

and

$$z = \rho \cos \beta.$$

Thus, the point (3, 4, 5) in Cartesian coordinates may be expressed in spherical coordinates as follows:

$$3 = \rho \sin \beta \cos \alpha, \tag{1}$$

$$4 = \rho \sin \beta \sin \alpha, \tag{2}$$

and

$$5 = \rho \cos \beta. \tag{3}$$

Dividing (2) by (1) we have $\tan \alpha = \frac{4}{3}$ or $\alpha = \tan^{-1} \frac{4}{3}$, so $\sin \alpha = \frac{4}{5}$.

Dividing (2) by (3), after substituting $\sin \alpha = \frac{4}{5}$, we have $\tan \beta = 1$, or $\beta = 45°$. Substituting $\cos \beta = 1/\sqrt{2}$ in (3), we obtain $\rho = 5\sqrt{2}$. Consequently, the spherical coordinates of the point are $(5\sqrt{2}, \tan^{-1} \frac{4}{3}, 45°)$.

Another system of coordinates used quite often in scientific practice is known as *cylindrical coordinates*. In this system a point P is determined by coordinates α, r, and z, where α has the same significance as in spherical

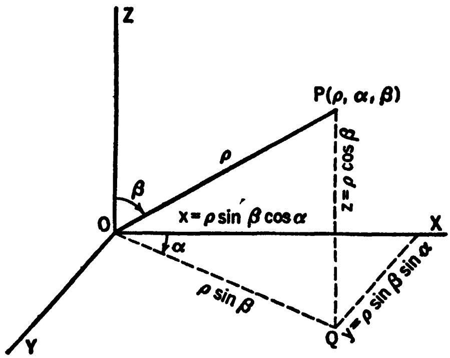


FIG. 95

coordinates, r is the projection of OP on the xy plane, and z has the same significance as in Cartesian coordinates.

Hence, if the Cartesian coordinates of P are x, y, z, we have, from Figure 96,

$$x = r \cos \alpha,$$

$$y = r \sin \alpha,$$

and

$$z = z.$$

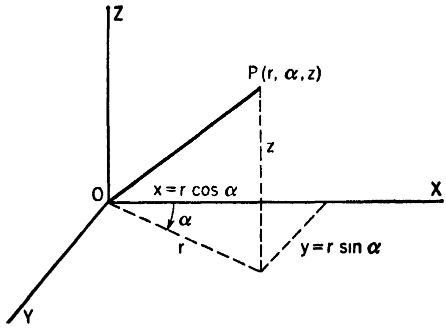


FIG. 96

Thus, a point (3, 4, 5) may be expressed in cylindrical coordinates as follows:

$$3 = r \cos \alpha,$$
$$4 = r \sin \alpha,$$

and

$$z = 5.$$

Solving, we get $\alpha = \tan^{-1} \frac{4}{3}$, $r = 5$.
Hence, the cylindrical coordinates of P are $(5, \tan^{-1} \frac{4}{3}, 5)$.

EXERCISES 41

1. (*a*) Find the equation of the sphere $x^2 + y^2 + z^2 = 4$ in spherical coordinates.
(*b*) In cylindrical coordinates.

2. (*a*) Find the equation of $x^2 + y^2 = 2ax$ in spherical coordinates.
(*b*) In cylindrical coordinates.

3. (*a*) Find the equation of $x^2 + y^2 = 2az$ in spherical coordinates.
(*b*) In cylindrical coordinates.

4. Transform $x^2 + y^2 = z^2$ into cylindrical coordinates; into spherical coordinates.

5. Transform $x^2 + y^2 + 2z^2 = 4$ into cylindrical coordinates; into spherical coordinates.

6. Transform the following from spherical to rectangular coordinates: (*a*) $\rho = 8$; (*b*) $\rho \sin \alpha \sin \beta = 8$; $\rho = 3 \cos \beta$

100 — Five-Place Common Logarithms — 150

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

100 — Five-Place Common Logarithms — 150

150 — Five-Place Common Logarithms — 200

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Parts

150 — Five-Place Common Logarithms — 200

200 — Five-Place Common Logarithms — 250

N	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20	19	18	17
1	2.0	1.9	1.8	1.7
2	4.0	3.8	3.6	3.4
3	6.0	5.7	5.4	5.1
4	8.0	7.6	7.2	6.8
5	10.0	9.5	9.0	8.5
6	12.0	11.4	10.8	10.2
7	14.0	13.3	12.6	11.9
8	16.0	15.2	14.4	13.6
9	18.0	17.1	16.2	15.3

Prop. Parts

250 — Five-Place Common Logarithms — 300

N	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

300 — Five-Place Common Logarithms — 350

N	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

350 — Five-Place Common Logarithms — 400

N	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	13	12	11	10
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

Prop. Parts

350 — Five-Place Common Logarithms — 400

400 — Five-Place Common Logarithms — 450

N	0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

400 — Five-Place Common Logarithms — 450

450 — Five-Place Common Logarithms — 500

N	0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

450 — Five-Place Common Logarithms — 500

500 — Five-Place Common Logarithms — 550

N	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

500 — Five-Place Common Logarithms — 550

550 — Five-Place Common Logarithms — 600

N	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

550 — Five-Place Common Logarithms — 600

600 — Five-Place Common Logarithms — 650

N	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Parts

650 — Five-Place Common Logarithms — 700

N	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Parts

650 — Five-Place Common Logarithms — 700

700 — Five-Place Common Logarithms — 750

N	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

700 — Five-Place Common Logarithms — 750

750 — Five-Place Common Logarithms — 800

N	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

800 — Five-Place Common Logarithms — 850

N	0	1	2	3	4	5	6	7	8	9	Prop. Parts
800	90 309	314	320	325	331	336	342	347	352	358	
801	363	369	374	380	385	390	396	401	407	412	
802	417	423	428	434	439	445	450	455	461	466	
803	472	477	482	488	493	499	504	509	515	520	
804	526	531	536	542	547	553	558	563	569	574	
805	580	585	590	596	601	607	612	617	623	628	
806	634	639	644	650	655	660	666	671	677	682	
807	687	693	698	703	709	714	720	725	730	736	
808	741	747	752	757	763	768	773	779	784	789	
809	795	800	806	811	816	822	827	832	838	843	
810	849	854	859	865	870	875	881	886	891	897	
811	902	907	913	918	924	929	934	940	945	950	
812	956	961	966	972	977	982	988	993	998	*004	
813	91 009	014	020	025	030	036	041	046	052	057	6
814	062	068	073	078	084	089	094	100	105	110	1 0.6 2 1.2 3 1.8 4 2.4 5 3.0 6 3.6 7 4.2 8 4.8 9 5.4
815	116	121	126	132	137	142	148	153	158	164	
816	169	174	180	185	190	196	201	206	212	217	
817	222	228	233	238	243	249	254	259	265	270	
818	275	281	286	291	297	302	307	312	318	323	
819	328	334	339	344	350	355	360	365	371	376	
820	381	387	392	397	403	408	413	418	424	429	
821	434	440	445	450	455	461	466	471	477	482	
822	487	492	498	503	508	514	519	524	529	535	
823	540	545	551	556	561	566	572	577	582	587	
824	593	598	603	609	614	619	624	630	635	640	
825	645	651	656	661	666	672	677	682	687	693	
826	698	703	709	714	719	724	730	735	740	745	
827	751	756	761	766	772	777	782	787	793	798	
828	803	808	814	819	824	829	834	840	845	850	
829	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	950	955	
831	960	965	971	976	981	986	991	997	*002	*007	
832	92 012	018	023	028	033	038	044	049	054	059	5
833	065	070	075	080	085	091	096	101	106	111	1 0.5 2 1.0 3 1.5 4 2.0 5 2.5 6 3.0 7 3.5 8 4.0 9 4.5
834	117	122	127	132	137	143	148	153	158	163	
835	169	174	179	184	189	195	200	205	210	215	
836	221	226	231	236	241	247	252	257	262	267	
837	273	278	283	288	293	298	304	309	314	319	
838	324	330	335	340	345	350	355	361	366	371	
839	376	381	387	392	397	402	407	412	418	423	
840	428	433	438	443	449	454	459	464	469	474	
841	480	485	490	495	500	505	511	516	521	526	
842	531	536	542	547	552	557	562	567	572	578	
843	583	588	593	598	603	609	614	619	624	629	
844	634	639	645	650	655	660	665	670	675	681	
845	686	691	696	701	706	711	716	722	727	732	
846	737	742	747	752	758	763	768	773	778	783	
847	788	793	799	804	809	814	819	824	829	834	
848	840	845	850	855	860	865	870	875	881	886	
849	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N	0	1	2	3	4	5	6	7	8	9	Prop. Parts

800 — Five-Place Common Logarithms — 850

850 — Five-Place Common Logarithms — 900

N	0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

900 — Five-Place Common Logarithms — 950

N	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

950 — Five-Place Common Logarithms — 1000

N	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

TABLE 2

Natural Trigonometric Functions

0°

′	Sin	Tan	Ctn	Cos	′
0	.00000	.00000	—	1.0000	**60**
1	.00029	.00029	3437.7	1.0000	59
2	.00058	.00058	1718.9	1.0000	58
3	.00087	.00087	1145.9	1.0000	57
4	.00116	.00116	859.44	1.0000	56
5	.00145	.00145	687.55	1.0000	**55**
6	.00175	.00175	572.96	1.0000	54
7	.00204	.00204	491.11	1.0000	53
8	.00233	.00233	429.72	1.0000	52
9	.00262	.00262	381.97	1.0000	51
10	.00291	.00291	343.77	1.0000	**50**
11	.00320	.00320	312.52	.99999	49
12	.00349	.00349	286.48	.99999	48
13	.00378	.00378	264.44	.99999	47
14	.00407	.00407	245.55	.99999	46
15	.00436	.00436	229.18	.99999	**45**
16	.00465	.00465	214.86	.99999	44
17	.00495	.00495	202.22	.99999	43
18	.00524	.00524	190.98	.99999	42
19	.00553	.00553	180.93	.99998	41
20	.00582	.00582	171.89	.99998	**40**
21	.00611	.00611	163.70	.99998	39
22	.00640	.00640	156.26	.99998	38
23	.00669	.00669	149.47	.99998	37
24	.00698	.00698	143.24	.99998	36
25	.00727	.00727	137.51	.99997	**35**
26	.00756	.00756	132.22	.99997	34
27	.00785	.00785	127.32	.99997	33
28	.00814	.00815	122.77	.99997	32
29	.00844	.00844	118.54	.99996	31
30	.00873	.00873	114.59	.99996	**30**
31	.00902	.00902	110.89	.99996	29
32	.00931	.00931	107.43	.99996	28
33	.00960	.00960	104.17	.99995	27
34	.00989	.00989	101.11	.99995	26
35	.01018	.01018	98.218	.99995	**25**
36	.01047	.01047	95.489	.99995	24
37	.01076	.01076	92.908	.99994	23
38	.01105	.01105	90.463	.99994	22
39	.01134	.01135	88.144	.99994	21
40	.01164	.01164	85.940	.99993	**20**
41	.01193	.01193	83.844	.99993	19
42	.01222	.01222	81.847	.99993	18
43	.01251	.01251	79.943	.99992	17
44	.01280	.01280	78.126	.99992	16
45	.01309	.01309	76.390	.99991	**15**
46	.01338	.01338	74.729	.99991	14
47	.01367	.01367	73.139	.99991	13
48	.01396	.01396	71.615	.99990	12
49	.01425	.01425	70.153	.99990	11
50	.01454	.01455	68.750	.99989	**10**
51	.01483	.01484	67.402	.99989	9
52	.01513	.01513	66.105	.99989	8
53	.01542	.01542	64.858	.99988	7
54	.01571	.01571	63.657	.99988	6
55	.01600	.01600	62.499	.99987	**5**
56	.01629	.01629	61.383	.99987	4
57	.01658	.01658	60.306	.99986	3
58	.01687	.01687	59.266	.99986	2
59	.01716	.01716	58.261	.99985	1
60	.01745	.01746	57.290	.99985	**0**
′	Cos	Ctn	Tan	Sin	′

89°

1°

′	Sin	Tan	Ctn	Cos	′
0	.01745	.01746	57.290	.99985	**60**
1	.01774	.01775	56.351	.99984	59
2	.01803	.01804	55.442	.99984	58
3	.01832	.01833	54.561	.99983	57
4	.01862	.01862	53.709	.99983	56
5	.01891	.01891	52.882	.99982	**55**
6	.01920	.01920	52.081	.99982	54
7	.01949	.01949	51.303	.99981	53
8	.01978	.01978	50.549	.99980	52
9	.02007	.02007	49.816	.99980	51
10	.02036	.02036	49.104	.99979	**50**
11	.02065	.02066	48.412	.99979	49
12	.02094	.02095	47.740	.99978	48
13	.02123	.02124	47.085	.99977	47
14	.02152	.02153	46.449	.99977	46
15	.02181	.02182	45.829	.99976	**45**
16	.02211	.02211	45.226	.99976	44
17	.02240	.02240	44.639	.99975	43
18	.02269	.02269	44.066	.99974	42
19	.02298	.02298	43.508	.99974	41
20	.02327	.02328	42.964	.99973	**40**
21	.02356	.02357	42.433	.99972	39
22	.02385	.02386	41.916	.99972	38
23	.02414	.02415	41.411	.99971	37
24	.02443	.02444	40.917	.99970	36
25	.02472	.02473	40.436	.99969	**35**
26	.02501	.02502	39.965	.99969	34
27	.02530	.02531	39.506	.99968	33
28	.02560	.02560	39.057	.99967	32
29	.02589	.02589	38.618	.99966	31
30	.02618	.02619	38.188	.99966	**30**
31	.02647	.02648	37.769	.99965	29
32	.02676	.02677	37.358	.99964	28
33	.02705	.02706	36.956	.99963	27
34	.02734	.02735	36.563	.99963	26
35	.02763	.02764	36.178	.99962	**25**
36	.02792	.02793	35.801	.99961	24
37	.02821	.02822	35.431	.99960	23
38	.02850	.02851	35.070	.99959	22
39	.02879	.02881	34.715	.99959	21
40	.02908	.02910	34.368	.99958	**20**
41	.02938	.02939	34.027	.99957	19
42	.02967	.02968	33.694	.99956	18
43	.02996	.02997	33.366	.99955	17
44	.03025	.03026	33.045	.99954	16
45	.03054	.03055	32.730	.99953	**15**
46	.03083	.03084	32.421	.99952	14
47	.03112	.03114	32.118	.99952	13
48	.03141	.03143	31.821	.99951	12
49	.03170	.03172	31.528	.99950	11
50	.03199	.03201	31.242	.99949	**10**
51	.03228	.03230	30.960	.99948	9
52	.03257	.03259	30.683	.99947	8
53	.03286	.03288	30.412	.99946	7
54	.03316	.03317	30.145	.99945	6
55	.03345	.03346	29.882	.99944	**5**
56	.03374	.03376	29.624	.99943	4
57	.03403	.03405	29.371	.99942	3
58	.03432	.03434	29.122	.99941	2
59	.03461	.03463	28.877	.99940	1
60	.03490	.03492	28.636	.99939	**0**
′	Cos	Ctn	Tan	Sin	′

88°

Natural Trigonometric Functions

2°

′	Sin	Tan	Ctn	Cos	′
0	.03490	.03492	28.636	.99939	60
1	.03519	.03521	28.399	.99938	59
2	.03548	.03550	28.166	.99937	58
3	.03577	.03579	27.937	.99936	57
4	.03606	.03609	27.712	.99935	56
5	.03635	.03638	27.490	.99934	55
6	.03664	.03667	27.271	.99933	54
7	.03693	.03696	27.057	.99932	53
8	.03723	.03725	26.845	.99931	52
9	.03752	.03754	26.637	.99930	51
10	.03781	.03783	26.432	.99929	50
11	.03810	.03812	26.230	.99927	49
12	.03839	.03842	26.031	.99926	48
13	.03868	.03871	25.835	.99925	47
14	.03897	.03900	25.642	.99924	46
15	.03926	.03929	25.452	.99923	45
16	.03955	.03958	25.264	.99922	44
17	.03984	.03987	25.080	.99921	43
18	.04013	.04016	24.898	.99919	42
19	.04042	.04046	24.719	.99918	41
20	.04071	.04075	24.542	.99917	40
21	.04100	.04104	24.368	.99916	39
22	.04129	.04133	24.196	.99915	38
23	.04159	.04162	24.026	.99913	37
24	.04188	.04191	23.859	.99912	36
25	.04217	.04220	23.695	.99911	35
26	.04246	.04250	23.532	.99910	34
27	.04275	.04279	23.372	.99909	33
28	.04304	.04308	23.214	.99907	32
29	.04333	.04337	23.058	.99906	31
30	.04362	.04366	22.904	.99905	30
31	.04391	.04395	22.752	.99904	29
32	.04420	.04424	22.602	.99902	28
33	.04449	.04454	22.454	.99901	27
34	.04478	.04483	22.308	.99900	26
35	.04507	.04512	22.164	.99898	25
36	.04536	.04541	22.022	.99897	24
37	.04565	.04570	21.881	.99896	23
38	.04594	.04599	21.743	.99894	22
39	.04623	.04628	21.606	.99893	21
40	.04653	.04658	21.470	.99892	20
41	.04682	.04687	21.337	.99890	19
42	.04711	.04716	21.205	.99889	18
43	.04740	.04745	21.075	.99888	17
44	.04769	.04774	20.946	.99886	16
45	.04798	.04803	20.819	.99885	15
46	.04827	.04833	20.693	.99883	14
47	.04856	.04862	20.569	.99882	13
48	.04885	.04891	20.446	.99881	12
49	.04914	.04920	20.325	.99879	11
50	.04943	.04949	20.206	.99878	10
51	.04972	.04978	20.087	.99876	9
52	.05001	.05007	19.970	.99875	8
53	.05030	.05037	19.855	.99873	7
54	.05059	.05066	19.740	.99872	6
55	.05088	.05095	19.627	.99870	5
56	.05117	.05124	19.516	.99869	4
57	.05146	.05153	19.405	.99867	3
58	.05175	.05182	19.296	.99866	2
59	.05205	.05212	19.188	.99864	1
60	.05234	.05241	19.081	.99863	0
′	Cos	Ctn	Tan	Sin	′

87°

3°

′	Sin	Tan	Ctn	Cos	′
0	.05234	.05241	19.081	.99863	60
1	.05263	.05270	18.976	.99861	59
2	.05292	.05299	18.871	.99860	58
3	.05321	.05328	18.768	.99858	57
4	.05350	.05357	18.666	.99857	56
5	.05379	.05387	18.564	.99855	55
6	.05408	.05416	18.464	.99854	54
7	.05437	.05445	18.366	.99852	53
8	.05466	.05474	18.268	.99851	52
9	.05495	.05503	18.171	.99849	51
10	.05524	.05533	18.075	.99847	50
11	.05553	.05562	17.980	.99846	49
12	.05582	.05591	17.886	.99844	48
13	.05611	.05620	17.793	.99842	47
14	.05640	.05649	17.702	.99841	46
15	.05669	.05678	17.611	.99839	45
16	.05698	.05708	17.521	.99838	44
17	.05727	.05737	17.431	.99836	43
18	.05756	.05766	17.343	.99834	42
19	.05785	.05795	17.256	.99833	41
20	.05814	.05824	17.169	.99831	40
21	.05844	.05854	17.084	.99829	39
22	.05873	.05883	16.999	.99827	38
23	.05902	.05912	16.915	.99826	37
24	.05931	.05941	16.832	.99824	36
25	.05960	.05970	16.750	.99822	35
26	.05989	.05999	16.668	.99821	34
27	.06018	.06029	16.587	.99819	33
28	.06047	.06058	16.507	.99817	32
29	.06076	.06087	16.428	.99815	31
30	.06105	.06116	16.350	.99813	30
31	.06134	.06145	16.272	.99812	29
32	.06163	.06175	16.195	.99810	28
33	.06192	.06204	16.119	.99808	27
34	.06221	.06233	16.043	.99806	26
35	.06250	.06262	15.969	.99804	25
36	.06279	.06291	15.895	.99803	24
37	.06308	.06321	15.821	.99801	23
38	.06337	.06350	15.748	.99799	22
39	.06366	.06379	15.676	.99797	21
40	.06395	.06408	15.605	.99795	20
41	.06424	.06438	15.534	.99793	19
42	.06453	.06467	15.464	.99792	18
43	.06482	.06496	15.394	.99790	17
44	.06511	.06525	15.325	.99788	16
45	.06540	.06554	15.257	.99786	15
46	.06569	.06584	15.189	.99784	14
47	.06598	.06613	15.122	.99782	13
48	.06627	.06642	15.056	.99780	12
49	.06656	.06671	14.990	.99778	11
50	.06685	.06700	14.924	.99776	10
51	.06714	.06730	14.860	.99774	9
52	.06743	.06759	14.795	.99772	8
53	.06773	.06788	14.732	.99770	7
54	.06802	.06817	14.669	.99768	6
55	.06831	.06847	14.606	.99766	5
56	.06860	.06876	14.544	.99764	4
57	.06889	.06905	14.482	.99762	3
58	.06918	.06934	14.421	.99760	2
59	.06947	.06963	14.361	.99758	1
60	.06976	.06993	14.301	.99756	0
′	Cos	Ctn	Tan	Sin	′

86°

Natural Trigonometric Functions

4°

′	Sin	Tan	Ctn	Cos	′
0	.06976	.06993	14.301	.99756	**60**
1	.07005	.07022	14.241	.99754	59
2	.07034	.07051	14.182	.99752	58
3	.07063	.07080	14.124	.99750	57
4	.07092	.07110	14.065	.99748	56
5	.07121	.07139	14.008	.99746	**55**
6	.07150	.07168	13.951	.99744	54
7	.07179	.07197	13.894	.99742	53
8	.07208	.07227	13.838	.99740	52
9	.07237	.07256	13.782	.99738	51
10	.07266	.07285	13.727	.99736	**50**
11	.07295	.07314	13.672	.99734	49
12	.07324	.07344	13.617	.99731	48
13	.07353	.07373	13.563	.99729	47
14	.07382	.07402	13.510	.99727	46
15	.07411	.07431	13.457	.99725	**45**
16	.07440	.07461	13.404	.99723	44
17	.07469	.07490	13.352	.99721	43
18	.07498	.07519	13.300	.99719	42
19	.07527	.07548	13.248	.99716	41
20	.07556	.07578	13.197	.99714	**40**
21	.07585	.07607	13.146	.99712	39
22	.07614	.07636	13.096	.99710	38
23	.07643	.07665	13.046	.99708	37
24	.07672	.07695	12.996	.99705	36
25	.07701	.07724	12.947	.99703	**35**
26	.07730	.07753	12.898	.99701	34
27	.07759	.07782	12.850	.99699	33
28	.07788	.07812	12.801	.99696	32
29	.07817	.07841	12.754	.99694	31
30	.07846	.07870	12.706	.99692	**30**
31	.07875	.07899	12.659	.99689	29
32	.07904	.07929	12.612	.99687	28
33	.07933	.07958	12.566	.99685	27
34	.07962	.07987	12.520	.99683	26
35	.07991	.08017	12.474	.99680	**25**
36	.08020	.08046	12.429	.99678	24
37	.08049	.08075	12.384	.99676	23
38	.08078	.08104	12.339	.99673	22
39	.08107	.08134	12.295	.99671	21
40	.08136	.08163	12.251	.99668	**20**
41	.08165	.08192	12.207	.99666	19
42	.08194	.08221	12.163	.99664	18
43	.08223	.08251	12.120	.99661	17
44	.08252	.08280	12.077	.99659	16
45	.08281	.08309	12.035	.99657	**15**
46	.08310	.08339	11.992	.99654	14
47	.08339	.08368	11.950	.99652	13
48	.08368	.08397	11.909	.99649	12
49	.08397	.08427	11.867	.99647	11
50	.08426	.08456	11.826	.99644	**10**
51	.08455	.08485	11.785	.99642	9
52	.08484	.08514	11.745	.99639	8
53	.08513	.08544	11.705	.99637	7
54	.08542	.08573	11.664	.99635	6
55	.08571	.08602	11.625	.99632	**5**
56	.08600	.08632	11.585	.99630	4
57	.08629	.08661	11.546	.99627	3
58	.08658	.08690	11.507	.99625	2
59	.08687	.08720	11.468	.99622	1
60	.08716	.08749	11.430	.99619	**0**
′	Cos	Ctn	Tan	Sin	′

85°

5°

′	Sin	Tan	Ctn	Cos	′
0	.08716	.08749	11.430	.99619	**60**
1	.08745	.08778	11.392	.99617	59
2	.08774	.08807	11.354	.99614	58
3	.08803	.08837	11.316	.99612	57
4	.08831	.08866	11.279	.99609	56
5	.08860	.08895	11.242	.99607	**55**
6	.08889	.08925	11.205	.99604	54
7	.08918	.08954	11.168	.99602	53
8	.08947	.08983	11.132	.99599	52
9	.08976	.09013	11.095	.99596	51
10	.09005	.09042	11.059	.99594	**50**
11	.09034	.09071	11.024	.99591	49
12	.09063	.09101	10.988	.99588	48
13	.09092	.09130	10.953	.99586	47
14	.09121	.09159	10.918	.99583	46
15	.09150	.09189	10.883	.99580	**45**
16	.09179	.09218	10.848	.99578	44
17	.09208	.09247	10.814	.99575	43
18	.09237	.09277	10.780	.99572	42
19	.09266	.09306	10.746	.99570	41
20	.09295	.09335	10.712	.99567	**40**
21	.09324	.09365	10.678	.99564	39
22	.09353	.09394	10.645	.99562	38
23	.09382	.09423	10.612	.99559	37
24	.09411	.09453	10.579	.99556	36
25	.09440	.09482	10.546	.99553	**35**
26	.09469	.09511	10.514	.99551	34
27	.09498	.09541	10.481	.99548	33
28	.09527	.09570	10.449	.99545	32
29	.09556	.09600	10.417	.99542	31
30	.09585	.09629	10.385	.99540	**30**
31	.09614	.09658	10.354	.99537	29
32	.09642	.09688	10.322	.99534	28
33	.09671	.09717	10.291	.99531	27
34	.09700	.09746	10.260	.99528	26
35	.09729	.09776	10.229	.99526	**25**
36	.09758	.09805	10.199	.99523	24
37	.09787	.09834	10.168	.99520	23
38	.09816	.09864	10.138	.99517	22
39	.09845	.09893	10.108	.99514	21
40	.09874	.09923	10.078	.99511	**20**
41	.09903	.09952	10.048	.99508	19
42	.09932	.09981	10.019	.99506	18
43	.09961	.10011	9.9893	.99503	17
44	.09990	.10040	9.9601	.99500	16
45	.10019	.10069	9.9310	.99497	**15**
46	.10048	.10099	9.9021	.99494	14
47	.10077	.10128	9.8734	.99491	13
48	.10106	.10158	9.8448	.99488	12
49	.10135	.10187	9.8164	.99485	11
50	.10164	.10216	9.7882	.99482	**10**
51	.10192	.10246	9.7601	.99479	9
52	.10221	.10275	9.7322	.99476	8
53	.10250	.10305	9.7044	.99473	7
54	.10279	.10334	9.6768	.99470	6
55	.10308	.10363	9.6493	.99467	**5**
56	.10337	.10393	9.6220	.99464	4
57	.10366	.10422	9.5949	.99461	3
58	.10395	.10452	9.5679	.99458	2
59	.10424	.10481	9.5411	.99455	1
60	.10453	.10510	9.5144	.99452	**0**
′	Cos	Ctn	Tan	Sin	′

84°

Natural Trigonometric Functions

6°

′	Sin	Tan	Ctn	Cos	′
0	.10453	.10510	9.5144	.99452	**60**
1	.10482	.10540	9.4878	.99449	59
2	.10511	.10569	9.4614	.99446	58
3	.10540	.10599	9.4352	.99443	57
4	.10569	.10628	9.4090	.99440	56
5	.10597	.10657	9.3831	.99437	**55**
6	.10626	.10687	9.3572	.99434	54
7	.10655	.10716	9.3315	.99431	53
8	.10684	.10746	9.3060	.99428	52
9	.10713	.10775	9.2806	.99424	51
10	.10742	.10805	9.2553	.99421	**50**
11	.10771	.10834	9.2302	.99418	49
12	.10800	.10863	9.2052	.99415	48
13	.10829	.10893	9.1803	.99412	47
14	.10858	.10922	9.1555	.99409	46
15	.10887	.10952	9.1309	.99406	**45**
16	.10916	.10981	9.1065	.99402	44
17	.10945	.11011	9.0821	.99399	43
18	.10973	.11040	9.0579	.99396	42
19	.11002	.11070	9.0338	.99393	41
20	.11031	.11099	9.0098	.99390	**40**
21	.11060	.11128	8.9860	.99386	39
22	.11089	.11158	8.9623	.99383	38
23	.11118	.11187	8.9387	.99380	37
24	.11147	.11217	8.9152	.99377	36
25	.11176	.11246	8.8919	.99374	**35**
26	.11205	.11276	8.8686	.99370	34
27	.11234	.11305	8.8455	.99367	33
28	.11263	.11335	8.8225	.99364	32
29	.11291	.11364	8.7996	.99360	31
30	.11320	.11394	8.7769	.99357	**30**
31	.11349	.11423	8.7542	.99354	29
32	.11378	.11452	8.7317	.99351	28
33	.11407	.11482	8.7093	.99347	27
34	.11436	.11511	8.6870	.99344	26
35	.11465	.11541	8.6648	.99341	**25**
36	.11494	.11570	8.6427	.99337	24
37	.11523	.11600	8.6208	.99334	23
38	.11552	.11629	8.5989	.99331	22
39	.11580	.11659	8.5772	.99327	21
40	.11609	.11688	8.5555	.99324	**20**
41	.11638	.11718	8.5340	.99320	19
42	.11667	.11747	8.5126	.99317	18
43	.11696	.11777	8.4913	.99314	17
44	.11725	.11806	8.4701	.99310	16
45	.11754	.11836	8.4490	.99307	**15**
46	.11783	.11865	8.4280	.99303	14
47	.11812	.11895	8.4071	.99300	13
48	.11840	.11924	8.3863	.99297	12
49	.11869	.11954	8.3656	.99293	11
50	.11898	.11983	8.3450	.99290	**10**
51	.11927	.12013	8.3245	.99286	9
52	.11956	.12042	8.3041	.99283	8
53	.11985	.12072	8.2838	.99279	7
54	.12014	.12101	8.2636	.99276	6
55	.12043	.12131	8.2434	.99272	**5**
56	.12071	.12160	8.2234	.99269	4
57	.12100	.12190	8.2035	.99265	3
58	.12129	.12219	8.1837	.99262	2
59	.12158	.12249	8.1640	.99258	1
60	.12187	.12278	8.1443	.99255	**0**
′	Cos	Ctn	Tan	Sin	′

83°

7°

′	Sin	Tan	Ctn	Cos	′
0	.12187	.12278	8.1443	.99255	**60**
1	.12216	.12308	8.1248	.99251	59
2	.12245	.12338	8.1054	.99248	58
3	.12274	.12367	8.0860	.99244	57
4	.12302	.12397	8.0667	.99240	56
5	.12331	.12426	8.0476	.99237	**55**
6	.12360	.12456	8.0285	.99233	54
7	.12389	.12485	8.0095	.99230	53
8	.12418	.12515	7.9906	.99226	52
9	.12447	.12544	7.9718	.99222	51
10	.12476	.12574	7.9530	.99219	**50**
11	.12504	.12603	7.9344	.99215	49
12	.12533	.12633	7.9158	.99211	48
13	.12562	.12662	7.8973	.99208	47
14	.12591	.12692	7.8789	.99204	46
15	.12620	.12722	7.8606	.99200	**45**
16	.12649	.12751	7.8424	.99197	44
17	.12678	.12781	7.8243	.99193	43
18	.12706	.12810	7.8062	.99189	42
19	.12735	.12840	7.7882	.99186	41
20	.12764	.12869	7.7704	.99182	**40**
21	.12793	.12899	7.7525	.99178	39
22	.12822	.12929	7.7348	.99175	38
23	.12851	.12958	7.7171	.99171	37
24	.12880	.12988	7.6996	.99167	36
25	.12908	.13017	7.6821	.99163	**35**
26	.12937	.13047	7.6647	.99160	34
27	.12966	.13076	7.6473	.99156	33
28	.12995	.13106	7.6301	.99152	32
29	.13024	.13136	7.6129	.99148	31
30	.13053	.13165	7.5958	.99144	**30**
31	.13081	.13195	7.5787	.99141	29
32	.13110	.13224	7.5618	.99137	28
33	.13139	.13254	7.5449	.99133	27
34	.13168	.13284	7.5281	.99129	26
35	.13197	.13313	7.5113	.99125	**25**
36	.13226	.13343	7.4947	.99122	24
37	.13254	.13372	7.4781	.99118	23
38	.13283	.13402	7.4615	.99114	22
39	.13312	.13432	7.4451	.99110	21
40	.13341	.13461	7.4287	.99106	**20**
41	.13370	.13491	7.4124	.99102	19
42	.13399	.13521	7.3962	.99098	18
43	.13427	.13550	7.3800	.99094	17
44	.13456	.13580	7.3639	.99091	16
45	.13485	.13609	7.3479	.99087	**15**
46	.13514	.13639	7.3319	.99083	14
47	.13543	.13669	7.3160	.99079	13
48	.13572	.13698	7.3002	.99075	12
49	.13600	.13728	7.2844	.99071	11
50	.13629	.13758	7.2687	.99067	**10**
51	.13658	.13787	7.2531	.99063	9
52	.13687	.13817	7.2375	.99059	8
53	.13716	.13846	7.2220	.99055	7
54	.13744	.13876	7.2066	.99051	6
55	.13773	.13906	7.1912	.99047	**5**
56	.13802	.13935	7.1759	.99043	4
57	.13831	.13965	7.1607	.99039	3
58	.13860	.13995	7.1455	.99035	2
59	.13889	.14024	7.1304	.99031	1
60	.13917	.14054	7.1154	.99027	**0**
′	Cos	Ctn	Tan	Sin	′

82°

Natural Trigonometric Functions

8°

′	Sin	Tan	Ctn	Cos	′
0	.13917	.14054	7.1154	.99027	**60**
1	.13946	.14084	7.1004	.99023	59
2	.13975	.14113	7.0855	.99019	58
3	.14004	.14143	7.0706	.99015	57
4	.14033	.14173	7.0558	.99011	56
5	.14061	.14202	7.0410	.99006	**55**
6	.14090	.14232	7.0264	.99002	54
7	.14119	.14262	7.0117	.98998	53
8	.14148	.14291	6.9972	.98994	52
9	.14177	.14321	6.9827	.98990	51
10	.14205	.14351	6.9682	.98986	**50**
11	.14234	.14381	6.9538	.98982	49
12	.14263	.14410	6.9395	.98978	48
13	.14292	.14440	6.9252	.98973	47
14	.14320	.14470	6.9110	.98969	46
15	.14349	.14499	6.8969	.98965	**45**
16	.14378	.14529	6.8828	.98961	44
17	.14407	.14559	6.8687	.98957	43
18	.14436	.14588	6.8548	.98953	42
19	.14464	.14618	6.8408	.98948	41
20	.14493	.14648	6.8269	.98944	**40**
21	.14522	.14678	6.8131	.98940	39
22	.14551	.14707	6.7994	.98936	38
23	.14580	.14737	6.7856	.98931	37
24	.14608	.14767	6.7720	.98927	36
25	.14637	.14796	6.7584	.98923	**35**
26	.14666	.14826	6.7448	.98919	34
27	.14695	.14856	6.7313	.98914	33
28	.14723	.14886	6.7179	.98910	32
29	.14752	.14915	6.7045	.98906	31
30	.14781	.14945	6.6912	.98902	**30**
31	.14810	.14975	6.6779	.98897	29
32	.14838	.15005	6.6646	.98893	28
33	.14867	.15034	6.6514	.98889	27
34	.14896	.15064	6.6383	.98884	26
35	.14925	.15094	6.6252	.98880	**25**
36	.14954	.15124	6.6122	.98876	24
37	.14982	.15153	6.5992	.98871	23
38	.15011	.15183	6.5863	.98867	22
39	.15040	.15213	6.5734	.98863	21
40	.15069	.15243	6.5606	.98858	**20**
41	.15097	.15272	6.5478	.98854	19
42	.15126	.15302	6.5350	.98849	18
43	.15155	.15332	6.5223	.98845	17
44	.15184	.15362	6.5097	.98841	16
45	.15212	.15391	6.4971	.98836	**15**
46	.15241	.15421	6.4846	.98832	14
47	.15270	.15451	6.4721	.98827	13
48	.15299	.15481	6.4596	.98823	12
49	.15327	.15511	6.4472	.98818	11
50	.15356	.15540	6.4348	.98814	**10**
51	.15385	.15570	6.4225	.98809	9
52	.15414	.15600	6.4103	.98805	8
53	.15442	.15630	6.3980	.98800	7
54	.15471	.15660	6.3859	.98796	6
55	.15500	.15689	6.3737	.98791	**5**
56	.15529	.15719	6.3617	.98787	4
57	.15557	.15749	6.3496	.98782	3
58	.15586	.15779	6.3376	.98778	2
59	.15615	.15809	6.3257	.98773	1
60	.15643	.15838	6.3138	.98769	**0**
′	Cos	Ctn	Tan	Sin	′

81°

9°

′	Sin	Tan	Ctn	Cos	′
0	.15643	.15838	6.3138	.98769	**60**
1	.15672	.15868	6.3019	.98764	59
2	.15701	.15898	6.2901	.98760	58
3	.15730	.15928	6.2783	.98755	57
4	.15758	.15958	6.2666	.98751	56
5	.15787	.15988	6.2549	.98746	**55**
6	.15816	.16017	6.2432	.98741	54
7	.15845	.16047	6.2316	.98737	53
8	.15873	.16077	6.2200	.98732	52
9	.15902	.16107	6.2085	.98728	51
10	.15931	.16137	6.1970	.98723	**50**
11	.15959	.16167	6.1856	.98718	49
12	.15988	.16196	6.1742	.98714	48
13	.16017	.16226	6.1628	.98709	47
14	.16046	.16256	6.1515	.98704	46
15	.16074	.16286	6.1402	.98700	**45**
16	.16103	.16316	6.1290	.98695	44
17	.16132	.16346	6.1178	.98690	43
18	.16160	.16376	6.1066	.98686	42
19	.16189	.16405	6.0955	.98681	41
20	.16218	.16435	6.0844	.98676	**40**
21	.16246	.16465	6.0734	.98671	39
22	.16275	.16495	6.0624	.98667	38
23	.16304	.16525	6.0514	.98662	37
24	.16333	.16555	6.0405	.98657	36
25	.16361	.16585	6.0296	.98652	**35**
26	.16390	.16615	6.0188	.98648	34
27	.16419	.16645	6.0080	.98643	33
28	.16447	.16674	5.9972	.98638	32
29	.16476	.16704	5.9865	.98633	31
30	.16505	.16734	5.9758	.98629	**30**
31	.16533	.16764	5.9651	.98624	29
32	.16562	.16794	5.9545	.98619	28
33	.16591	.16824	5.9439	.98614	27
34	.16620	.16854	5.9333	.98609	26
35	.16648	.16884	5.9228	.98604	**25**
36	.16677	.16914	5.9124	.98600	24
37	.16706	.16944	5.9019	.98595	23
38	.16734	.16974	5.8915	.98590	22
39	.16763	.17004	5.8811	.98585	21
40	.16792	.17033	5.8708	.98580	**20**
41	.16820	.17063	5.8605	.98575	19
42	.16849	.17093	5.8502	.98570	18
43	.16878	.17123	5.8400	.98565	17
44	.16906	.17153	5.8298	.98561	16
45	.16935	.17183	5.8197	.98556	**15**
46	.16964	.17213	5.8095	.98551	14
47	.16992	.17243	5.7994	.98546	13
48	.17021	.17273	5.7894	.98541	12
49	.17050	.17303	5.7794	.98536	11
50	.17078	.17333	5.7694	.98531	**10**
51	.17107	.17363	5.7594	.98526	9
52	.17136	.17393	5.7495	.98521	8
53	.17164	.17423	5.7396	.98516	7
54	.17193	.17453	5.7297	.98511	6
55	.17222	.17483	5.7199	.98506	**5**
56	.17250	.17513	5.7101	.98501	4
57	.17279	.17543	5.7004	.98496	3
58	.17308	.17573	5.6906	.98491	2
59	.17336	.17603	5.6809	.98486	1
60	.17365	.17633	5.6713	.98481	**0**
′	Cos	Ctn	Tan	Sin	′

80°

Natural Trigonometric Functions

10°

′	Sin	Tan	Ctn	Cos	′
0	.17365	.17633	5.6713	.98481	60
1	.17393	.17663	5.6617	.98476	59
2	.17422	.17693	5.6521	.98471	58
3	.17451	.17723	5.6425	.98466	57
4	.17479	.17753	5.6329	.98461	56
5	.17508	.17783	5.6234	.98455	55
6	.17537	.17813	5.6140	.98450	54
7	.17565	.17843	5.6045	.98445	53
8	.17594	.17873	5.5951	.98440	52
9	.17623	.17903	5.5857	.98435	51
10	.17651	.17933	5.5764	.98430	50
11	.17680	.17963	5.5671	.98425	49
12	.17708	.17993	5.5578	.98420	48
13	.17737	.18023	5.5485	.98414	47
14	.17766	.18053	5.5393	.98409	46
15	.17794	.18083	5.5301	.98404	45
16	.17823	.18113	5.5209	.98399	44
17	.17852	.18143	5.5118	.98394	43
18	.17880	.18173	5.5026	.98389	42
19	.17909	.18203	5.4936	.98383	41
20	.17937	.18233	5.4845	.98378	40
21	.17966	.18263	5.4755	.98373	39
22	.17995	.18293	5.4665	.98368	38
23	.18023	.18323	5.4575	.98362	37
24	.18052	.18353	5.4486	.98357	36
25	.18081	.18384	5.4397	.98352	35
26	.18109	.18414	5.4308	.98347	34
27	.18138	.18444	5.4219	.98341	33
28	.18166	.18474	5.4131	.98336	32
29	.18195	.18504	5.4043	.98331	31
30	.18224	.18534	5.3955	.98325	30
31	.18252	.18564	5.3868	.98320	29
32	.18281	.18594	5.3781	.98315	28
33	.18309	.18624	5.3694	.98310	27
34	.18338	.18654	5.3607	.98304	26
35	.18367	.18684	5.3521	.98299	25
36	.18395	.18714	5.3435	.98294	24
37	.18424	.18745	5.3349	.98288	23
38	.18452	.18775	5.3263	.98283	22
39	.18481	.18805	5.3178	.98277	21
40	.18509	.18835	5.3093	.98272	20
41	.18538	.18865	5.3008	.98267	19
42	.18567	.18895	5.2924	.98261	18
43	.18595	.18925	5.2839	.98256	17
44	.18624	.18955	5.2755	.98250	16
45	.18652	.18986	5.2672	.98245	15
46	.18681	.19016	5.2588	.98240	14
47	.18710	.19046	5.2505	.98234	13
48	.18738	.19076	5.2422	.98229	12
49	.18767	.19106	5.2339	.98223	11
50	.18795	.19136	5.2257	.98218	10
51	.18824	.19166	5.2174	.98212	9
52	.18852	.19197	5.2092	.98207	8
53	.18881	.19227	5.2011	.98201	7
54	.18910	.19257	5.1929	.98196	6
55	.18938	.19287	5.1848	.98190	5
56	.18967	.19317	5.1767	.98185	4
57	.18995	.19347	5.1686	.98179	3
58	.19024	.19378	5.1606	.98174	2
59	.19052	.19408	5.1526	.98168	1
60	.19081	.19438	5.1446	.98163	0
′	Cos	Ctn	Tan	Sin	′

79°

11°

′	Sin	Tan	Ctn	Cos	′
0	.19081	.19438	5.1446	.98163	60
1	.19109	.19468	5.1366	.98157	59
2	.19138	.19498	5.1286	.98152	58
3	.19167	.19529	5.1207	.98146	57
4	.19195	.19559	5.1128	.98140	56
5	.19224	.19589	5.1049	.98135	55
6	.19252	.19619	5.0970	.98129	54
7	.19281	.19649	5.0892	.98124	53
8	.19309	.19680	5.0814	.98118	52
9	.19338	.19710	5.0736	.98112	51
10	.19366	.19740	5.0658	.98107	50
11	.19395	.19770	5.0581	.98101	49
12	.19423	.19801	5.0504	.98096	48
13	.19452	.19831	5.0427	.98090	47
14	.19481	.19861	5.0350	.98084	46
15	.19509	.19891	5.0273	.98079	45
16	.19538	.19921	5.0197	.98073	44
17	.19566	.19952	5.0121	.98067	43
18	.19595	.19982	5.0045	.98061	42
19	.19623	.20012	4.9969	.98056	41
20	.19652	.20042	4.9894	.98050	40
21	.19680	.20073	4.9819	.98044	39
22	.19709	.20103	4.9744	.98039	38
23	.19737	.20133	4.9669	.98033	37
24	.19766	.20164	4.9594	.98027	36
25	.19794	.20194	4.9520	.98021	35
26	.19823	.20224	4.9446	.98016	34
27	.19851	.20254	4.9372	.98010	33
28	.19880	.20285	4.9298	.98004	32
29	.19908	.20315	4.9225	.97998	31
30	.19937	.20345	4.9152	.97992	30
31	.19965	.20376	4.9078	.97987	29
32	.19994	.20406	4.9006	.97981	28
33	.20022	.20436	4.8933	.97975	27
34	.20051	.20466	4.8860	.97969	26
35	.20079	.20497	4.8788	.97963	25
36	.20108	.20527	4.8716	.97958	24
37	.20136	.20557	4.8644	.97952	23
38	.20165	.20588	4.8573	.97946	22
39	.20193	.20618	4.8501	.97940	21
40	.20222	.20648	4.8430	.97934	20
41	.20250	.20679	4.8359	.97928	19
42	.20279	.20709	4.8288	.97922	18
43	.20307	.20739	4.8218	.97916	17
44	.20336	.20770	4.8147	.97910	16
45	20364	.20800	4.8077	.97905	15
46	.20393	.20830	4.8007	.97899	14
47	.20421	.20861	4.7937	.97893	13
48	.20450	.20891	4.7867	.97887	12
49	.20478	.20921	4.7798	.97881	11
50	.20507	.20952	4.7729	.97875	10
51	.20535	.20982	4.7659	.97869	9
52	.20563	.21013	4.7591	.97863	8
53	.20592	.21043	4.7522	.97857	7
54	.20620	.21073	4.7453	.97851	6
55	.20649	.21104	4.7385	.97845	5
56	.20677	.21134	4.7317	.97839	4
57	.20706	.21164	4.7249	.97833	3
58	.20734	.21195	4.7181	.97827	2
59	.20763	.21225	4.7114	.97821	1
60	.20791	.21256	4.7046	.97815	0
′	Cos	Ctn	Tan	Sin	′

78°

Natural Trigonometric Functions

12°

′	Sin	Tan	Ctn	Cos	′
0	.20791	.21256	4.7046	.97815	60
1	.20820	.21286	4.6979	.97809	59
2	.20848	.21316	4.6912	.97803	58
3	.20877	.21347	4.6845	.97797	57
4	.20905	.21377	4.6779	.97791	56
5	.20933	.21408	4.6712	.97784	55
6	.20962	.21438	4.6646	.97778	54
7	.20990	.21469	4.6580	.97772	53
8	.21019	.21499	4.6514	.97766	52
9	.21047	.21529	4.6448	.97760	51
10	.21076	.21560	4.6382	.97754	50
11	.21104	.21590	4.6317	.97748	49
12	.21132	.21621	4.6252	.97742	48
13	.21161	.21651	4.6187	.97735	47
14	.21189	.21682	4.6122	.97729	46
15	.21218	.21712	4.6057	.97723	45
16	.21246	.21743	4.5993	.97717	44
17	.21275	.21773	4.5928	.97711	43
18	.21303	.21804	4.5864	.97705	42
19	.21331	.21834	4.5800	.97698	41
20	.21360	.21864	4.5736	.97692	40
21	.21388	.21895	4.5673	.97686	39
22	.21417	.21925	4.5609	.97680	38
23	.21445	.21956	4.5546	.97673	37
24	.21474	.21986	4.5483	.97667	36
25	.21502	.22017	4.5420	.97661	35
26	.21530	.22047	4.5357	.97655	34
27	.21559	.22078	4.5294	.97648	33
28	.21587	.22108	4.5232	.97642	32
29	.21616	.22139	4.5169	.97636	31
30	.21644	.22169	4.5107	.97630	30
31	.21672	.22200	4.5045	.97623	29
32	.21701	.22231	4.4983	.97617	28
33	.21729	.22261	4.4922	.97611	27
34	.21758	.22292	4.4860	.97604	26
35	.21786	.22322	4.4799	.97598	25
36	.21814	.22353	4.4737	.97592	24
37	.21843	.22383	4.4676	.97585	23
38	.21871	.22414	4.4615	.97579	22
39	.21899	.22444	4.4555	.97573	21
40	.21928	.22475	4.4494	.97566	20
41	.21956	.22505	4.4434	.97560	19
42	.21985	.22536	4.4373	.97553	18
43	.22013	.22567	4.4313	.97547	17
44	.22041	.22597	4.4253	.97541	16
45	.22070	.22628	4.4194	.97534	15
46	.22098	.22658	4.4134	.97528	14
47	.22126	.22689	4.4075	.97521	13
48	.22155	.22719	4.4015	.97515	12
49	.22183	.22750	4.3956	.97508	11
50	.22212	.22781	4.3897	.97502	10
51	.22240	.22811	4.3838	.97496	9
52	.22268	.22842	4.3779	.97489	8
53	.22297	.22872	4.3721	.97483	7
54	.22325	.22903	4.3662	.97476	6
55	.22353	.22934	4.3604	.97470	5
56	.22382	.22964	4.3546	.97463	4
57	.22410	.22995	4.3488	.97457	3
58	.22438	.23026	4.3430	.97450	2
59	.22467	.23056	4.3372	.97444	1
60	.22495	.23087	4.3315	.97437	0
′	Cos	Ctn	Tan	Sin	′

77°

13°

′	Sin	Tan	Ctn	Cos	′
0	.22495	.23087	4.3315	.97437	60
1	.22523	.23117	4.3257	.97430	59
2	.22552	.23148	4.3200	.97424	58
3	.22580	.23179	4.3143	.97417	57
4	.22608	.23209	4.3086	.97411	56
5	.22637	.23240	4.3029	.97404	55
6	.22665	.23271	4.2972	.97398	54
7	.22693	.23301	4.2916	.97391	53
8	.22722	.23332	4.2859	.97384	52
9	.22750	.23363	4.2803	.97378	51
10	.22778	.23393	4.2747	.97371	50
11	.22807	.23424	4.2691	.97365	49
12	.22835	.23455	4.2635	.97358	48
13	.22863	.23485	4.2580	.97351	47
14	.22892	.23516	4.2524	.97345	46
15	.22920	.23547	4.2468	.97338	45
16	.22948	.23578	4.2413	.97331	44
17	.22977	.23608	4.2358	.97325	43
18	.23005	.23639	4.2303	.97318	42
19	.23033	.23670	4.2248	.97311	41
20	.23062	.23700	4.2193	.97304	40
21	.23090	.23731	4.2139	.97298	39
22	.23118	.23762	4.2084	.97291	38
23	.23146	.23793	4.2030	.97284	37
24	.23175	.23823	4.1976	.97278	36
25	.23203	.23854	4.1922	.97271	35
26	.23231	.23885	4.1868	.97264	34
27	.23260	.23916	4.1814	.97257	33
28	.23288	.23946	4.1760	.97251	32
29	.23316	.23977	4.1706	.97244	31
30	.23345	.24008	4.1653	.97237	30
31	.23373	.24039	4.1600	.97230	29
32	.23401	.24069	4.1547	.97223	28
33	.23429	.24100	4.1493	.97217	27
34	.23458	.24131	4.1441	.97210	26
35	.23486	.24162	4.1388	.97203	25
36	.23514	.24193	4.1335	.97196	24
37	.23542	.24223	4.1282	.97189	23
38	.23571	.24254	4.1230	.97182	22
39	.23599	.24285	4.1178	.97176	21
40	.23627	.24316	4.1126	.97169	20
41	.23656	.24347	4.1074	.97162	19
42	.23684	.24377	4.1022	.97155	18
43	.23712	.24408	4.0970	.97148	17
44	.23740	.24439	4.0918	.97141	16
45	.23769	.24470	4.0867	.97134	15
46	.23797	.24501	4.0815	.97127	14
47	.23825	.24532	4.0764	.97120	13
48	.23853	.24562	4.0713	.97113	12
49	.23882	.24593	4.0662	.97106	11
50	.23910	.24624	4.0611	.97100	10
51	.23938	.24655	4.0560	.97093	9
52	.23966	.24686	4.0509	.97086	8
53	.23995	.24717	4.0459	.97079	7
54	.24023	.24747	4.0408	.97072	6
55	.24051	.24778	4.0358	.97065	5
56	.24079	.24809	4.0308	.97058	4
57	.24108	.24840	4.0257	.97051	3
58	.24136	.24871	4.0207	.97044	2
59	.24164	.24902	4.0158	.97037	1
60	.24192	.24933	4.0108	.97030	0
′	Cos	Ctn	Tan	Sin	′

76°

Natural Trigonometric Functions

14°

′	Sin	Tan	Ctn	Cos	′
0	.24192	.24933	4.0108	.97030	60
1	.24220	.24964	4.0058	.97023	59
2	.24249	.24995	4.0009	.97015	58
3	.24277	.25026	3.9959	.97008	57
4	.24305	.25056	3.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	.24362	.25118	3.9812	.96987	54
7	.24390	.25149	3.9763	.96980	53
8	.24418	.25180	3.9714	.96973	52
9	.24446	.25211	3.9665	.96966	51
10	.24474	.25242	3.9617	.96959	50
11	.24503	.25273	3.9568	.96952	49
12	.24531	.25304	3.9520	.96945	48
13	.24559	.25335	3.9471	.96937	47
14	.24587	.25366	3.9423	.96930	46
15	.24615	.25397	3.9375	.96923	45
16	.24644	.25428	3.9327	.96916	44
17	.24672	.25459	3.9279	.96909	43
18	.24700	.25490	3.9232	.96902	42
19	.24728	.25521	3.9184	.96894	41
20	.24756	.25552	3.9136	.96887	40
21	.24784	.25583	3.9089	.96880	39
22	.24813	.25614	3.9042	.96873	38
23	.24841	.25645	3.8995	.96866	37
24	.24869	.25676	3.8947	.96858	36
25	.24897	.25707	3.8900	.96851	35
26	.24925	.25738	3.8854	.96844	34
27	.24954	.25769	3.8807	.96837	33
28	.24982	.25800	3.8760	.96829	32
29	.25010	.25831	3.8714	.96822	31
30	.25038	.25862	3.8667	.96815	30
31	.25066	.25893	3.8621	.96807	29
32	.25094	.25924	3.8575	.96800	28
33	.25122	.25955	3.8528	.96793	27
34	.25151	.25986	3.8482	.96786	26
35	.25179	.26017	3.8436	.96778	25
36	.25207	.26048	3.8391	.96771	24
37	.25235	.26079	3.8345	.96764	23
38	.25263	.26110	3.8299	.96756	22
39	.25291	.26141	3.8254	.96749	21
40	.25320	.26172	3.8208	.96742	20
41	.25348	.26203	3.8163	.96734	19
42	.25376	.26235	3.8118	.96727	18
43	.25404	.26266	3.8073	.96719	17
44	.25432	.26297	3.8028	.96712	16
45	.25460	.26328	3.7983	.96705	15
46	.25488	.26359	3.7938	.96697	14
47	.25516	.26390	3.7893	.96690	13
48	.25545	.26421	3.7848	.96682	12
49	.25573	.26452	3.7804	.96675	11
50	.25601	.26483	3.7760	.96667	10
51	.25629	.26515	3.7715	.96660	9
52	.25657	.26546	3.7671	.96653	8
53	.25685	.26577	3.7627	.96645	7
54	.25713	.26608	3.7583	.96638	6
55	.25741	.26639	3.7539	.96630	5
56	.25769	.26670	3.7495	.96623	4
57	.25798	.26701	3.7451	.96615	3
58	.25826	.26733	3.7408	.96608	2
59	.25854	.26764	3.7364	.96600	1
60	.25882	.26795	3.7321	.96593	0
′	Cos	Ctn	Tan	Sin	′

75°

15°

′	Sin	Tan	Ctn	Cos	′
0	.25882	.26795	3.7321	.96593	60
1	.25910	.26826	3.7277	.96585	59
2	.25938	.26857	3.7234	.96578	58
3	.25966	.26888	3.7191	.96570	57
4	.25994	.26920	3.7148	.96562	56
5	.26022	.26951	3.7105	.96555	55
6	.26050	.26982	3.7062	.96547	54
7	.26079	.27013	3.7019	.96540	53
8	.26107	.27044	3.6976	.96532	52
9	.26135	.27076	3.6933	.96524	51
10	.26163	.27107	3.6891	.96517	50
11	.26191	.27138	3.6848	.96509	49
12	.26219	.27169	3.6806	.96502	48
13	.26247	.27201	3.6764	.96494	47
14	.26275	.27232	3.6722	.96486	46
15	.26303	.27263	3.6680	.96479	45
16	.26331	.27294	3.6638	.96471	44
17	.26359	.27326	3.6596	.96463	43
18	.26387	.27357	3.6554	.96456	42
19	.26415	.27388	3.6512	.96448	41
20	.26443	.27419	3.6470	.96440	40
21	.26471	.27451	3.6429	.96433	39
22	.26500	.27482	3.6387	.96425	38
23	.26528	.27513	3.6346	.96417	37
24	.26556	.27545	3.6305	.96410	36
25	.26584	.27576	3.6264	.96402	35
26	.26612	.27607	3.6222	.96394	34
27	.26640	.27638	3.6181	.96386	33
28	.26668	.27670	3.6140	.96379	32
29	.26696	.27701	3.6100	.96371	31
30	.26724	.27732	3.6059	.96363	30
31	.26752	.27764	3.6018	.96355	29
32	.26780	.27795	3.5978	.96347	28
33	.26808	.27826	3.5937	.96340	27
34	.26836	.27858	3.5897	.96332	26
35	.26864	.27889	3.5856	.96324	25
36	.26892	.27921	3.5816	.96316	24
37	.26920	.27952	3.5776	.96308	23
38	.26948	.27983	3.5736	.96301	22
39	.26976	.28015	3.5696	.96293	21
40	.27004	.28046	3.5656	.96285	20
41	.27032	.28077	3.5616	.96277	19
42	.27060	.28109	3.5576	.96269	18
43	.27088	.28140	3.5536	.96261	17
44	.27116	.28172	3.5497	.96253	16
45	.27144	.28203	3.5457	.96246	15
46	.27172	.28234	3.5418	.96238	14
47	.27200	.28266	3.5379	.96230	13
48	.27228	.28297	3.5339	.96222	12
49	.27256	.28329	3.5300	.96214	11
50	.27284	.28360	3.5261	.96206	10
51	.27312	.28391	3.5222	.96198	9
52	.27340	.28423	3.5183	.96190	8
53	.27368	.28454	3.5144	.96182	7
54	.27396	.28486	3.5105	.96174	6
55	.27424	.28517	3.5067	.96166	5
56	.27452	.28549	3.5028	.96158	4
57	.27480	.28580	3.4989	.96150	3
58	.27508	.28612	3.4951	.96142	2
59	.27536	.28643	3.4912	.96134	1
60	.27564	.28675	3.4874	.96126	0
′	Cos	Ctn	Tan	Sin	′

74°

Natural Trigonometric Functions

16°

′	Sin	Tan	Ctn	Cos	′
0	.27564	.28675	3.4874	.96126	60
1	.27592	.28706	3.4836	.96118	59
2	.27620	.28738	3.4798	.96110	58
3	.27648	.28769	3.4760	.96102	57
4	.27676	.28801	3.4722	.96094	56
5	.27704	.28832	3.4684	.96086	55
6	.27731	.28864	3.4646	.96078	54
7	.27759	.28895	3.4608	.96070	53
8	.27787	.28927	3.4570	.96062	52
9	.27815	.28958	3.4533	.96054	51
10	.27843	.28990	3.4495	.96046	50
11	.27871	.29021	3.4458	.96037	49
12	.27899	.29053	3.4420	.96029	48
13	.27927	.29084	3.4383	.96021	47
14	.27955	.29116	3.4346	.96013	46
15	.27983	.29147	3.4308	.96005	45
16	.28011	.29179	3.4271	.95997	44
17	.28039	.29210	3.4234	.95989	43
18	.28067	.29242	3.4197	.95981	42
19	.28095	.29274	3.4160	.95972	41
20	.28123	.29305	3.4124	.95964	40
21	.28150	.29337	3.4087	.95956	39
22	.28178	.29368	3.4050	.95948	38
23	.28206	.29400	3.4014	.95940	37
24	.28234	.29432	3.3977	.95931	36
25	.28262	.29463	3.3941	.95923	35
26	.28290	.29495	3.3904	.95915	34
27	.28318	.29526	3.3868	.95907	33
28	.28346	.29558	3.3832	.95898	32
29	.28374	.29590	3.3796	.95890	31
30	.28402	.29621	3.3759	.95882	30
31	.28429	.29653	3.3723	.95874	29
32	.28457	.29685	3.3687	.95865	28
33	.28485	.29716	3.3652	.95857	27
34	.28513	.29748	3.3616	.95849	26
35	.28541	.29780	3.3580	.95841	25
36	.28569	.29811	3.3544	.95832	24
37	.28597	.29843	3.3509	.95824	23
38	.28625	.29875	3.3473	.95816	22
39	.28652	.29906	3.3438	.95807	21
40	.28680	.29938	3.3402	.95799	20
41	.28708	.29970	3.3367	.95791	19
42	.28736	.30001	3.3332	.95782	18
43	.28764	.30033	3.3297	.95774	17
44	.28792	.30065	3.3261	.95766	16
45	.28820	.30097	3.3226	.95757	15
46	.28847	.30128	3.3191	.95749	14
47	.28875	.30160	3.3156	.95740	13
48	.28903	.30192	3.3122	.95732	12
49	.28931	.30224	3.3087	.95724	11
50	.28959	.30255	3.3052	.95715	10
51	.28987	.30287	3.3017	.95707	9
52	.29015	.30319	3.2983	.95698	8
53	.29042	.30351	3.2948	.95690	7
54	.29070	.30382	3.2914	.95681	6
55	.29098	.30414	3.2879	.95673	5
56	.29126	.30446	3.2845	.95664	4
57	.29154	.30478	3.2811	.95656	3
58	.29182	.30509	3.2777	.95647	2
59	.29209	.30541	3.2743	.95639	1
60	.29237	.30573	3.2709	.95630	0
′	Cos	Ctn	Tan	Sin	′

73°

17°

′	Sin	Tan	Ctn	Cos	′
0	.29237	.30573	3.2709	.95630	60
1	.29265	.30605	3.2675	.95622	59
2	.29293	.30637	3.2641	.95613	58
3	.29321	.30669	3.2607	.95605	57
4	.29348	.30700	3.2573	.95596	56
5	.29376	.30732	3.2539	.95588	55
6	.29404	.30764	3.2506	.95579	54
7	.29432	.30796	3.2472	.95571	53
8	.29460	.30828	3.2438	.95562	52
9	.29487	.30860	3.2405	.95554	51
10	.29515	.30891	3.2371	.95545	50
11	.29543	.30923	3.2338	.95536	49
12	.29571	.30955	3.2305	.95528	48
13	.29599	.30987	3.2272	.95519	47
14	.29626	.31019	3.2238	.95511	46
15	.29654	.31051	3.2205	.95502	45
16	.29682	.31083	3.2172	.95493	44
17	.29710	.31115	3.2139	.95485	43
18	.29737	.31147	3.2106	.95476	42
19	.29765	.31178	3.2073	.95467	41
20	.29793	.31210	3.2041	.95459	40
21	.29821	.31242	3.2008	.95450	39
22	.29849	.31274	3.1975	.95441	38
23	.29876	.31306	3.1943	.95433	37
24	.29904	.31338	3.1910	.95424	36
25	.29932	.31370	3.1878	.95415	35
26	.29960	.31402	3.1845	.95407	34
27	.29987	.31434	3.1813	.95398	33
28	.30015	.31466	3.1780	.95389	32
29	.30043	.31498	3.1748	.95380	31
30	.30071	.31530	3.1716	.95372	30
31	.30098	.31562	3.1684	.95363	29
32	.30126	.31594	3.1652	.95354	28
33	.30154	.31626	3.1620	.95345	27
34	.30182	.31658	3.1588	.95337	26
35	.30209	.31690	3.1556	.95328	25
36	.30237	.31722	3.1524	.95319	24
37	.30265	.31754	3.1492	.95310	23
38	.30292	.31786	3.1460	.95301	22
39	.30320	.31818	3.1429	.95293	21
40	.30348	.31850	3.1397	.95284	20
41	.30376	.31882	3.1366	.95275	19
42	.30403	.31914	3.1334	.95266	18
43	.30431	.31946	3.1303	.95257	17
44	.30459	.31978	3.1271	.95248	16
45	.30486	.32010	3.1240	.95240	15
46	.30514	.32042	3.1209	.95231	14
47	.30542	.32074	3.1178	.95222	13
48	.30570	.32106	3.1146	.95213	12
49	.30597	.32139	3.1115	.95204	11
50	.30625	.32171	3.1084	.95195	10
51	.30653	.32203	3.1053	.95186	9
52	.30680	.32235	3.1022	.95177	8
53	.30708	.32267	3.0991	.95168	7
54	.30736	.32299	3.0961	.95159	6
55	.30763	.32331	3.0930	.95150	5
56	.30791	.32363	3.0899	.95142	4
57	.30819	.32396	3.0868	.95133	3
58	.30846	.32428	3.0838	.95124	2
59	.30874	.32460	3.0807	.95115	1
60	.30902	.32492	3.0777	.95106	0
′	Cos	Ctn	Tan	Sin	′

72°

Natural Trigonometric Functions

18°

′	Sin	Tan	Ctn	Cos	′
0	.30902	.32492	3.0777	.95106	60
1	.30929	.32524	3.0746	.95097	59
2	.30957	.32556	3.0716	.95088	58
3	.30985	.32588	3.0686	.95079	57
4	.31012	.32621	3.0655	.95070	56
5	.31040	.32653	3.0625	.95061	55
6	.31068	.32685	3.0595	.95052	54
7	.31095	.32717	3.0565	.95043	53
8	.31123	.32749	3.0535	.95033	52
9	.31151	.32782	3.0505	.95024	51
10	.31178	.32814	3.0475	.95015	50
11	.31206	.32846	3.0445	.95006	49
12	.31233	.32878	3.0415	.94997	48
13	.31261	.32911	3.0385	.94988	47
14	.31289	.32943	3.0356	.94979	46
15	.31316	.32975	3.0326	.94970	45
16	.31344	.33007	3.0296	.94961	44
17	.31372	.33040	3.0267	.94952	43
18	.31399	.33072	3.0237	.94943	42
19	.31427	.33104	3.0208	.94933	41
20	.31454	.33136	3.0178	.94924	40
21	.31482	.33169	3.0149	.94915	39
22	.31510	.33201	3.0120	.94906	38
23	.31537	.33233	3.0090	.94897	37
24	.31565	.33266	3.0061	.94888	36
25	.31593	.33298	3.0032	.94878	35
26	.31620	.33330	3.0003	.94869	34
27	.31648	.33363	2.9974	.94860	33
28	.31675	.33395	2.9945	.94851	32
29	.31703	.33427	2.9916	.94842	31
30	.31730	.33460	2.9887	.94832	30
31	.31758	.33492	2.9858	.94823	29
32	.31786	.33524	2.9829	.94814	28
33	.31813	.33557	2.9800	.94805	27
34	.31841	.33589	2.9772	.94795	26
35	.31868	.33621	2.9743	.94786	25
36	.31896	.33654	2.9714	.94777	24
37	.31923	.33686	2.9686	.94768	23
38	.31951	.33718	2.9657	.94758	22
39	.31979	.33751	2.9629	.94749	21
40	.32006	.33783	2.9600	.94740	20
41	.32034	.33816	2.9572	.94730	19
42	.32061	.33848	2.9544	.94721	18
43	.32089	.33881	2.9515	.94712	17
44	.32116	.33913	2.9487	.94702	16
45	.32144	.33945	2.9459	.94693	15
46	.32171	.33978	2.9431	.94684	14
47	.32199	.34010	2.9403	.94674	13
48	.32227	.34043	2.9375	.94665	12
49	.32254	.34075	2.9347	.94656	11
50	.32282	.34108	2.9319	.94646	10
51	.32309	.34140	2.9291	.94637	9
52	.32337	.34173	2.9263	.94627	8
53	.32364	.34205	2.9235	.94618	7
54	.32392	.34238	2.9208	.94609	6
55	.32419	.34270	2.9180	.94599	5
56	.32447	.34303	2.9152	.94590	4
57	.32474	.34335	2.9125	.94580	3
58	.32502	.34368	2.9097	.94571	2
59	.32529	.34400	2.9070	.94561	1
60	.32557	.34433	2.9042	.94552	0
′	Cos	Ctn	Tan	Sin	′

71°

19°

′	Sin	Tan	Ctn	Cos	′
0	.32557	.34433	2.9042	.94552	60
1	.32584	.34465	2.9015	.94542	59
2	.32612	.34498	2.8987	.94533	58
3	.32639	.34530	2.8960	.94523	57
4	.32667	.34563	2.8933	.94514	56
5	.32694	.34596	2.8905	.94504	55
6	.32722	.34628	2.8878	.94495	54
7	.32749	.34661	2.8851	.94485	53
8	.32777	.34693	2.8824	.94476	52
9	.32804	.34726	2.8797	.94466	51
10	.32832	.34758	2.8770	.94457	50
11	.32859	.34791	2.8743	.94447	49
12	.32887	.34824	2.8716	.94438	48
13	.32914	.34856	2.8689	.94428	47
14	.32942	.34889	2.8662	.94418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	.34954	2.8609	.94399	44
17	.33024	.34987	2.8582	.94390	43
18	.33051	.35020	2.8556	.94380	42
19	.33079	.35052	2.8529	.94370	41
20	.33106	.35085	2.8502	.94361	40
21	.33134	.35118	2.8476	.94351	39
22	.33161	.35150	2.8449	.94342	38
23	.33189	.35183	2.8423	.94332	37
24	.33216	.35216	2.8397	.94322	36
25	.33244	.35248	2.8370	.94313	35
26	.33271	.35281	2.8344	.94303	34
27	.33298	.35314	2.8318	.94293	33
28	.33326	.35346	2.8291	.94284	32
29	.33353	.35379	2.8265	.94274	31
30	.33381	.35412	2.8239	.94264	30
31	.33408	.35445	2.8213	.94254	29
32	.33436	.35477	2.8187	.94245	28
33	.33463	.35510	2.8161	.94235	27
34	.33490	.35543	2.8135	.94225	26
35	.33518	.35576	2.8109	.94215	25
36	.33545	.35608	2.8083	.94206	24
37	.33573	.35641	2.8057	.94196	23
38	.33600	.35674	2.8032	.94186	22
39	.33627	.35707	2.8006	.94176	21
40	.33655	.35740	2.7980	.94167	20
41	.33682	.35772	2.7955	.94157	19
42	.33710	.35805	2.7929	.94147	18
43	.33737	.35838	2.7903	.94137	17
44	.33764	.35871	2.7878	.94127	16
45	.33792	.35904	2.7852	.94118	15
46	.33819	.35937	2.7827	.94108	14
47	.33846	.35969	2.7801	.94098	13
48	.33874	.36002	2.7776	.94088	12
49	.33901	.36035	2.7751	.94078	11
50	33929	36068	2 7725	.94068	10
51	.33956	.36101	2.7700	.94058	9
52	.33983	.36134	2.7675	.94049	8
53	.34011	.36167	2.7650	.94039	7
54	.34038	.36199	2.7625	.94029	6
55	.34065	.36232	2.7600	.94019	5
56	.34093	.36265	2.7575	.94009	4
57	.34120	.36298	2.7550	.93999	3
58	.34147	.36331	2.7525	.93989	2
59	.34175	.36364	2.7500	.93979	1
60	.34202	.36397	2.7475	.93969	0
′	Cos	Ctn	Tan	Sin	′

70°

Natural Trigonometric Functions

20°

′	Sin	Tan	Ctn	Cos	′
0	.34202	.36397	2.7475	.93969	60
1	.34229	.36430	2.7450	.93959	59
2	.34257	.36463	2.7425	.93949	58
3	.34284	.36496	2.7400	.93939	57
4	.34311	.36529	2.7376	.93929	56
5	.34339	.36562	2.7351	.93919	55
6	.34366	.36595	2.7326	.93909	54
7	.34393	.36628	2.7302	.93899	53
8	.34421	.36661	2.7277	.93889	52
9	.34448	.36694	2.7253	.93879	51
10	.34475	.36727	2.7228	.93869	50
11	.34503	.36760	2.7204	.93859	49
12	.34530	.36793	2.7179	.93849	48
13	.34557	.36826	2.7155	.93839	47
14	.34584	.36859	2.7130	.93829	46
15	.34612	.36892	2.7106	.93819	45
16	.34639	.36925	2.7082	.93809	44
17	.34666	.36958	2.7058	.93799	43
18	.34694	.36991	2.7034	.93789	42
19	.34721	.37024	2.7009	.93779	41
20	.34748	.37057	2.6985	.93769	40
21	.34775	.37090	2.6961	.93759	39
22	.34803	.37123	2.6937	.93748	38
23	.34830	.37157	2.6913	.93738	37
24	.34857	.37190	2.6889	.93728	36
25	.34884	.37223	2.6865	.93718	35
26	.34912	.37256	2.6841	.93708	34
27	.34939	.37289	2.6818	.93698	33
28	.34966	.37322	2.6794	.93688	32
29	.34993	.37355	2.6770	.93677	31
30	.35021	.37388	2.6746	.93667	30
31	.35048	.37422	2.6723	.93657	29
32	.35075	.37455	2.6699	.93647	28
33	.35102	.37488	2.6675	.93637	27
34	.35130	.37521	2.6652	.93626	26
35	.35157	.37554	2.6628	.93616	25
36	.35184	.37588	2.6605	.93606	24
37	.35211	.37621	2.6581	.93596	23
38	.35239	.37654	2.6558	.93585	22
39	.35266	.37687	2.6534	.93575	21
40	.35293	.37720	2.6511	.93565	20
41	.35320	.37754	2.6488	.93555	19
42	.35347	.37787	2.6464	.93544	18
43	.35375	.37820	2.6441	.93534	17
44	.35402	.37853	2.6418	.93524	16
45	.35429	.37887	2.6395	.93514	15
46	.35456	.37920	2.6371	.93503	14
47	.35484	.37953	2.6348	.93493	13
48	.35511	.37986	2.6325	.93483	12
49	.35538	.38020	2.6302	.93472	11
50	.35565	.38053	2.6279	.93462	10
51	.35592	.38086	2.6256	.93452	9
52	.35619	.38120	2.6233	.93441	8
53	.35647	.38153	2.6210	.93431	7
54	.35674	.38186	2.6187	.93420	6
55	.35701	.38220	2.6165	.93410	5
56	.35728	.38253	2.6142	.93400	4
57	.35755	.38286	2.6119	.93389	3
58	.35782	.38320	2.6096	.93379	2
59	.35810	.38353	2.6074	.93368	1
60	.35837	.38386	2.6051	.93358	0
′	Cos	Ctn	Tan	Sin	′

69°

21°

′	Sin	Tan	Ctn	Cos	′
0	.35837	.38386	2.6051	.93358	60
1	.35864	.38420	2.6028	.93348	59
2	.35891	.38453	2.6006	.93337	58
3	.35918	.38487	2.5983	.93327	57
4	.35945	.38520	2.5961	.93316	56
5	.35973	.38553	2.5938	.93306	55
6	.36000	.38587	2.5916	.93295	54
7	.36027	.38620	2.5893	.93285	53
8	.36054	.38654	2.5871	.93274	52
9	.36081	.38687	2.5848	.93264	51
10	.36108	.38721	2.5826	.93253	50
11	.36135	.38754	2.5804	.93243	49
12	.36162	.38787	2.5782	.93232	48
13	.36190	.38821	2.5759	.93222	47
14	.36217	.38854	2.5737	.93211	46
15	.36244	.38888	2.5715	.93201	45
16	.36271	.38921	2.5693	.93190	44
17	.36298	.38955	2.5671	.93180	43
18	.36325	.38988	2.5649	.93169	42
19	.36352	.39022	2.5627	.93159	41
20	.36379	.39055	2.5605	.93148	40
21	.36406	.39089	2.5583	.93137	39
22	.36434	.39122	2.5561	.93127	38
23	.36461	.39156	2.5539	.93116	37
24	.36488	.39190	2.5517	.93106	36
25	.36515	.39223	2.5495	.93095	35
26	.36542	.39257	2.5473	.93084	34
27	.36569	.39290	2.5452	.93074	33
28	.36596	.39324	2.5430	.93063	32
29	.36623	.39357	2.5408	.93052	31
30	.36650	.39391	2.5386	.93042	30
31	.36677	.39425	2.5365	.93031	29
32	.36704	.39458	2.5343	.93020	28
33	.36731	.39492	2.5322	.93010	27
34	.36758	.39526	2.5300	.92999	26
35	.36785	.39559	2.5279	.92988	25
36	.36812	.39593	2.5257	.92978	24
37	.36839	.39626	2.5236	.92967	23
38	.36867	.39660	2.5214	.92956	22
39	.36894	.39694	2.5193	.92945	21
40	.36921	.39727	2.5172	.92935	20
41	.36948	.39761	2.5150	.92924	19
42	.36975	.39795	2.5129	.92913	18
43	.37002	.39829	2.5108	.92902	17
44	.37029	.39862	2.5086	.92892	16
45	.37056	.39896	2.5065	.92881	15
46	.37083	.39930	2.5044	.92870	14
47	.37110	.39963	2.5023	.92859	13
48	.37137	.39997	2.5002	.92849	12
49	.37164	.40031	2.4981	.92838	11
50	.37191	.40065	2.4960	.92827	10
51	.37218	.40098	2.4939	.92816	9
52	.37245	.40132	2.4918	.92805	8
53	.37272	.40166	2.4897	.92794	7
54	.37299	.40200	2.4876	.92784	6
55	.37326	.40234	2.4855	.92773	5
56	.37353	.40267	2.4834	.92762	4
57	.37380	.40301	2.4813	.92751	3
58	.37407	.40335	2.4792	.92740	2
59	.37434	.40369	2.4772	.92729	1
60	.37461	.40403	2.4751	.92718	0
′	Cos	Ctn	Tan	Sin	′

68°

Natural Trigonometric Functions

22°

′	Sin	Tan	Ctn	Cos	′
0	.37461	.40403	2.4751	.92718	**60**
1	.37488	.40436	2.4730	.92707	59
2	.37515	.40470	2.4709	.92697	58
3	.37542	.40504	2.4689	.92686	57
4	.37569	.40538	2.4668	.92675	56
5	.37595	.40572	2.4648	.92664	**55**
6	.37622	.40606	2.4627	.92653	54
7	.37649	.40640	2.4606	.92642	53
8	.37676	.40674	2.4586	.92631	52
9	.37703	.40707	2.4566	.92620	51
10	.37730	.40741	2.4545	.92609	**50**
11	.37757	.40775	2.4525	.92598	49
12	.37784	.40809	2.4504	.92587	48
13	.37811	.40843	2.4484	.92576	47
14	.37838	.40877	2.4464	.92565	46
15	.37865	.40911	2.4443	.92554	**45**
16	.37892	.40945	2.4423	.92543	44
17	.37919	.40979	2.4403	.92532	43
18	.37946	.41013	2.4383	.92521	42
19	.37973	.41047	2.4362	.92510	41
20	.37999	.41081	2.4342	.92499	**40**
21	.38026	.41115	2.4322	.92488	39
22	.38053	.41149	2.4302	.92477	38
23	.38080	.41183	2.4282	.92466	37
24	.38107	.41217	2.4262	.92455	36
25	.38134	.41251	2.4242	.92444	**35**
26	.38161	.41285	2.4222	.92432	34
27	.38188	.41319	2.4202	.92421	33
28	.38215	.41353	2.4182	.92410	32
29	.38241	.41387	2.4162	.92399	31
30	.38268	.41421	2.4142	.92388	**30**
31	.38295	.41455	2.4122	.92377	29
32	.38322	.41490	2.4102	.92366	28
33	.38349	.41524	2.4083	.92355	27
34	.38376	.41558	2.4063	.92343	26
35	.38403	.41592	2.4043	.92332	**25**
36	.38430	.41626	2.4023	.92321	24
37	.38456	.41660	2.4004	.92310	23
38	.38483	.41694	2.3984	.92299	22
39	.38510	.41728	2.3964	.92287	21
40	.38537	.41763	2.3945	.92276	**20**
41	.38564	.41797	2.3925	.92265	19
42	.38591	.41831	2.3906	.92254	18
43	.38617	.41865	2.3886	.92243	17
44	.38644	.41899	2.3867	.92231	16
45	.38671	.41933	2.3847	.92220	**15**
46	.38698	.41968	2.3828	.92209	14
47	.38725	.42002	2.3808	.92198	13
48	.38752	.42036	2.3789	.92186	12
49	.38778	.42070	2.3770	.92175	11
50	.38805	.42105	2.3750	.92164	**10**
51	.38832	.42139	2.3731	.92152	9
52	.38859	.42173	2.3712	.92141	8
53	.38886	.42207	2.3693	.92130	7
54	.38912	.42242	2.3673	.92119	6
55	.38939	.42276	2.3654	.92107	**5**
56	.38966	.42310	2.3635	.92096	4
57	.38993	.42345	2.3616	.92085	3
58	.39020	.42379	2.3597	.92073	2
59	.39046	.42413	2.3578	.92062	1
60	.39073	.42447	2.3559	.92050	**0**
′	Cos	Ctn	Tan	Sin	′

67°

23°

′	Sin	Tan	Ctn	Cos	′
0	.39073	.42447	2.3559	.92050	**60**
1	.39100	.42482	2.3539	.92039	59
2	.39127	.42516	2.3520	.92028	58
3	.39153	.42551	2.3501	.92016	57
4	.39180	.42585	2.3483	.92005	56
5	.39207	.42619	2.3464	.91994	**55**
6	.39234	.42654	2.3445	.91982	54
7	.39260	.42688	2.3426	.91971	53
8	.39287	.42722	2.3407	.91959	52
9	.39314	.42757	2.3388	.91948	51
10	.39341	.42791	2.3369	.91936	**50**
11	.39367	.42826	2.3351	.91925	49
12	.39394	.42860	2.3332	.91914	48
13	.39421	.42894	2.3313	.91902	47
14	.39448	.42929	2.3294	.91891	46
15	.39474	.42963	2.3276	.91879	**45**
16	.39501	.42998	2.3257	.91868	44
17	.39528	.43032	2.3238	.91856	43
18	.39555	.43067	2.3220	.91845	42
19	.39581	.43101	2.3201	.91833	41
20	.39608	.43136	2.3183	.91822	**40**
21	.39635	.43170	2.3164	.91810	39
22	.39661	.43205	2.3146	.91799	38
23	.39688	.43239	2.3127	.91787	37
24	.39715	.43274	2.3109	.91775	36
25	.39741	.43308	2.3090	.91764	**35**
26	.39768	.43343	2.3072	.91752	34
27	.39795	.43378	2.3053	.91741	33
28	.39822	.43412	2.3035	.91729	32
29	.39848	.43447	2.3017	.91718	31
30	.39875	.43481	2.2998	.91706	**30**
31	.39902	.43516	2.2980	.91694	29
32	.39928	.43550	2.2962	.91683	28
33	.39955	.43585	2.2944	.91671	27
34	.39982	.43620	2.2925	.91660	26
35	.40008	.43654	2.2907	.91648	**25**
36	.40035	.43689	2.2889	.91636	24
37	.40062	.43724	2.2871	.91625	23
38	.40088	.43758	2.2853	.91613	22
39	.40115	.43793	2.2835	.91601	21
40	.40141	.43828	2.2817	.91590	**20**
41	.40168	.43862	2.2799	.91578	19
42	.40195	.43897	2.2781	.91566	18
43	.40221	.43932	2.2763	.91555	17
44	.40248	.43966	2.2745	.91543	16
45	.40275	.44001	2.2727	.91531	**15**
46	.40301	.44036	2.2709	.91519	14
47	.40328	.44071	2.2691	.91508	13
48	.40355	.44105	2.2673	.91496	12
49	.40381	.44140	2.2655	.91484	11
50	.40408	.44175	2.2637	.91472	**10**
51	.40434	.44210	2.2620	.91461	9
52	.40461	.44244	2.2602	.91449	8
53	.40488	.44279	2.2584	.91437	7
54	.40514	.44314	2.2566	.91425	6
55	.40541	.44349	2.2549	.91414	**5**
56	.40567	.44384	2.2531	.91402	4
57	.40594	.44418	2.2513	.91390	3
58	.40621	.44453	2.2496	.91378	2
59	.40647	.44488	2.2478	.91366	1
60	.40674	.44523	2.2460	.91355	**0**
′	Cos	Ctn	Tan	Sin	′

66°

Natural Trigonometric Functions

24°

′	Sin	Tan	Ctn	Cos	′
0	.40674	.44523	2.2460	.91355	60
1	.40700	.44558	2.2443	.91343	59
2	.40727	.44593	2.2425	.91331	58
3	.40753	.44627	2.2408	.91319	57
4	.40780	.44662	2.2390	.91307	56
5	.40806	.44697	2.2373	.91295	55
6	.40833	.44732	2.2355	.91283	54
7	.40860	.44767	2.2338	.91272	53
8	.40886	.44802	2.2320	.91260	52
9	.40913	.44837	2.2303	.91248	51
10	.40939	.44872	2.2286	.91236	50
11	.40966	.44907	2.2268	.91224	49
12	.40992	.44942	2.2251	.91212	48
13	.41019	.44977	2.2234	.91200	47
14	.41045	.45012	2.2216	.91188	46
15	.41072	.45047	2.2199	.91176	45
16	.41098	.45082	2.2182	.91164	44
17	.41125	.45117	2.2165	.91152	43
18	.41151	.45152	2.2148	.91140	42
19	.41178	.45187	2.2130	91128	41
20	.41204	.45222	2.2113	.91116	40
21	.41231	.45257	2.2096	.91104	39
22	.41257	.45292	2.2079	.91092	38
23	.41284	.45327	2.2062	.91080	37
24	.41310	.45362	2.2045	.91068	36
25	.41337	.45397	2.2028	.91056	35
26	.41363	.45432	2.2011	.91044	34
27	.41390	.45467	2.1994	.91032	33
28	.41416	.45502	2.1977	.91020	32
29	.41443	.45538	2.1960	.91008	31
30	.41469	.45573	2.1943	.90996	30
31	.41496	.45608	2.1926	.90984	29
32	.41522	.45643	2.1909	.90972	28
33	.41549	.45678	2.1892	.90960	27
34	.41575	.45713	2.1876	.90948	26
35	.41602	.45748	2.1859	.90936	25
36	.41628	.45784	2.1842	.90924	24
37	.41655	.45819	2.1825	.90911	23
38	.41681	.45854	2.1808	.90899	22
39	.41707	.45889	2.1792	.90887	21
40	.41734	.45924	2.1775	.90875	20
41	.41760	.45960	2.1758	.90863	19
42	.41787	.45995	2.1742	.90851	18
43	.41813	.46030	2.1725	.90839	17
44	.41840	.46065	2.1708	.90826	16
45	.41866	.46101	2.1692	.90814	15
46	.41892	.46136	2.1675	.90802	14
47	.41919	.46171	2.1659	.90790	13
48	.41945	.46206	2.1642	.90778	12
49	.41972	.46242	2.1625	.90766	11
50	.41998	.46277	2.1609	.90753	10
51	.42024	.46312	2.1592	.90741	9
52	.42051	.46348	2.1576	.90729	8
53	.42077	.46383	2.1560	.90717	7
54	.42104	.46418	2.1543	.90704	6
55	.42130	.46454	2.1527	.90692	5
56	.42156	.46489	2.1510	.90680	4
57	.42183	.46525	2.1494	.90668	3
58	.42209	.46560	2.1478	.90655	2
59	.42235	.46595	2.1461	.90643	1
60	.42262	.46631	2.1445	.90631	0
′	Cos	Ctn	Tan	Sin	′

65°

25°

′	Sin	Tan	Ctn	Cos	′
0	.42262	.46631	2.1445	.90631	60
1	.42288	.46666	2.1429	.90618	59
2	.42315	.46702	2.1413	.90606	58
3	.42341	.46737	2.1396	.90594	57
4	.42367	.46772	2.1380	.90582	56
5	.42394	.46808	2.1364	.90569	55
6	.42420	.46843	2.1348	.90557	54
7	.42446	.46879	2.1332	.90545	53
8	.42473	.46914	2.1315	.90532	52
9	.42499	.46950	2.1299	.90520	51
10	.42525	.46985	2.1283	.90507	50
11	.42552	.47021	2.1267	.90495	49
12	.42578	.47056	2.1251	.90483	48
13	.42604	.47092	2.1235	.90470	47
14	.42631	.47128	2.1219	.90458	46
15	.42657	.47163	2.1203	.90446	45
16	.42683	.47199	2.1187	.90433	44
17	.42709	.47234	2.1171	.90421	43
18	.42736	.47270	2.1155	.90408	42
19	.42762	.47305	2.1139	.90396	41
20	.42788	.47341	2.1123	.90383	40
21	.42815	.47377	2.1107	.90371	39
22	.42841	.47412	2.1092	.90358	38
23	.42867	.47448	2.1076	.90346	37
24	.42894	.47483	2.1060	.90334	36
25	.42920	.47519	2.1044	.90321	35
26	.42946	.47555	2.1028	.90309	34
27	.42972	.47590	2.1013	.90296	33
28	.42999	.47626	2.0997	.90284	32
29	.43025	.47662	2.0981	.90271	31
30	.43051	.47698	2.0965	.90259	30
31	.43077	.47733	2.0950	.90246	29
32	.43104	.47769	2.0934	.90233	28
33	.43130	.47805	2.0918	.90221	27
34	.43156	.47840	2.0903	.90208	26
35	.43182	.47876	2.0887	.90196	25
36	.43209	.47912	2.0872	.90183	24
37	.43235	.47948	2.0856	.90171	23
38	.43261	.47984	2.0840	.90158	22
39	.43287	.48019	2.0825	.90146	21
40	.43313	.48055	2.0809	.90133	20
41	.43340	.48091	2.0794	.90120	19
42	.43366	.48127	2.0778	.90108	18
43	.43392	.48163	2.0763	.90095	17
44	.43418	.48198	2.0748	.90082	16
45	.43445	.48234	2.0732	.90070	15
46	.43471	.48270	2.0717	.90057	14
47	.43497	.48306	2.0701	.90045	13
48	.43523	.48342	2.0686	.90032	12
49	.43549	.48378	2.0671	.90019	11
50	.43575	.48414	2.0655	.90007	10
51	.43602	.48450	2.0640	.89994	9
52	.43628	.48486	2.0625	.89981	8
53	.43654	.48521	2.0609	.89968	7
54	.43680	.48557	2.0594	.89956	6
55	.43706	.48593	2.0579	.89943	5
56	.43733	.48629	2.0564	.89930	4
57	.43759	.48665	2.0549	.89918	3
58	.43785	.48701	2.0533	.89905	2
59	.43811	.48737	2.0518	.89892	1
60	.43837	.48773	2.0503	.89879	0
′	Cos	Ctn	Tan	Sin	′

64°

Natural Trigonometric Functions

26°

′	Sin	Tan	Ctn	Cos	′
0	.43837	.48773	2.0503	.89879	60
1	.43863	.48809	2.0488	.89867	59
2	.43889	.48845	2.0473	.89854	58
3	.43916	.48881	2.0458	.89841	57
4	.43942	.48917	2.0443	.89828	56
5	.43968	.48953	2.0428	.89816	55
6	.43994	.48989	2.0413	.89803	54
7	.44020	.49026	2.0398	.89790	53
8	.44046	.49062	2.0383	.89777	52
9	.44072	.49098	2.0368	.89764	51
10	.44098	.49134	2.0353	.89752	50
11	.44124	.49170	2.0338	.89739	49
12	.44151	.49206	2.0323	.89726	48
13	.44177	.49242	2.0308	.89713	47
14	.44203	.49278	2.0293	.89700	46
15	.44229	.49315	2.0278	.89687	45
16	.44255	.49351	2.0263	.89674	44
17	.44281	.49387	2.0248	.89662	43
18	.44307	.49423	2.0233	.89649	42
19	.44333	.49459	2.0219	.89636	41
20	.44359	.49495	2.0204	.89623	40
21	.44385	.49532	2.0189	.89610	39
22	.44411	.49568	2.0174	.89597	38
23	.44437	.49604	2.0160	.89584	37
24	.44464	.49640	2.0145	.89571	36
25	.44490	.49677	2.0130	.89558	35
26	.44516	.49713	2.0115	.89545	34
27	.44542	.49749	2 0101	.89532	33
28	.44568	.49786	2.0086	.89519	32
29	.44594	.49822	2.0072	.89506	31
30	.44620	.49858	2.0057	.89493	30
31	.44646	.49894	2.0042	.89480	29
32	.44672	.49931	2.0028	.89467	28
33	.44698	.49967	2.0013	.89454	27
34	.44724	.50004	1.9999	.89441	26
35	.44750	.50040	1.9984	.89428	25
36	.44776	.50076	1.9970	.89415	24
37	.44802	.50113	1.9955	.89402	23
38	.44828	.50149	1.9941	.89389	22
39	.44854	.50185	1.9926	.89376	21
40	.44880	.50222	1.9912	.89363	20
41	.44906	.50258	1.9897	.89350	19
42	.44932	.50295	1.9883	.89337	18
43	.44958	.50331	1.9868	.89324	17
44	.44984	.50368	1.9854	.89311	16
45	.45010	.50404	1.9840	.89298	15
46	.45036	.50441	1.9825	.89285	14
47	.45062	.50477	1.9811	.89272	13
48	.45088	.50514	1.9797	.89259	12
49	.45114	.50550	1.9782	.89245	11
50	.45140	.50587	1.9768	.89232	10
51	.45166	.50623	1.9754	.89219	9
52	.45192	.50660	1.9740	.89206	8
53	.45218	.50696	1.9725	.89193	7
54	.45243	.50733	1.9711	.89180	6
55	.45269	.50769	1.9697	.89167	5
56	.45295	.50806	1.9683	.89153	4
57	.45321	.50843	1.9669	.89140	3
58	.45347	.50879	1.9654	.89127	2
59	.45373	.50916	1.9640	.89114	1
60	.45399	.50953	1.9626	.89101	0
′	Cos	Ctn	Tan	Sin	′

63°

27°

′	Sin	Tan	Ctn	Cos	′
0	.45399	.50953	1.9626	.89101	60
1	.45425	.50989	1.9612	.89087	59
2	.45451	.51026	1.9598	.89074	58
3	.45477	.51063	1.9584	.89061	57
4	.45503	.51099	1.9570	.89048	56
5	.45529	.51136	1.9556	.89035	55
6	.45554	.51173	1.9542	.89021	54
7	.45580	.51209	1.9528	.89008	53
8	.45606	.51246	1.9514	.88995	52
9	.45632	.51283	1.9500	.88981	51
10	.45658	.51319	1.9486	.88968	50
11	.45684	.51356	1.9472	.88955	49
12	.45710	.51393	1.9458	.88942	48
13	.45736	.51430	1.9444	.88928	47
14	.45762	.51467	1.9430	.88915	46
15	.45787	.51503	1.9416	.88902	45
16	.45813	.51540	1.9402	.88888	44
17	.45839	.51577	1.9388	.88875	43
18	.45865	.51614	1.9375	.88862	42
19	.45891	.51651	1.9361	.88848	41
20	.45917	.51688	1.9347	.88835	40
21	.45942	.51724	1.9333	.88822	39
22	.45968	.51761	1.9319	.88808	38
23	.45994	.51798	1.9306	.88795	37
24	.46020	.51835	1.9292	.88782	36
25	.46046	.51872	1.9278	.88768	35
26	.46072	.51909	1.9265	.88755	34
27	.46097	.51946	1.9251	.88741	33
28	.46123	.51983	1.9237	.88728	32
29	.46149	.52020	1.9223	.88715	31
30	.46175	.52057	1.9210	.88701	30
31	.46201	.52094	1.9196	.88688	29
32	.46226	.52131	1.9183	.88674	28
33	.46252	.52168	1.9169	.88661	27
34	.46278	.52205	1.9155	.88647	26
35	.46304	.52242	1.9142	.88634	25
36	.46330	.52279	1.9128	.88620	24
37	.46355	.52316	1.9115	.88607	23
38	.46381	.52353	1.9101	.88593	22
39	.46407	.52390	1.9088	.88580	21
40	.46433	.52427	1.9074	.88566	20
41	.46458	.52464	1.9061	.88553	19
42	.46484	.52501	1.9047	.88539	18
43	.46510	.52538	1.9034	.88526	17
44	.46536	.52575	1.9020	.88512	16
45	.46561	.52613	1.9007	.88499	15
46	.46587	.52650	1.8993	.88485	14
47	.46613	.52687	1.8980	.88472	13
48	.46639	.52724	1.8967	.88458	12
49	.46664	.52761	1.8953	.88445	11
50	.46690	.52798	1.8940	.88431	10
51	.46716	.52836	1.8927	.88417	9
52	.46742	.52873	1.8913	.88404	8
53	.46767	.52910	1.8900	.88390	7
54	.46793	.52947	1.8887	.88377	6
55	.46819	.52985	1.8873	.88363	5
56	.46844	.53022	1.8860	.88349	4
57	.46870	.53059	1.8847	.88336	3
58	.46896	.53096	1.8834	.88322	2
59	.46921	.53134	1.8820	.88308	1
60	.46947	.53171	1.8807	.88295	0
′	Cos	Ctn	Tan	Sin	′

62°

Natural Trigonometric Functions

28°

′	Sin	Tan	Ctn	Cos	′
0	.46947	.53171	1.8807	.88295	60
1	.46973	.53208	1.8794	.88281	59
2	.46999	.53246	1.8781	.88267	58
3	.47024	.53283	1.8768	.88254	57
4	.47050	.53320	1.8755	.88240	56
5	.47076	.53358	1.8741	.88226	55
6	.47101	.53395	1.8728	.88213	54
7	.47127	.53432	1.8715	.88199	53
8	.47153	.53470	1.8702	.88185	52
9	.47178	.53507	1.8689	.88172	51
10	.47204	.53545	1.8676	.88158	50
11	.47229	.53582	1.8663	.88144	49
12	.47255	.53620	1.8650	.88130	48
13	.47281	.53657	1.8637	.88117	47
14	.47306	.53694	1.8624	.88103	46
15	.47332	.53732	1.8611	.88089	45
16	.47358	.53769	1.8598	.88075	44
17	.47383	.53807	1.8585	.88062	43
18	.47409	.53844	1.8572	.88048	42
19	.47434	.53882	1.8559	.88034	41
20	.47460	.53920	1.8546	.88020	40
21	.47486	.53957	1.8533	.88006	39
22	.47511	.53995	1.8520	.87993	38
23	.47537	.54032	1.8507	.87979	37
24	.47562	.54070	1.8495	.87965	36
25	.47588	.54107	1.8482	.87951	35
26	.47614	.54145	1.8469	.87937	34
27	.47639	.54183	1.8456	.87923	33
28	.47665	.54220	1.8443	.87909	32
29	.47690	.54258	1.8430	.87896	31
30	.47716	.54296	1.8418	.87882	30
31	.47741	.54333	1.8405	.87868	29
32	.47767	.54371	1.8392	.87854	28
33	.47793	.54409	1.8379	.87840	27
34	.47818	.54446	1.8367	.87826	26
35	.47844	.54484	1.8354	.87812	25
36	.47869	.54522	1.8341	.87798	24
37	.47895	.54560	1.8329	.87784	23
38	.47920	.54597	1.8316	.87770	22
39	.47946	.54635	1.8303	.87756	21
40	.47971	.54673	1.8291	.87743	20
41	.47997	.54711	1.8278	.87729	19
42	.48022	.54748	1.8265	.87715	18
43	.48048	.54786	1.8253	.87701	17
44	.48073	.54824	1.8240	.87687	16
45	.48099	.54862	1.8228	.87673	15
46	.48124	.54900	1.8215	.87659	14
47	.48150	.54938	1.8202	.87645	13
48	.48175	.54975	1.8190	.87631	12
49	.48201	.55013	1.8177	.87617	11
50	.48226	.55051	1.8165	.87603	10
51	.48252	.55089	1.8152	.87589	9
52	.48277	.55127	1.8140	.87575	8
53	.48303	.55165	1.8127	.87561	7
54	.48328	.55203	1.8115	.87546	6
55	.48354	.55241	1.8103	.87532	5
56	.48379	.55279	1.8090	.87518	4
57	.48405	.55317	1.8078	.87504	3
58	.48430	.55355	1.8065	.87490	2
59	.48456	.55393	1.8053	.87476	1
60	.48481	.55431	1.8040	.87462	0
′	Cos	Ctn	Tan	Sin	′

61°

29°

′	Sin	Tan	Ctn	Cos	′
0	.48481	.55431	1.8040	.87462	60
1	.48506	.55469	1.8028	.87448	59
2	.48532	.55507	1.8016	.87434	58
3	.48557	.55545	1.8003	.87420	57
4	.48583	.55583	1.7991	.87406	56
5	.48608	.55621	1.7979	.87391	55
6	.48634	.55659	1.7966	.87377	54
7	.48659	.55697	1.7954	.87363	53
8	.48684	.55736	1.7942	.87349	52
9	.48710	.55774	1.7930	.87335	51
10	.48735	.55812	1.7917	.87321	50
11	.48761	.55850	1.7905	.87306	49
12	.48786	.55888	1.7893	.87292	48
13	.48811	.55926	1.7881	.87278	47
14	.48837	.55964	1.7868	.87264	46
15	.48862	.56003	1.7856	.87250	45
16	.48888	.56041	1.7844	.87235	44
17	.48913	.56079	1.7832	.87221	43
18	.48938	.56117	1.7820	.87207	42
19	.48964	.56156	1.7808	.87193	41
20	.48989	.56194	1.7796	.87178	40
21	.49014	.56232	1.7783	.87164	39
22	.49040	.56270	1.7771	.87150	38
23	.49065	.56309	1.7759	.87136	37
24	.49090	.56347	1.7747	.87121	36
25	.49116	.56385	1.7735	.87107	35
26	.49141	.56424	1.7723	.87093	34
27	.49166	.56462	1.7711	.87079	33
28	.49192	.56501	1.7699	.87064	32
29	.49217	.56539	1.7687	.87050	31
30	.49242	.56577	1.7675	.87036	30
31	.49268	.56616	1.7663	.87021	29
32	.49293	.56654	1.7651	.87007	28
33	.49318	.56693	1.7639	.86993	27
34	.49344	.56731	1.7627	.86978	26
35	.49369	.56769	1.7615	.86964	25
36	.49394	.56808	1.7603	.86949	24
37	.49419	.56846	1.7591	.86935	23
38	.49445	.56885	1.7579	.86921	22
39	.49470	.56923	1.7567	.86906	21
40	.49495	.56962	1.7556	.86892	20
41	.49521	.57000	1.7544	.86878	19
42	.49546	.57039	1.7532	.86863	18
43	.49571	.57078	1.7520	.86849	17
44	.49596	.57116	1.7508	.86834	16
45	.49622	.57155	1.7496	.86820	15
46	.49647	.57193	1.7485	.86805	14
47	.49672	.57232	1.7473	.86791	13
48	.49697	.57271	1.7461	.86777	12
49	.49723	.57309	1.7449	.86762	11
50	.49748	.57348	1.7437	.86748	10
51	.49773	.57386	1.7426	.86733	9
52	.49798	.57425	1.7414	.86719	8
53	.49824	.57464	1.7402	.86704	7
54	.49849	.57503	1.7391	.86690	6
55	.49874	.57541	1.7379	.86675	5
56	.49899	.57580	1.7367	.86661	4
57	.49924	.57619	1.7355	.86646	3
58	.49950	.57657	1.7344	.86632	2
59	.49975	.57696	1.7332	.86617	1
60	.50000	.57735	1.7321	.86603	0
′	Cos	Ctn	Tan	Sin	′

60°

Natural Trigonometric Functions

30°

′	Sin	Tan	Ctn	Cos	′
0	.50000	.57735	1.7321	.86603	**60**
1	.50025	.57774	1.7309	.86588	59
2	.50050	.57813	1.7297	.86573	58
3	.50076	.57851	1.7286	.86559	57
4	.50101	.57890	1.7274	.86544	56
5	.50126	.57929	1.7262	.86530	**55**
6	.50151	.57968	1.7251	.86515	54
7	.50176	.58007	1.7239	.86501	53
8	.50201	.58046	1.7228	.86486	52
9	.50227	.58085	1.7216	.86471	51
10	.50252	.58124	1.7205	.86457	**50**
11	.50277	.58162	1.7193	.86442	49
12	.50302	.58201	1.7182	.86427	48
13	.50327	.58240	1.7170	.86413	47
14	.50352	.58279	1.7159	.86398	46
15	.50377	.58318	1.7147	.86384	**45**
16	.50403	.58357	1.7136	.86369	44
17	.50428	.58396	1.7124	.86354	43
18	.50453	.58435	1.7113	.86340	42
19	.50478	.58474	1.7102	.86325	41
20	.50503	.58513	1.7090	.86310	**40**
21	.50528	.58552	1.7079	.86295	39
22	.50553	.58591	1.7067	.86281	38
23	.50578	.58631	1.7056	.86266	37
24	.50603	.58670	1.7045	.86251	36
25	.50628	.58709	1.7033	.86237	**35**
26	.50654	.58748	1.7022	.86222	34
27	.50679	.58787	1.7011	.86207	33
28	.50704	.58826	1.6999	.86192	32
29	.50729	.58865	1.6988	.86178	31
30	.50754	.58905	1.6977	.86163	**30**
31	.50779	.58944	1.6965	.86148	29
32	.50804	.58983	1.6954	.86133	28
33	.50829	.59022	1.6943	.86119	27
34	.50854	.59061	1.6932	.86104	26
35	.50879	.59101	1.6920	.86089	**25**
36	.50904	.59140	1.6909	.86074	24
37	.50929	.59179	1.6898	.86059	23
38	.50954	.59218	1.6887	.86045	22
39	.50979	.59258	1.6875	.86030	21
40	.51004	.59297	1.6864	.86015	**20**
41	.51029	.59336	1.6853	.86000	19
42	.51054	.59376	1.6842	.85985	18
43	.51079	.59415	1.6831	.85970	17
44	.51104	.59454	1.6820	.85956	16
45	.51129	.59494	1.6808	.85941	**15**
46	.51154	.59533	1.6797	.85926	14
47	.51179	.59573	1.6786	.85911	13
48	.51204	.59612	1.6775	.85896	12
49	.51229	.59651	1.6764	.85881	11
50	.51254	.59691	1.6753	.85866	**10**
51	.51279	.59730	1.6742	.85851	9
52	.51304	.59770	1.6731	.85836	8
53	.51329	.59809	1.6720	.85821	7
54	.51354	.59849	1.6709	.85806	6
55	.51379	.59888	1.6698	.85792	**5**
56	.51404	.59928	1.6687	.85777	4
57	.51429	.59967	1.6676	.85762	3
58	.51454	.60007	1.6665	.85747	2
59	.51479	.60046	1.6654	.85732	1
60	.51504	.60086	1.6643	.85717	**0**
′	Cos	Ctn	Tan	Sin	′

59°

31°

′	Sin	Tan	Ctn	Cos	′
0	.51504	.60086	1.6643	.85717	**60**
1	.51529	.60126	1.6632	.85702	59
2	.51554	.60165	1.6621	.85687	58
3	.51579	.60205	1.6610	.85672	57
4	.51604	.60245	1.6599	.85657	56
5	.51628	.60284	1.6588	.85642	**55**
6	.51653	.60324	1.6577	.85627	54
7	.51678	.60364	1.6566	.85612	53
8	.51703	.60403	1.6555	.85597	52
9	.51728	.60443	1.6545	.85582	51
10	.51753	.60483	1.6534	.85567	**50**
11	.51778	.60522	1.6523	.85551	49
12	.51803	.60562	1.6512	.85536	48
13	.51828	.60602	1.6501	.85521	47
14	.51852	.60642	1.6490	.85506	46
15	.51877	.60681	1.6479	.85491	**45**
16	.51902	.60721	1.6469	.85476	44
17	.51927	.60761	1.6458	.85461	43
18	.51952	.60801	1.6447	.85446	42
19	.51977	.60841	1.6436	.85431	41
20	.52002	.60881	1.6426	.85416	**40**
21	.52026	.60921	1.6415	.85401	39
22	.52051	.60960	1.6404	.85385	38
23	.52076	.61000	1.6393	.85370	37
24	.52101	.61040	1.6383	.85355	36
25	.52126	.61080	1.6372	.85340	**35**
26	.52151	.61120	1.6361	.85325	34
27	.52175	.61160	1.6351	.85310	33
28	.52200	.61200	1.6340	.85294	32
29	.52225	.61240	1.6329	.85279	31
30	.52250	.61280	1.6319	.85264	**30**
31	.52275	.61320	1.6308	.85249	29
32	.52299	.61360	1.6297	.85234	28
33	.52324	.61400	1.6287	.85218	27
34	.52349	.61440	1.6276	.85203	26
35	.52374	.61480	1.6265	.85188	**25**
36	.52399	.61520	1.6255	.85173	24
37	.52423	.61561	1.6244	.85157	23
38	.52448	.61601	1.6234	.85142	22
39	.52473	.61641	1.6223	.85127	21
40	.52498	.61681	1.6212	.85112	**20**
41	.52522	.61721	1.6202	.85096	19
42	.52547	.61761	1.6191	.85081	18
43	.52572	.61801	1.6181	.85066	17
44	.52597	.61842	1.6170	.85051	16
45	.52621	.61882	1.6160	.85035	**15**
46	.52646	.61922	1.6149	.85020	14
47	.52671	.61962	1.6139	.85005	13
48	.52696	.62003	1.6128	.84989	12
49	.52720	.62043	1.6118	.84974	11
50	.52745	.62083	1.6107	.84959	**10**
51	.52770	.62124	1.6097	.84943	9
52	.52794	.62164	1.6087	.84928	8
53	.52819	.62204	1.6076	.84913	7
54	.52844	.62245	1.6066	.84897	6
55	.52869	.62285	1.6055	.84882	**5**
56	.52893	.62325	1.6045	.84866	4
57	.52918	.62366	1.6034	.84851	3
58	.52943	.62406	1.6024	.84836	2
59	.52967	.62446	1.6014	.84820	1
60	.52992	.62487	1.6003	.84805	**0**
′	Cos	Ctn	Tan	Sin	′

58°

Natural Trigonometric Functions

32°

′	Sin	Tan	Ctn	Cos	′
0	.52992	.62487	1.6003	.84805	**60**
1	.53017	.62527	1.5993	.84789	59
2	.53041	.62568	1.5983	.84774	58
3	.53066	.62608	1.5972	.84759	57
4	.53091	.62649	1.5962	.84743	56
5	.53115	.62689	1.5952	.84728	**55**
6	.53140	.62730	1.5941	.84712	54
7	.53164	.62770	1.5931	.84697	53
8	.53189	.62811	1.5921	.84681	52
9	.53214	.62852	1.5911	.84666	51
10	.53238	.62892	1.5900	.84650	**50**
11	.53263	.62933	1.5890	.84635	49
12	.53288	.62973	1.5880	.84619	48
13	.53312	.63014	1.5869	.84604	47
14	.53337	.63055	1.5859	.84588	46
15	.53361	.63095	1.5849	.84573	**45**
16	.53386	.63136	1.5839	.84557	44
17	.53411	.63177	1.5829	.84542	43
18	.53435	.63217	1.5818	.84526	42
19	.53460	.63258	1.5808	.84511	41
20	.53484	.63299	1.5798	.84495	**40**
21	.53509	.63340	1.5788	.84480	39
22	.53534	.63380	1.5778	.84464	38
23	.53558	.63421	1.5768	.84448	37
24	.53583	.63462	1.5757	.84433	36
25	.53607	.63503	1.5747	.84417	**35**
26	.53632	.63544	1.5737	.84402	34
27	.53656	.63584	1.5727	.84386	33
28	.53681	.63625	1.5717	.84370	32
29	.53705	.63666	1.5707	.84355	31
30	.53730	.63707	1.5697	.84339	**30**
31	.53754	.63748	1.5687	.84324	29
32	.53779	.63789	1.5677	.84308	28
33	.53804	.63830	1.5667	.84292	27
34	.53828	.63871	1.5657	.84277	26
35	.53853	.63912	1.5647	.84261	**25**
36	.53877	.63953	1.5637	.84245	24
37	.53902	.63994	1.5627	.84230	23
38	.53926	.64035	1.5617	.84214	22
39	.53951	.64076	1.5607	.84198	21
40	.53975	.64117	1.5597	.84182	**20**
41	.54000	.64158	1.5587	.84167	19
42	.54024	.64199	1.5577	.84151	18
43	.54049	.64240	1.5567	.84135	17
44	.54073	.64281	1.5557	.84120	16
45	.54097	.64322	1.5547	.84104	**15**
46	.54122	.64363	1.5537	.84088	14
47	.54146	.64404	1.5527	.84072	13
48	.54171	.64446	1.5517	.84057	12
49	.54195	.64487	1.5507	.84041	11
50	.54220	.64528	1.5497	.84025	**10**
51	.54244	.64569	1.5487	.84009	9
52	.54269	.64610	1.5477	.83994	8
53	.54293	.64652	1.5468	.83978	7
54	.54317	.64693	1.5458	.83962	6
55	.54342	.64734	1.5448	.83946	**5**
56	.54366	.64775	1.5438	.83930	4
57	.54391	.64817	1.5428	.83915	3
58	.54415	.64858	1.5418	.83899	2
59	.54440	.64899	1.5408	.83883	1
60	.54464	.64941	1.5399	.83867	**0**
′	Cos	Ctn	Tan	Sin	′

57°

33°

′	Sin	Tan	Ctn	Cos	′
0	.54464	.64941	1.5399	.83867	**60**
1	.54488	.64982	1.5389	.83851	59
2	.54513	.65024	1.5379	.83835	58
3	.54537	.65065	1.5369	.83819	57
4	.54561	.65106	1.5359	.83804	56
5	.54586	.65148	1.5350	.83788	**55**
6	.54610	.65189	1.5340	.83772	54
7	.54635	.65231	1.5330	.83756	53
8	.54659	.65272	1.5320	.83740	52
9	.54683	.65314	1.5311	.83724	51
10	.54708	.65355	1.5301	.83708	**50**
11	.54732	.65397	1.5291	.83692	49
12	.54756	.65438	1.5282	.83676	48
13	.54781	.65480	1.5272	.83660	47
14	.54805	.65521	1.5262	.83645	46
15	.54829	.65563	1.5253	.83629	**45**
16	.54854	.65604	1.5243	.83613	44
17	.54878	.65646	1.5233	.83597	43
18	.54902	.65688	1.5224	.83581	42
19	.54927	.65729	1.5214	.83565	41
20	.54951	.65771	1.5204	.83549	**40**
21	.54975	.65813	1.5195	.83533	39
22	.54999	.65854	1.5185	.83517	38
23	.55024	.65896	1.5175	.83501	37
24	.55048	.65938	1.5166	.83485	36
25	.55072	.65980	1.5156	.83469	**35**
26	.55097	.66021	1.5147	.83453	34
27	.55121	.66063	1.5137	.83437	33
28	.55145	.66105	1.5127	.83421	32
29	.55169	.66147	1.5118	.83405	31
30	.55194	.66189	1.5108	.83389	**30**
31	.55218	.66230	1.5099	.83373	29
32	.55242	.66272	1.5089	83356	28
33	.55266	.66314	1.5080	.83340	27
34	.55291	.66356	1.5070	.83324	26
35	.55315	.66398	1.5061	.83308	**25**
36	.55339	.66440	1.5051	.83292	24
37	.55363	.66482	1.5042	.83276	23
38	.55388	.66524	1.5032	.83260	22
39	.55412	.66566	1.5023	.83244	21
40	.55436	.66608	1.5013	.83228	**20**
41	.55460	.66650	1.5004	.83212	19
42	.55484	.66692	1 4994	.83195	18
43	.55509	.66734	1.4985	.83179	17
44	.55533	.66776	1.4975	.83163	16
45	.55557	.66818	1.4966	.83147	**15**
46	55581	.66860	1.4957	.83131	14
47	.55605	.66902	1.4947	.83115	13
48	.55630	.66944	1.4938	.83098	12
49	.55654	.66986	1.4928	.83082	11
50	.55678	.67028	1.4919	.83066	**10**
51	.55702	.67071	1.4910	.83050	9
52	.55726	.67113	1.4900	.83034	8
53	.55750	.67155	1.4891	.83017	7
54	.55775	.67197	1.4882	.83001	6
55	.55799	.67239	1.4872	.82985	**5**
56	.55823	.67282	1.4863	.82969	4
57	.55847	.67324	1.4854	.82953	3
58	.55871	.67366	1.4844	.82936	2
59	.55895	.67409	1.4835	.82920	1
60	.55919	.67451	1.4826	.82904	**0**
′	Cos	Ctn	Tan	Sin	′

56°

Natural Trigonometric Functions

34°

′	Sin	Tan	Ctn	Cos	′
0	.55919	.67451	1.4826	.82904	60
1	.55943	.67493	1.4816	.82887	59
2	.55968	.67536	1.4807	.82871	58
3	.55992	.67578	1.4798	.82855	57
4	.56016	.67620	1.4788	.82839	56
5	.56040	.67663	1.4779	.82822	55
6	.56064	.67705	1.4770	.82806	54
7	.56088	.67748	1.4761	.82790	53
8	.56112	.67790	1.4751	.82773	52
9	.56136	.67832	1.4742	.82757	51
10	.56160	.67875	1.4733	.82741	50
11	.56184	.67917	1.4724	.82724	49
12	.56208	.67960	1.4715	.82708	48
13	.56232	.68002	1.4705	.82692	47
14	.56256	.68045	1.4696	.82675	46
15	.56280	.68088	1.4687	.82659	45
16	.56305	.68130	1.4678	.82643	44
17	.56329	.68173	1.4669	.82626	43
18	.56353	.68215	1.4659	.82610	42
19	.56377	.68258	1.4650	.82593	41
20	.56401	.68301	1.4641	.82577	40
21	.56425	.68343	1.4632	.82561	39
22	.56449	.68386	1.4623	.82544	38
23	.56473	.68429	1.4614	.82528	37
24	.56497	.68471	1.4605	.82511	36
25	.56521	.68514	1.4596	.82495	35
26	.56545	.68557	1.4586	.82478	34
27	.56569	.68600	1.4577	.82462	33
28	.56593	.68642	1.4568	.82446	32
29	.56617	.68685	1.4559	.82429	31
30	.56641	.68728	1.4550	.82413	30
31	.56665	.68771	1.4541	.82396	29
32	.56689	.68814	1.4532	.82380	28
33	.56713	.68857	1.4523	.82363	27
34	.56736	.68900	1.4514	.82347	26
35	.56760	.68942	1.4505	.82330	25
36	.56784	.68985	1.4496	.82314	24
37	.56808	.69028	1.4487	.82297	23
38	.56832	.69071	1.4478	.82281	22
39	.56856	.69114	1.4469	.82264	21
40	.56880	.69157	1.4460	.82248	20
41	.56904	.69200	1.4451	.82231	19
42	.56928	.69243	1.4442	.82214	18
43	.56952	.69286	1.4433	.82198	17
44	.56976	.69329	1.4424	.82181	16
45	.57000	.69372	1.4415	.82165	15
46	.57024	.69416	1.4406	.82148	14
47	.57047	.69459	1.4397	.82132	13
48	.57071	.69502	1.4388	.82115	12
49	.57095	.69545	1.4379	.82098	11
50	.57119	.69588	1.4370	.82082	10
51	.57143	.69631	1.4361	.82065	9
52	.57167	.69675	1.4352	.82048	8
53	.57191	.69718	1.4344	.82032	7
54	.57215	.69761	1.4335	.82015	6
55	.57238	.69804	1.4326	.81999	5
56	.57262	.69847	1.4317	.81982	4
57	.57286	.69891	1.4308	.81965	3
58	.57310	.69934	1.4299	.81949	2
59	.57334	.69977	1.4290	.81932	1
60	.57358	.70021	1.4281	.81915	0
′	Cos	Ctn	Tan	Sin	′

55°

35°

′	Sin	Tan	Ctn	Cos	′
0	.57358	.70021	1.4281	.81915	60
1	.57381	.70064	1.4273	.81899	59
2	.57405	.70107	1.4264	.81882	58
3	.57429	.70151	1.4255	.81865	57
4	.57453	.70194	1.4246	.81848	56
5	.57477	.70238	1.4237	.81832	55
6	.57501	.70281	1.4229	.81815	54
7	.57524	.70325	1.4220	.81798	53
8	.57548	.70368	1.4211	.81782	52
9	.57572	.70412	1.4202	.81765	51
10	.57596	.70455	1.4193	.81748	50
11	.57619	.70499	1.4185	.81731	49
12	.57643	.70542	1.4176	.81714	48
13	.57667	.70586	1.4167	.81698	47
14	.57691	.70629	1.4158	.81681	46
15	.57715	.70673	1.4150	.81664	45
16	.57738	.70717	1.4141	.81647	44
17	.57762	.70760	1.4132	.81631	43
18	.57786	.70804	1.4124	.81614	42
19	.57810	.70848	1.4115	.81597	41
20	.57833	.70891	1.4106	.81580	40
21	.57857	.70935	1.4097	.81563	39
22	.57881	.70979	1.4089	.81546	38
23	.57904	.71023	1.4080	.81530	37
24	.57928	.71066	1.4071	.81513	36
25	.57952	.71110	1.4063	.81496	35
26	.57976	.71154	1.4054	.81479	34
27	.57999	.71198	1.4045	.81462	33
28	.58023	.71242	1.4037	.81445	32
29	.58047	.71285	1.4028	.81428	31
30	.58070	.71329	1.4019	.81412	30
31	.58094	.71373	1.4011	.81395	29
32	.58118	.71417	1.4002	.81378	28
33	.58141	.71461	1.3994	.81361	27
34	.58165	.71505	1.3985	.81344	26
35	.58189	.71549	1.3976	.81327	25
36	.58212	.71593	1.3968	.81310	24
37	.58236	.71637	1.3959	.81293	23
38	.58260	.71681	1.3951	.81276	22
39	.58283	.71725	1.3942	.81259	21
40	.58307	.71769	1.3934	.81242	20
41	.58330	.71813	1.3925	.81225	19
42	.58354	.71857	1.3916	.81208	18
43	.58378	.71901	1.3908	.81191	17
44	.58401	.71946	1.3899	.81174	16
45	.58425	.71990	1.3891	.81157	15
46	.58449	.72034	1.3882	.81140	14
47	.58472	.72078	1.3874	.81123	13
48	.58496	.72122	1.3865	.81106	12
49	.58519	.72167	1.3857	.81089	11
50	.58543	.72211	1.3848	.81072	10
51	.58567	.72255	1.3840	.81055	9
52	.58590	.72299	1.3831	.81038	8
53	.58614	.72344	1.3823	.81021	7
54	.58637	.72388	1.3814	.81004	6
55	.58661	.72432	1.3806	.80987	5
56	.58684	.72477	1.3798	.80970	4
57	.58708	.72521	1.3789	.80953	3
58	.58731	.72565	1.3781	.80936	2
59	.58755	.72610	1.3772	.80919	1
60	.58779	.72654	1.3764	.80902	0
′	Cos	Ctn	Tan	Sin	′

54°

Natural Trigonometric Functions

36°

′	Sin	Tan	Ctn	Cos	′
0	.58779	.72654	1.3764	.80902	60
1	.58802	.72699	1.3755	.80885	59
2	.58826	.72743	1.3747	.80867	58
3	.58849	.72788	1.3739	.80850	57
4	.58873	.72832	1.3730	.80833	56
5	.58896	.72877	1.3722	.80816	55
6	.58920	.72921	1.3713	.80799	54
7	.58943	.72966	1.3705	.80782	53
8	.58967	.73010	1.3697	.80765	52
9	.58990	.73055	1.3688	.80748	51
10	.59014	.73100	1.3680	.80730	50
11	.59037	.73144	1.3672	.80713	49
12	.59061	.73189	1.3663	.80696	48
13	.59084	.73234	1.3655	.80679	47
14	.59108	.73278	1.3647	.80662	46
15	.59131	.73323	1.3638	.80644	45
16	.59154	.73368	1.3630	.80627	44
17	.59178	.73413	1.3622	.80610	43
18	.59201	.73457	1.3613	.80593	42
19	.59225	.73502	1.3605	.80576	41
20	.59248	.73547	1.3597	.80558	40
21	.59272	.73592	1.3588	.80541	39
22	.59295	.73637	1.3580	.80524	38
23	.59318	.73681	1.3572	.80507	37
24	.59342	.73726	1.3564	.80489	36
25	.59365	.73771	1.3555	.80472	35
26	.59389	.73816	1.3547	.80455	34
27	.59412	.73861	1.3539	.80438	33
28	.59436	.73906	1.3531	.80420	32
29	.59459	.73951	1.3522	.80403	31
30	.59482	.73996	1.3514	.80386	30
31	.59506	.74041	1.3506	.80368	29
32	.59529	.74086	1.3498	.80351	28
33	.59552	.74131	1.3490	.80334	27
34	.59576	.74176	1.3481	.80316	26
35	.59599	.74221	1.3473	.80299	25
36	.59622	.74267	1.3465	.80282	24
37	.59646	.74312	1.3457	.80264	23
38	.59669	.74357	1.3449	.80247	22
39	.59693	.74402	1.3440	.80230	21
40	.59716	.74447	1.3432	.80212	20
41	.59739	.74492	1.3424	.80195	19
42	.59763	.74538	1.3416	.80178	18
43	.59786	.74583	1.3408	.80160	17
44	.59809	.74628	1.3400	.80143	16
45	.59832	.74674	1.3392	.80125	15
46	.59856	.74719	1.3384	.80108	14
47	.59879	.74764	1.3376	.80091	13
48	.59902	.74810	1.3367	.80073	12
49	.59926	.74855	1.3359	.80056	11
50	.59949	.74900	1.3351	.80038	10
51	.59972	.74946	1.3343	.80021	9
52	.59995	.74991	1.3335	.80003	8
53	.60019	.75037	1.3327	.79986	7
54	.60042	.75082	1.3319	.79968	6
55	.60065	.75128	1.3311	.79951	5
56	.60089	.75173	1.3303	.79934	4
57	.60112	.75219	1.3295	.79916	3
58	.60135	.75264	1.3287	.79899	2
59	.60158	.75310	1.3278	.79881	1
60	.60182	.75355	1.3270	.79864	0
′	Cos	Ctn	Tan	Sin	′

53°

37°

′	Sin	Tan	Ctn	Cos	′
0	.60182	.75355	1.3270	.79864	60
1	.60205	.75401	1.3262	.79846	59
2	.60228	.75447	1.3254	.79829	58
3	.60251	.75492	1.3246	.79811	57
4	.60274	.75538	1.3238	.79793	56
5	.60298	.75584	1.3230	.79776	55
6	.60321	.75629	1.3222	.79758	54
7	.60344	.75675	1.3214	.79741	53
8	.60367	.75721	1.3206	.79723	52
9	.60390	.75767	1.3198	.79706	51
10	.60414	.75812	1.3190	.79688	50
11	.60437	.75858	1.3182	.79671	49
12	.60460	.75904	1.3175	.79653	48
13	.60483	.75950	1.3167	.79635	47
14	.60506	.75996	1.3159	.79618	46
15	.60529	.76042	1.3151	.79600	45
16	.60553	.76088	1.3143	.79583	44
17	.60576	.76134	1.3135	.79565	43
18	.60599	.76180	1.3127	.79547	42
19	.60622	.76226	1.3119	.79530	41
20	.60645	.76272	1.3111	.79512	40
21	.60668	.76318	1.3103	.79494	39
22	.60691	.76364	1.3095	.79477	38
23	.60714	.76410	1.3087	.79459	37
24	.60738	.76456	1.3079	.79441	36
25	.60761	.76502	1.3072	.79424	35
26	.60784	.76548	1.3064	.79406	34
27	.60807	.76594	1.3056	.79388	33
28	.60830	.76640	1.3048	.79371	32
29	.60853	.76686	1.3040	.79353	31
30	.60876	.76733	1.3032	.79335	30
31	.60899	.76779	1.3024	.79318	29
32	.60922	.76825	1.3017	.79300	28
33	.60945	.76871	1.3009	.79282	27
34	.60968	.76918	1.3001	.79264	26
35	.60991	.76964	1.2993	.79247	25
36	.61015	.77010	1.2985	.79229	24
37	.61038	.77057	1.2977	.79211	23
38	.61061	.77103	1.2970	.79193	22
39	.61084	.77149	1.2962	.79176	21
40	.61107	.77196	1.2954	.79158	20
41	.61130	.77242	1.2946	.79140	19
42	.61153	.77289	1.2938	.79122	18
43	.61176	.77335	1.2931	.79105	17
44	.61199	.77382	1.2923	.79087	16
45	.61222	.77428	1.2915	.79069	15
46	.61245	.77475	1.2907	.79051	14
47	.61268	.77521	1.2900	.79033	13
48	.61291	.77568	1.2892	.79016	12
49	.61314	.77615	1.2884	.78998	11
50	.61337	.77661	1.2876	.78980	10
51	.61360	.77708	1.2869	.78962	9
52	.61383	.77754	1.2861	.78944	8
53	.61406	.77801	1.2853	.78926	7
54	.61429	.77848	1.2846	.78908	6
55	.61451	.77895	1.2838	.78891	5
56	.61474	.77941	1.2830	.78873	4
57	.61497	.77988	1.2822	.78855	3
58	.61520	.78035	1.2815	.78837	2
59	.61543	.78082	1.2807	.78819	1
60	.61566	.78129	1.2799	.78801	0
′	Cos	Ctn	Tan	Sin	′

52°

Natural Trigonometric Functions

38°

′	Sin	Tan	Ctn	Cos	′
0	.61566	.78129	1.2799	.78801	60
1	.61589	.78175	1.2792	.78783	59
2	.61612	.78222	1.2784	.78765	58
3	.61635	.78269	1.2776	.78747	57
4	.61658	.78316	1.2769	.78729	56
5	.61681	.78363	1.2761	.78711	55
6	.61704	.78410	1.2753	.78694	54
7	.61726	.78457	1.2746	.78676	53
8	.61749	.78504	1.2738	.78658	52
9	.61772	.78551	1.2731	.78640	51
10	.61795	.78598	1.2723	.78622	50
11	.61818	.78645	1.2715	.78604	49
12	.61841	.78692	1.2708	.78586	48
13	.61864	.78739	1.2700	.78568	47
14	.61887	.78786	1.2693	.78550	46
15	.61909	.78834	1.2685	.78532	45
16	.61932	.78881	1.2677	.78514	44
17	.61955	.78928	1.2670	.78496	43
18	.61978	.78975	1.2662	.78478	42
19	.62001	.79022	1.2655	.78460	41
20	.62024	.79070	1.2647	.78442	40
21	.62046	.79117	1.2640	.78424	39
22	.62069	.79164	1.2632	.78405	38
23	.62092	.79212	1.2624	.78387	37
24	.62115	.79259	1.2617	.78369	36
25	.62138	.79306	1.2609	.78351	35
26	.62160	.79354	1.2602	.78333	34
27	.62183	.79401	1.2594	.78315	33
28	.62206	.79449	1.2587	.78297	32
29	.62229	.79496	1.2579	.78279	31
30	.62251	.79544	1.2572	.78261	30
31	.62274	.79591	1.2564	.78243	29
32	.62297	.79639	1.2557	.78225	28
33	.62320	.79686	1.2549	.78206	27
34	.62342	.79734	1.2542	.78188	26
35	.62365	.79781	1.2534	.78170	25
36	.62388	.79829	1.2527	.78152	24
37	.62411	.79877	1.2519	.78134	23
38	.62433	.79924	1.2512	.78116	22
39	.62456	.79972	1.2504	.78098	21
40	.62479	.80020	1.2497	.78079	20
41	.62502	.80067	1.2489	.78061	19
42	.62524	.80115	1.2482	.78043	18
43	.62547	.80163	1.2475	.78025	17
44	.62570	.80211	1.2467	.78007	16
45	.62592	.80258	1.2460	.77988	15
46	.62615	.80306	1.2452	.77970	14
47	.62638	.80354	1.2445	.77952	13
48	.62660	.80402	1.2437	.77934	12
49	.62683	.80450	1.2430	.77916	11
50	.62706	.80498	1.2423	.77897	10
51	.62728	.80546	1.2415	.77879	9
52	.62751	.80594	1.2408	.77861	8
53	.62774	.80642	1.2401	.77843	7
54	.62796	.80690	1.2393	.77824	6
55	.62819	.80738	1.2386	.77806	5
56	.62842	.80786	1.2378	.77788	4
57	.62864	.80834	1.2371	.77769	3
58	.62887	.80882	1.2364	.77751	2
59	.62909	.80930	1.2356	.77733	1
60	.62932	.80978	1.2349	.77715	0
′	Cos	Ctn	Tan	Sin	′

51°

39°

′	Sin	Tan	Ctn	Cos	′
0	.62932	.80978	1.2349	.77715	60
1	.62955	.81027	1.2342	.77696	59
2	.62977	.81075	1.2334	.77678	58
3	.63000	.81123	1.2327	.77660	57
4	.63022	.81171	1.2320	.77641	56
5	.63045	.81220	1.2312	.77623	55
6	.63068	.81268	1.2305	.77605	54
7	.63090	.81316	1.2298	.77586	53
8	.63113	.81364	1.2290	.77568	52
9	.63135	.81413	1.2283	.77550	51
10	.63158	.81461	1.2276	.77531	50
11	.63180	.81510	1.2268	.77513	49
12	.63203	.81558	1.2261	.77494	48
13	.63225	.81606	1.2254	.77476	47
14	.63248	.81655	1.2247	.77458	46
15	.63271	.81703	1.2239	.77439	45
16	.63293	.81752	1.2232	.77421	44
17	.63316	.81800	1.2225	.77402	43
18	.63338	.81849	1.2218	.77384	42
19	.63361	.81898	1.2210	.77366	41
20	.63383	.81946	1.2203	.77347	40
21	.63406	.81995	1.2196	.77329	39
22	.63428	.82044	1.2189	.77310	38
23	.63451	.82092	1.2181	.77292	37
24	.63473	.82141	1.2174	.77273	36
25	.63496	.82190	1.2167	.77255	35
26	.63518	.82238	1.2160	.77236	34
27	.63540	.82287	1.2153	.77218	33
28	.63563	.82336	1.2145	.77199	32
29	.63585	.82385	1.2138	.77181	31
30	.63608	.82434	1.2131	.77162	30
31	.63630	.82483	1.2124	.77144	29
32	.63653	.82531	1.2117	.77125	28
33	.63675	.82580	1.2109	.77107	27
34	.63698	.82629	1.2102	.77088	26
35	.63720	.82678	1.2095	.77070	25
36	.63742	.82727	1.2088	.77051	24
37	.63765	.82776	1.2081	.77033	23
38	.63787	.82825	1.2074	.77014	22
39	.63810	.82874	1.2066	.76996	21
40	.63832	.82923	1.2059	.76977	20
41	.63854	.82972	1.2052	.76959	19
42	.63877	.83022	1.2045	.76940	18
43	.63899	.83071	1.2038	.76921	17
44	.63922	.83120	1.2031	.76903	16
45	.63944	.83169	1.2024	.76884	15
46	.63966	.83218	1.2017	.76866	14
47	.63989	.83268	1.2009	.76847	13
48	.64011	.83317	1.2002	.76828	12
49	.64033	.83366	1.1995	.76810	11
50	.64056	.83415	1.1988	.76791	10
51	.64078	.83465	1.1981	.76772	9
52	.64100	.83514	1.1974	.76754	8
53	.64123	.83564	1.1967	.76735	7
54	.64145	.83613	1.1960	.76717	6
55	.64167	.83662	1.1953	.76698	5
56	.64190	.83712	1.1946	.76679	4
57	.64212	.83761	1.1939	.76661	3
58	.64234	.83811	1.1932	.76642	2
59	.64256	.83860	1.1925	.76623	1
60	.64279	.83910	1.1918	.76604	0
′	Cos	Ctn	Tan	Sin	′

50°

Natural Trigonometric Functions

40°

′	Sin	Tan	Ctn	Cos	′
0	.64279	.83910	1.1918	.76604	**60**
1	.64301	.83960	1.1910	.76586	59
2	.64323	.84009	1.1903	.76567	58
3	.64346	.84059	1.1896	.76548	57
4	.64368	.84108	1.1889	.76530	56
5	.64390	.84158	1.1882	.76511	**55**
6	.64412	.84208	1.1875	.76492	54
7	.64435	.84258	1.1868	.76473	53
8	.64457	.84307	1.1861	.76455	52
9	.64479	.84357	1.1854	.76436	51
10	.64501	.84407	1.1847	.76417	**50**
11	.64524	.84457	1.1840	.76398	49
12	.64546	.84507	1.1833	.76380	48
13	.64568	.84556	1.1826	.76361	47
14	.64590	.84606	1.1819	.76342	46
15	.64612	.84656	1.1812	.76323	**45**
16	.64635	.84706	1.1806	.76304	44
17	.64657	.84756	1.1799	.76286	43
18	.64679	.84806	1.1792	.76267	42
19	.64701	.84856	1.1785	.76248	41
20	.64723	.84906	1.1778	.76229	**40**
21	.64746	.84956	1.1771	.76210	39
22	.64768	.85006	1.1764	.76192	38
23	.64790	.85057	1.1757	.76173	37
24	.64812	.85107	1.1750	.76154	36
25	.64834	.85157	1.1743	.76135	**35**
26	.64856	.85207	1.1736	.76116	34
27	.64878	.85257	1.1729	.76097	33
28	.64901	.85308	1.1722	.76078	32
29	.64923	.85358	1.1715	.76059	31
30	.64945	.85408	1.1708	.76041	**30**
31	.64967	.85458	1.1702	.76022	29
32	.64989	.85509	1.1695	.76003	28
33	.65011	.85559	1.1688	.75984	27
34	.65033	.85609	1.1681	.75965	26
35	.65055	.85660	1.1674	.75946	**25**
36	.65077	.85710	1.1667	.75927	24
37	.65100	.85761	1.1660	.75908	23
38	.65122	.85811	1.1653	.75889	22
39	.65144	.85862	1.1647	.75870	21
40	.65166	.85912	1.1640	.75851	**20**
41	.65188	.85963	1.1633	.75832	19
42	.65210	.86014	1.1626	.75813	18
43	.65232	.86064	1.1619	.75794	17
44	.65254	.86115	1.1612	.75775	16
45	.65276	.86166	1.1606	.75756	**15**
46	.65298	.86216	1.1599	.75738	14
47	.65320	.86267	1.1592	.75719	13
48	.65342	.86318	1.1585	.75700	12
49	.65364	.86368	1.1578	.75680	11
50	.65386	.86419	1.1571	.75661	**10**
51	.65408	.86470	1.1565	.75642	9
52	.65430	.86521	1.1558	.75623	8
53	.65452	.86572	1.1551	.75604	7
54	.65474	.86623	1.1544	.75585	6
55	.65496	.86674	1.1538	.75566	**5**
56	.65518	.86725	1.1531	.75547	4
57	.65540	.86776	1.1524	.75528	3
58	.65562	.86827	1.1517	.75509	2
59	.65584	.86878	1.1510	.75490	1
60	.65606	.86929	1.1504	.75471	**0**
′	Cos	Ctn	Tan	Sin	′

49°

41°

′	Sin	Tan	Ctn	Cos	′
0	.65606	.86929	1.1504	.75471	**60**
1	.65628	.86980	1.1497	.75452	59
2	.65650	.87031	1.1490	.75433	58
3	.65672	.87082	1.1483	.75414	57
4	.65694	.87133	1.1477	.75395	56
5	.65716	.87184	1.1470	.75375	**55**
6	.65738	.87236	1.1463	.75356	54
7	.65759	.87287	1.1456	.75337	53
8	.65781	.87338	1.1450	.75318	52
9	.65803	.87389	1.1443	.75299	51
10	.65825	.87441	1.1436	.75280	**50**
11	.65847	.87492	1.1430	.75261	49
12	.65869	.87543	1.1423	.75241	48
13	.65891	.87595	1.1416	.75222	47
14	.65913	.87646	1.1410	.75203	46
15	.65935	.87698	1.1403	.75184	**45**
16	.65956	.87749	1.1396	.75165	44
17	.65978	.87801	1.1389	.75146	43
18	.66000	.87852	1.1383	.75126	42
19	.66022	.87904	1.1376	.75107	41
20	.66044	.87955	1.1369	.75088	**40**
21	.66066	.88007	1.1363	.75069	39
22	.66088	.88059	1.1356	.75050	38
23	.66109	.88110	1.1349	.75030	37
24	.66131	.88162	1.1343	.75011	36
25	.66153	.88214	1.1336	.74992	**35**
26	.66175	.88265	1.1329	.74973	34
27	.66197	.88317	1.1323	.74953	33
28	.66218	.88369	1.1316	.74934	32
29	.66240	.88421	1.1310	.74915	31
30	.66262	.88473	1.1303	.74896	**30**
31	.66284	.88524	1.1296	.74876	29
32	.66306	.88576	1.1290	.74857	28
33	.66327	.88628	1.1283	.74838	27
34	.66349	.88680	1.1276	.74818	26
35	.66371	.88732	1.1270	.74799	**25**
36	.66393	.88784	1.1263	.74780	24
37	.66414	.88836	1.1257	.74760	23
38	.66436	.88888	1.1250	.74741	22
39	.66458	.88940	1.1243	.74722	21
40	.66480	.88992	1.1237	.74703	**20**
41	.66501	.89045	1.1230	.74683	19
42	.66523	.89097	1.1224	.74664	18
43	.66545	.89149	1.1217	.74644	17
44	.66566	.89201	1.1211	.74625	16
45	.66588	.89253	1.1204	.74606	**15**
46	.66610	.89306	1.1197	.74586	14
47	.66632	.89358	1.1191	.74567	13
48	.66653	.89410	1.1184	.74548	12
49	.66675	.89463	1.1178	.74528	11
50	.66697	.89515	1.1171	.74509	**10**
51	.66718	.89567	1.1165	.74489	9
52	.66740	.89620	1.1158	.74470	8
53	.66762	.89672	1.1152	.74451	7
54	.66783	.89725	1.1145	.74431	6
55	.66805	.89777	1.1139	.74412	**5**
56	.66827	.89830	1.1132	.74392	4
57	.66848	.89883	1.1126	.74373	3
58	.66870	.89935	1.1119	.74353	2
59	.66891	.89988	1.1113	.74334	1
60	.66913	.90040	1.1106	.74314	**0**
′	Cos	Ctn	Tan	Sin	′

48°

Natural Trigonometric Functions

42°

′	Sin	Tan	Ctn	Cos	′
0	.66913	.90040	1.1106	.74314	60
1	.66935	.90093	1.1100	.74295	59
2	.66956	.90146	1.1093	.74276	58
3	.66978	.90199	1.1087	.74256	57
4	.66999	.90251	1.1080	.74237	56
5	.67021	.90304	1.1074	.74217	55
6	.67043	.90357	1.1067	.74198	54
7	.67064	.90410	1.1061	.74178	53
8	.67086	.90463	1.1054	.74159	52
9	.67107	.90516	1.1048	.74139	51
10	.67129	.90569	1.1041	.74120	50
11	.67151	.90621	1.1035	.74100	49
12	.67172	.90674	1.1028	.74080	48
13	.67194	.90727	1.1022	.74061	47
14	.67215	.90781	1.1016	.74041	46
15	.67237	.90834	1.1009	.74022	45
16	.67258	.90887	1.1003	.74002	44
17	.67280	.90940	1.0996	.73983	43
18	.67301	.90993	1.0990	.73963	42
19	.67323	.91046	1.0983	.73944	41
20	.67344	.91099	1.0977	.73924	40
21	.67366	.91153	1.0971	.73904	39
22	.67387	.91206	1.0964	.73885	38
23	.67409	.91259	1.0958	.73865	37
24	.67430	.91313	1.0951	.73846	36
25	.67452	.91366	1.0945	.73826	35
26	.67473	.91419	1.0939	.73806	34
27	.67495	.91473	1.0932	.73787	33
28	.67516	.91526	1.0926	.73767	32
29	.67538	.91580	1.0919	.73747	31
30	.67559	.91633	1.0913	.73728	30
31	.67580	.91687	1.0907	.73708	29
32	.67602	.91740	1.0900	.73688	28
33	.67623	.91794	1.0894	.73669	27
34	.67645	.91847	1.0888	.73649	26
35	.67666	.91901	1.0881	.73629	25
36	.67688	.91955	1.0875	.73610	24
37	.67709	.92008	1.0869	.73590	23
38	.67730	.92062	1.0862	.73570	22
39	.67752	.92116	1.0856	.73551	21
40	.67773	.92170	1.0850	.73531	20
41	.67795	.92224	1.0843	.73511	19
42	.67816	.92277	1.0837	.73491	18
43	.67837	.92331	1.0831	.73472	17
44	.67859	.92385	1.0824	.73452	16
45	.67880	.92439	1.0818	.73432	15
46	.67901	.92493	1.0812	.73413	14
47	.67923	.92547	1.0805	.73393	13
48	.67944	.92601	1.0799	.73373	12
49	.67965	.92655	1.0793	.73353	11
50	.67987	.92709	1.0786	.73333	10
51	.68008	.92763	1.0780	.73314	9
52	.68029	.92817	1.0774	.73294	8
53	.68051	.92872	1.0768	.73274	7
54	.68072	.92926	1.0761	.73254	6
55	68093	.92980	1.0755	.73234	5
56	.68115	.93034	1.0749	.73215	4
57	.68136	.93088	1.0742	.73195	3
58	.68157	.93143	1.0736	.73175	2
59	.68179	.93197	1.0730	.73155	1
60	.68200	.93252	1.0724	.73135	0
′	Cos	Ctn	Tan	Sin	′

47°

43°

′	Sin	Tan	Ctn	Cos	′
0	.68200	.93252	1.0724	.73135	60
1	.68221	.93306	1.0717	.73116	59
2	.68242	.93360	1.0711	.73096	58
3	.68264	.93415	1.0705	.73076	57
4	.68285	.93469	1.0699	.73056	56
5	.68306	.93524	1.0692	.73036	55
6	.68327	.93578	1.0686	.73016	54
7	.68349	.93633	1.0680	.72996	53
8	.68370	.93688	1.0674	.72976	52
9	.68391	.93742	1.0668	.72957	51
10	.68412	.93797	1.0661	.72937	50
11	.68434	.93852	1.0655	.72917	49
12	.68455	.93906	1.0649	.72897	48
13	.68476	.93961	1.0643	.72877	47
14	.68497	.94016	1.0637	.72857	46
15	.68518	.94071	1.0630	.72837	45
16	.68539	.94125	1.0624	.72817	44
17	.68561	.94180	1.0618	.72797	43
18	.68582	.94235	1.0612	.72777	42
19	.68603	.94290	1.0606	.72757	41
20	.68624	.94345	1.0599	.72737	40
21	.68645	.94400	1.0593	.72717	39
22	.68666	.94455	1.0587	.72697	38
23	.68688	.94510	1.0581	.72677	37
24	.68709	.94565	1.0575	.72657	36
25	.68730	.94620	1.0569	.72637	35
26	.68751	.94676	1.0562	.72617	34
27	.68772	.94731	1.0556	.72597	33
28	.68793	.94786	1.0550	.72577	32
29	.68814	.94841	1.0544	.72557	31
30	.68835	.94896	1.0538	.72537	30
31	.68857	.94952	1.0532	.72517	29
32	.68878	.95007	1.0526	.72497	28
33	.68899	.95062	1.0519	.72477	27
34	.68920	.95118	1.0513	.72457	26
35	.68941	.95173	1.0507	.72437	25
36	.68962	.95229	1.0501	.72417	24
37	.68983	.95284	1.0495	.72397	23
38	.69004	.95340	1.0489	.72377	22
39	.69025	.95395	1.0483	.72357	21
40	.69046	.95451	1.0477	.72337	20
41	.69067	.95506	1.0470	.72317	19
42	.69088	.95562	1.0464	.72297	18
43	.69109	.95618	1.0458	.72277	17
44	.69130	.95673	1.0452	.72257	16
45	.69151	.95729	1.0446	.72236	15
46	.69172	.95785	1.0440	.72216	14
47	.69193	.95841	1.0434	.72196	13
48	.69214	.95897	1.0428	.72176	12
49	.69235	.95952	1.0422	.72156	11
50	.69256	.96008	1.0416	.72136	10
51	.69277	.96064	1.0410	.72116	9
52	.69298	.96120	1.0404	.72095	8
53	.69319	.96176	1.0398	.72075	7
54	.69340	.96232	1.0392	.72055	6
55	.69361	.96288	1.0385	.72035	5
56	.69382	.96344	1.0379	.72015	4
57	.69403	.96400	1.0373	.71995	3
58	.69424	.96457	1.0367	.71974	2
59	.69445	.96513	1.0361	.71954	1
60	.69466	.96569	1.0355	.71934	0
′	Cos	Ctn	Tan	Sin	′

46°

Natural Trigonometric Functions

44°

′	Sin	Tan	Ctn	Cos	′
0	.69466	.96569	1.0355	.71934	60
1	.69487	.96625	1.0349	.71914	59
2	.69508	.96681	1.0343	.71894	58
3	.69529	.96738	1.0337	.71873	57
4	.69549	.96794	1.0331	.71853	56
5	.69570	.96850	1.0325	.71833	55
6	.69591	.96907	1.0319	.71813	54
7	.69612	.96963	1.0313	.71792	53
8	.69633	.97020	1.0307	.71772	52
9	.69654	.97076	1.0301	.71752	51
10	.69675	.97133	1.0295	.71732	50
11	.69696	.97189	1.0289	.71711	49
12	.69717	.97246	1.0283	.71691	48
13	.69737	.97302	1.0277	.71671	47
14	.69758	.97359	1.0271	.71650	46
15	.69779	.97416	1.0265	.71630	45
16	.69800	.97472	1.0259	.71610	44
17	.69821	.97529	1.0253	.71590	43
18	.69842	.97586	1.0247	.71569	42
19	.69862	.97643	1.0241	.71549	41
20	.69883	.97700	1.0235	.71529	40
21	.69904	.97756	1.0230	.71508	39
22	.69925	.97813	1.0224	.71488	38
23	.69946	.97870	1.0218	.71468	37
24	.69966	.97927	1.0212	.71447	36
25	.69987	.97984	1.0206	.71427	35
26	.70008	.98041	1.0200	.71407	34
27	.70029	.98098	1.0194	.71386	33
28	.70049	.98155	1.0188	.71366	32
29	.70070	.98213	1.0182	.71345	31
30	.70091	.98270	1.0176	.71325	30
31	.70112	.98327	1.0170	.71305	29
32	.70132	.98384	1.0164	.71284	28
33	.70153	.98441	1.0158	.71264	27
34	.70174	.98499	1.0152	.71243	26
35	.70195	.98556	1.0147	.71223	25
36	.70215	.98613	1.0141	.71203	24
37	.70236	.98671	1.0135	.71182	23
38	.70257	.98728	1.0129	.71162	22
39	.70277	.98786	1.0123	.71141	21
40	.70298	.98843	1.0117	.71121	20
41	.70319	.98901	1.0111	.71100	19
42	.70339	.98958	1.0105	.71080	18
43	.70360	.99016	1.0099	.71059	17
44	.70381	.99073	1.0094	.71039	16
45	.70401	.99131	1.0088	.71019	15
46	.70422	.99189	1.0082	.70998	14
47	.70443	.99247	1.0076	.70978	13
48	.70463	.99304	1.0070	.70957	12
49	.70484	.99362	1.0064	.70937	11
50	.70505	.99420	1.0058	.70916	10
51	.70525	.99478	1.0052	.70896	9
52	.70546	.99536	1.0047	.70875	8
53	.70567	.99594	1.0041	.70855	7
54	.70587	.99652	1.0035	.70834	6
55	.70608	.99710	1.0029	.70813	5
56	.70628	.99768	1.0023	.70793	4
57	.70649	.99826	1.0017	.70772	3
58	.70670	.99884	1.0012	.70752	2
59	.70690	.99942	1.0006	.70731	1
60	.70711	1.0000	1.0000	.70711	0
′	Cos	Ctn	Tan	Sin	′

45°

TABLE 3

0° — Log Sine — 0°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	—	5.46373	5.76476	5.94085	6.06579	6.16270	6.24188	6.30882	6.36682	6.41797	**60**
1	6.46373	6.50512	6.54291	6.57767	6.60985	6.63982	6.66785	6.69418	6.71900	6.74248	59
2	6.76476	6.78595	6.80615	6.82545	6.84394	6.86167	6.87870	6.89509	6.91088	6.92612	58
3	6.94085	6.95509	6.96888	6.98224	6.99520	7.00779	7.02003	7.03193	7.04351	7.05479	57
4	7.06579	7.07651	7.08698	7.09719	7.10718	7.11694	7.12648	7.13582	7.14497	7.15392	56
5	7.16270	7.17130	7.17973	7.18800	7.19612	7.20409	7.21191	7.21960	7.22715	7.23458	**55**
6	7.24188	7.24906	7.25612	7.26307	7.26991	7.27664	7.28327	7.28980	7.29623	7.30257	54
7	7.30882	7.31498	7.32106	7.32705	7.33296	7.33879	7.34454	7.35022	7.35582	7.36135	53
8	7.36682	7.37221	7.37754	7.38280	7.38800	7.39314	7.39822	7.40324	7.40821	7.41312	52
9	7.41797	7.42277	7.42751	7.43221	7.43685	7.44145	7.44600	7.45050	7.45495	7.45936	51
10	7.46373	7.46805	7.47233	7.47656	7.48076	7.48491	7.48903	7.49311	7.49715	7.50115	**50**
11	7.50512	7.50905	7.51294	7.51680	7.52063	7.52442	7.52818	7.53191	7.53561	7.53927	49
12	7.54291	7.54651	7.55009	7.55363	7.55715	7.56064	7.56410	7.56753	7.57094	7.57431	48
13	7.57767	7.58100	7.58430	7.58758	7.59083	7.59406	7.59726	7.60045	7.60360	7.60674	47
14	7.60985	7.61294	7.61601	7.61906	7.62209	7.62509	7.62808	7.63104	7.63399	7.63691	46
15	7.63982	7.64270	7.64557	7.64842	7.65125	7.65406	7.65685	7.65962	7.66238	7.66512	**45**
16	7.66784	7.67055	7.67324	7.67591	7.67857	7.68121	7.68383	7.68644	7.68903	7.69161	44
17	7.69417	7.69672	7.69925	7.70177	7.70427	7.70676	7.70924	7.71170	7.71414	7.71658	43
18	7.71900	7.72140	7.72380	7.72618	7.72854	7.73090	7.73324	7.73557	7.73788	7.74019	42
19	7.74248	7.74476	7.74703	7.74928	7.75153	7.75376	7.75598	7.75819	7.76039	7.76258	41
20	7.76475	7.76692	7.76907	7.77122	7.77335	7.77548	7.77759	7.77969	7.78179	7.78387	**40**
21	7.78594	7.78801	7.79006	7.79210	7.79414	7.79616	7.79818	7.80018	7.80218	7.80417	39
22	7.80615	7.80812	7.81008	7.81203	7.81397	7.81591	7.81783	7.81975	7.82166	7.82356	38
23	7.82545	7.82733	7.82921	7.83108	7.83294	7.83479	7.83663	7.83847	7.84030	7.84212	37
24	7.84393	7.84574	7.84754	7.84933	7.85111	7.85289	7.85466	7.85642	7.85817	7.85992	36
25	7.86166	7.86340	7.86512	7.86684	7.86856	7.87026	7.87196	7.87366	7.87534	7.87702	**35**
26	7.87870	7.88036	7.88202	7.88368	7.88533	7.88697	7.88860	7.89023	7.89186	7.89347	34
27	7.89509	7.89669	7.89829	7.89988	7.90147	7.90305	7.90463	7.90620	7.90777	7.90933	33
28	7.91088	7.91243	7.91397	7.91551	7.91704	7.91857	7.92009	7.92160	7.92311	7.92462	32
29	7.92612	7.92761	7.92910	7.93059	7.93207	7.93354	7.93501	7.93648	7.93794	7.93939	31
30	7.94084	7.94229	7.94373	7.94516	7.94659	7.94802	7.94944	7.95086	7.95227	7.95368	**30**
31	7.95508	7.95648	7.95787	7.95926	7.96065	7.96203	7.96341	7.96478	7.96615	7.96751	29
32	7.96887	7.97022	7.97158	7.97292	7.97426	7.97560	7.97694	7.97827	7.97959	7.98092	28
33	7.98223	7.98355	7.98486	7.98616	7.98747	7.98876	7.99006	7.99135	7.99264	7.99392	27
34	7.99520	7.99647	7.99775	7.99901	8.00028	8.00154	8.00279	8.00405	8.00530	8.00654	26
35	8.00779	8.00903	8.01026	8.01149	8.01272	8.01395	8.01517	8.01639	8.01760	8.01881	**25**
36	8.02002	8.02123	8.02243	8.02362	8.02482	8.02601	8.02720	8.02838	8.02957	8.03074	24
37	8.03192	8.03309	8.03426	8.03543	8.03659	8.03775	8.03891	8.04006	8.04121	8.04236	23
38	8.04350	8.04464	8.04578	8.04692	8.04805	8.04918	8.05030	8.05143	8.05255	8.05367	22
39	8.05478	8.05589	8.05700	8.05811	8.05921	8.06031	8.06141	8.06251	8.06360	8.06469	21
40	8.06578	8.06686	8.06794	8.06902	8.07010	8.07117	8.07224	8.07331	8.07438	8.07544	**20**
41	8.07650	8.07756	8.07861	8.07967	8.08072	8.08176	8.08281	8.08385	8.08489	8.08593	19
42	8.08696	8.08800	8.08903	8.09006	8.09108	8.09210	8.09312	8.09414	8.09516	8.09617	18
43	8.09718	8.09819	8.09920	8.10020	8.10120	8.10220	8.10320	8.10420	8.10519	8.10618	17
44	8.10717	8.10815	8.10914	8.11012	8.11110	8.11207	8.11305	8.11402	8.11499	8.11596	16
45	8.11693	8.11789	8.11885	8.11981	8.12077	8.12172	8.12268	8.12363	8.12458	8.12553	**15**
46	8.12647	8.12741	8.12836	8.12929	8.13023	8.13117	8.13210	8.13303	8.13396	8.13489	14
47	8.13581	8.13673	8.13765	8.13857	8.13949	8.14041	8.14132	8.14223	8.14314	8.14405	13
48	8.14495	8.14586	8.14676	8.14766	8.14856	8.14945	8.15035	8.15124	8.15213	8.15302	12
49	8.15391	8.15479	8.15568	8.15656	8.15744	8.15832	8.15919	8.16007	8.16094	8.16181	11
50	8.16268	8.16355	8.16441	8.16528	8.16614	8.16700	8.16786	8.16872	8.16957	8.17043	**10**
51	8.17128	8.17213	8.17298	8.17383	8.17467	8.17552	8.17636	8.17720	8.17804	8.17888	9
52	8.17971	8.18055	8.18138	8.18221	8.18304	8.18387	8.18469	8.18552	8.18634	8.18716	8
53	8.18798	8.18880	8.18962	8.19044	8.19125	8.19206	8.19287	8.19368	8.19449	8.19530	7
54	8.19610	8.19691	8.19771	8.19851	8.19931	8.20010	8.20090	8.20170	8.20249	8.20328	6
55	8.20407	8.20486	8.20565	8.20643	8.20722	8.20800	8.20878	8.20956	8.21034	8.21112	**5**
56	8.21189	8.21267	8.21344	8.21422	8.21499	8.21576	8.21652	8.21729	8.21805	8.21882	4
57	8.21958	8.22034	8.22110	8.22186	8.22262	8.22337	8.22413	8.22488	8.22563	8.22638	3
58	8.22713	8.22788	8.22863	8.22937	8.23012	8.23086	8.23160	8.23234	8.23308	8.23382	2
59	8.23456	8.23529	8.23603	8.23676	8.23749	8.23822	8.23895	8.23968	8.24041	8.24113	1
60	8.24186	8.24258	8.24330	8.24402	8.24474	8.24546	8.24618	8.24689	8.24761	8.24832	**0**
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

89° — Log Cosine — 89°

0° — Log Tan — 0°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	—	5.46373	5.76476	5.94085	6.06579	6.16270	6.24188	6.30882	6.36682	6.41797	60
1	6.46373	6.50512	6.54291	6.57767	6.60985	6.63982	6.66785	6.69418	6.71900	6.74248	59
2	6.76476	6.78595	6.80615	6.82545	6.84394	6.86167	6.87870	6.89509	6.91088	6.92612	58
3	6.94085	6.95509	6.96888	6.98224	6.99521	7.00779	7.02003	7.03193	7.04351	7.05479	57
4	7.06579	7.07651	7.08698	7.09719	7.10718	7.11694	7.12648	7.13582	7.14497	7.15392	56
5	7.16270	7.17130	7.17973	7.18800	7.19612	7.20409	7.21191	7.21960	7.22715	7.23458	55
6	7.24188	7.24906	7.25612	7.26307	7.26991	7.27664	7.28327	7.28980	7.29624	7.30258	54
7	7.30882	7.31499	7.32106	7.32705	7.33296	7.33879	7.34454	7.35022	7.35582	7.36135	53
8	7.36682	7.37221	7.37754	7.38281	7.38801	7.39315	7.39823	7.40325	7.40821	7.41312	52
9	7.41797	7.42277	7.42751	7.43221	7.43686	7.44145	7.44600	7.45050	7.45495	7.45936	51
10	7.46373	7.46805	7.47233	7.47656	7.48076	7.48492	7.48903	7.49311	7.49715	7.50115	50
11	7.50512	7.50905	7.51295	7.51681	7.52063	7.52443	7.52819	7.53191	7.53561	7.53927	49
12	7.54291	7.54651	7.55009	7.55363	7.55715	7.56064	7.56410	7.56753	7.57094	7.57432	48
13	7.57767	7.58100	7.58430	7.58758	7.59083	7.59406	7.59727	7.60045	7.60361	7.60674	47
14	7.60986	7.61295	7.61602	7.61906	7.62209	7.62510	7.62808	7.63105	7.63399	7.63692	46
15	7.63982	7.64271	7.64557	7.64842	7.65125	7.65406	7.65685	7.65963	7.66239	7.66513	45
16	7.66785	7.67056	7.67324	7.67592	7.67857	7.68121	7.68384	7.68645	7.68904	7.69162	44
17	7.69418	7.69673	7.69926	7.70178	7.70428	7.70677	7.70924	7.71170	7.71415	7.71658	43
18	7.71900	7.72141	7.72380	7.72618	7.72855	7.73090	7.73324	7.73557	7.73789	7.74019	42
19	7.74248	7.74476	7.74703	7.74929	7.75153	7.75377	7.75599	7.75820	7.76040	7.76258	41
20	7.76476	7.76693	7.76908	7.77123	7.77336	7.77549	7.77760	7.77970	7.78179	7.78388	40
21	7.78595	7.78801	7.79007	7.79211	7.79415	7.79617	7.79819	7.80019	7.80219	7.80418	39
22	7.80615	7.80812	7.81009	7.81204	7.81398	7.81591	7.81784	7.81976	7.82167	7.82357	38
23	7.82546	7.82734	7.82922	7.83109	7.83295	7.83480	7.83664	7.83848	7.84031	7.84213	37
24	7.84394	7.84575	7.84755	7.84934	7.85112	7.85290	7.85467	7.85643	7.85819	7.85993	36
25	7.86167	7.86341	7.86513	7.86685	7.86857	7.87027	7.87197	7.87367	7.87535	7.87703	35
26	7.87871	7.88037	7.88204	7.88369	7.88534	7.88698	7.88862	7.89025	7.89187	7.89349	34
27	7.89510	7.89670	7.89830	7.89990	7.90149	7.90307	7.90464	7.90622	7.90778	7.90934	33
28	7.91089	7.91244	7.91398	7.91552	7.91705	7.91858	7.92010	7.92162	7.92313	7.92463	32
29	7.92613	7.92763	7.92912	7.93060	7.93208	7.93356	7.93503	7.93649	7.93795	7.93941	31
30	7.94086	7.94230	7.94374	7.94518	7.94661	7.94804	7.94946	7.95088	7.95229	7.95370	30
31	7.95510	7.95650	7.95789	7.95928	7.96067	7.96205	7.96343	7.96480	7.96617	7.96753	29
32	7.96889	7.97024	7.97159	7.97294	7.97428	7.97562	7.97696	7.97829	7.97961	7.98094	28
33	7.98225	7.98357	7.98488	7.98618	7.98749	7.98878	7.99008	7.99137	7.99266	7.99394	27
34	7.99522	7.99649	7.99777	7.99903	8.00030	8.00156	8.00282	8.00407	8.00532	8.00657	26
35	8.00781	8.00905	8.01028	8.01152	8.01274	8.01397	8.01519	8.01641	8.01762	8.01884	25
36	8.02004	8.02125	8.02245	8.02365	8.02484	8.02604	8.02722	8.02841	8.02959	8.03077	24
37	8.03194	8.03312	8.03429	8.03545	8.03661	8.03777	8.03893	8.04008	8.04124	8.04238	23
38	8.04353	8.04467	8.04581	8.04694	8.04808	8.04921	8.05033	8.05146	8.05258	8.05369	22
39	8.05481	8.05592	8.05703	8.05814	8.05924	8.06034	8.06144	8.06254	8.06363	8.06472	21
40	8.06581	8.06689	8.06797	8.06905	8.07013	8.07120	8.07227	8.07334	8.07441	8.07547	20
41	8.07653	8.07759	8.07864	8.07970	8.08075	8.08180	8.08284	8.08388	8.08492	8.08596	19
42	8.08700	8.08803	8.08906	8.09009	8.09111	8.09214	8.09316	8.09418	8.09519	8.09621	18
43	8.09722	8.09823	8.09923	8.10024	8.10124	8.10224	8.10324	8.10423	8.10522	8.10621	17
44	8.10720	8.10819	8.10917	8.11015	8.11113	8.11211	8.11309	8.11406	8.11503	8.11600	16
45	8.11696	8.11793	8.11889	8.11985	8.12081	8.12176	8.12272	8.12367	8.12462	8.12556	15
46	8.12651	8.12745	8.12839	8.12933	8.13027	8.13121	8.13214	8.13307	8.13400	8.13493	14
47	8.13585	8.13677	8.13770	8.13861	8.13953	8.14045	8.14136	8.14227	8.14318	8.14409	13
48	8.14500	8.14590	8.14680	8.14770	8.14860	8.14950	8.15039	8.15128	8.15218	8.15306	12
49	8.15395	8.15484	8.15572	8.15660	8.15748	8.15836	8.15924	8.16011	8.16099	8.16186	11
50	8.16273	8.16359	8.16446	8.16533	8.16619	8.16705	8.16791	8.16877	8.16962	8.17048	10
51	8.17133	8.17218	8.17303	8.17388	8.17472	8.17557	8.17641	8.17725	8.17809	8.17893	9
52	8.17976	8.18060	8.18143	8.18226	8.18309	8.18392	8.18475	8.18557	8.18639	8.18722	8
53	8.18804	8.18886	8.18967	8.19049	8.19130	8.19211	8.19293	8.19374	8.19454	8.19535	7
54	8.19616	8.19696	8.19776	8.19856	8.19936	8.20016	8.20096	8.20175	8.20254	8.20334	6
55	8.20413	8.20491	8.20570	8.20649	8.20727	8.20806	8.20884	8.20962	8.21040	8.21118	5
56	8.21195	8.21273	8.21350	8.21427	8.21504	8.21581	8.21658	8.21735	8.21811	8.21888	4
57	8.21964	8.22040	8.22116	8.22192	8.22268	8.22343	8.22419	8.22494	8.22569	8.22645	3
58	8.22720	8.22794	8.22869	8.22944	8.23018	8.23092	8.23167	8.23241	8.23315	8.23388	2
59	8.23462	8.23536	8.23609	8.23682	8.23756	8.23829	8.23902	8.23974	8.24047	8.24120	1
60	8.24192	8.24264	8.24337	8.24409	8.24481	8.24553	8.24624	8.24696	8.24767	8.24839	0
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

89° — Log Ctn — 89°

1° — Log Sine — 1°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	8.24186	8.24258	8.24330	8.24402	8.24474	8.24546	8.24618	8.24689	8.24761	8.24832	60
1	8.24903	8.24974	8.25045	8.25116	8.25187	8.25258	8.25328	8.25399	8.25469	8.25539	59
2	8.25609	8.25679	8.25749	8.25819	8.25889	8.25958	8.26028	8.26097	8.26166	8.26235	58
3	8.26304	8.26373	8.26442	8.26511	8.26579	8.26648	8.26716	8.26784	8.26852	8.26920	57
4	8.26988	8.27056	8.27124	8.27191	8.27259	8.27326	8.27393	8.27460	8.27528	8.27595	56
5	8.27661	8.27728	8.27795	8.27861	8.27928	8.27994	8.28060	8.28127	8.28193	8.28258	55
6	8.28324	8.28390	8.28456	8.28521	8.28587	8.28652	8.28717	8.28782	8.28848	8.28912	54
7	8.28977	8.29042	8.29107	8.29171	8.29236	8.29300	8.29364	8.29429	8.29493	8.29557	53
8	8.29621	8.29684	8.29748	8.29812	8.29875	8.29939	8.30002	8.30065	8.30129	8.30192	52
9	8.30255	8.30317	8.30380	8.30443	8.30506	8.30568	8.30631	8.30693	8.30755	8.30817	51
10	8.30879	8.30941	8.31003	8.31065	8.31127	8.31188	8.31250	8.31311	8.31373	8.31434	50
11	8.31495	8.31556	8.31618	8.31678	8.31739	8.31800	8.31861	8.31921	8.31982	8.32042	49
12	8.32103	8.32163	8.32223	8.32283	8.32343	8.32403	8.32463	8.32523	8.32583	8.32642	48
13	8.32702	8.32761	8.32820	8.32880	8.32939	8.32998	8.33057	8.33116	8.33175	8.33234	47
14	8.33292	8.33351	8.33410	8.33468	8.33527	8.33585	8.33643	8.33701	8.33759	8.33817	46
15	8.33875	8.33933	8.33991	8.34049	8.34106	8.34164	8.34221	8.34279	8.34336	8.34393	45
16	8.34450	8.34508	8.34565	8.34621	8.34678	8.34735	8.34792	8.34849	8.34905	8.34962	44
17	8.35018	8.35074	8.35131	8.35187	8.35243	8.35299	8.35355	8.35411	8.35467	8.35523	43
18	8.35578	8.35634	8.35690	8.35745	8.35800	8.35856	8.35911	8.35966	8.36021	8.36076	42
19	8.36131	8.36186	8.36241	8.36296	8.36351	8.36405	8.36460	8.36515	8.36569	8.36623	41
20	8.36678	8.36732	8.36786	8.36840	8.36894	8.36948	8.37002	8.37056	8.37110	8.37163	40
21	8.37217	8.37271	8.37324	8.37378	8.37431	8.37484	8.37538	8.37591	8.37644	8.37697	39
22	8.37750	8.37803	8.37856	8.37908	8.37961	8.38014	8.38066	8.38119	8.38171	8.38224	38
23	8.38276	8.38328	8.38381	8.38433	8.38485	8.38537	8.38589	8.38641	8.38693	8.38744	37
24	8.38796	8.38848	8.38899	8.38951	8.39002	8.39054	8.39105	8.39157	8.39208	8.39259	36
25	8.39310	8.39361	8.39412	8.39463	8.39514	8.39565	8.39616	8.39666	8.39717	8.39767	35
26	8.39818	8.39868	8.39919	8.39969	8.40019	8.40070	8.40120	8.40170	8.40220	8.40270	34
27	8.40320	8.40370	8.40420	8.40469	8.40519	8.40569	8.40618	8.40668	8.40717	8.40767	33
28	8.40816	8.40865	8.40915	8.40964	8.41013	8.41062	8.41111	8.41160	8.41209	8.41258	32
29	8.41307	8.41356	8.41404	8.41453	8.41501	8.41550	8.41598	8.41647	8.41695	8.41744	31
30	8.41792	8.41840	8.41888	8.41936	8.41984	8.42032	8.42080	8.42128	8.42176	8.42224	30
31	8.42272	8.42319	8.42367	8.42415	8.42462	8.42510	8.42557	8.42604	8.42652	8.42699	29
32	8.42746	8.42793	8.42840	8.42888	8.42935	8.42982	8.43028	8.43075	8.43122	8.43169	28
33	8.43216	8.43262	8.43309	8.43355	8.43402	8.43448	8.43495	8.43541	8.43588	8.43634	27
34	8.43680	8.43726	8.43772	8.43818	8.43864	8.43910	8.43956	8.44002	8.44048	8.44094	26
35	8.44139	8.44185	8.44231	8.44276	8.44322	8.44367	8.44413	8.44458	8.44504	8.44549	25
36	8.44594	8.44639	8.44684	8.44730	8.44775	8.44820	8.44865	8.44910	8.44954	8.44999	24
37	8.45044	8.45089	8.45133	8.45178	8.45223	8.45267	8.45312	8.45356	8.45401	8.45445	23
38	8.45489	8.45534	8.45578	8.45622	8.45666	8.45710	8.45754	8.45798	8.45842	8.45886	22
39	8.45930	8.45974	8.46018	8.46061	8.46105	8.46149	8.46192	8.46236	8.46280	8.46323	21
40	8.46366	8.46410	8.46453	8.46497	8.46540	8.46583	8.46626	8.46669	8.46712	8.46755	20
41	8.46799	8.46841	8.46884	8.46927	8.46970	8.47013	8.47056	8.47098	8.47141	8.47184	19
42	8.47226	8.47269	8.47311	8.47354	8.47396	8.47439	8.47481	8.47523	8.47565	8.47608	18
43	8.47650	8.47692	8.47734	8.47776	8.47818	8.47860	8.47902	8.47944	8.47986	8.48028	17
44	8.48069	8.48111	8.48153	8.48194	8.48236	8.48278	8.48319	8.48361	8.48402	8.48443	16
45	8.48485	8.48526	8.48567	8.48609	8.48650	8.48691	8.48732	8.48773	8.48814	8.48855	15
46	8.48896	8.48937	8.48978	8.49019	8.49060	8.49101	8.49141	8.49182	8.49223	8.49263	14
47	8.49304	8.49345	8.49385	8.49426	8.49466	8.49506	8.49547	8.49587	8.49627	8.49668	13
48	8.49708	8.49748	8.49788	8.49828	8.49868	8.49908	8.49948	8.49988	8.50028	8.50068	12
49	8.50108	8.50148	8.50188	8.50227	8.50267	8.50307	8.50346	8.50386	8.50425	8.50465	11
50	8.50504	8.50544	8.50583	8.50623	8.50662	8.50701	8.50741	8.50780	8.50819	8.50858	10
51	8.50897	8.50936	8.50976	8.51015	8.51054	8.51092	8.51131	8.51170	8.51209	8.51248	9
52	8.51287	8.51325	8.51364	8.51403	8.51442	8.51480	8.51519	8.51557	8.51596	8.51634	8
53	8.51673	8.51711	8.51749	8.51788	8.51826	8.51864	8.51903	8.51941	8.51979	8.52017	7
54	8.52055	8.52093	8.52131	8.52169	8.52207	8.52245	8.52283	8.52321	8.52359	8.52397	6
55	8.52434	8.52472	8.52510	8.52547	8.52585	8.52623	8.52660	8.52698	8.52735	8.52773	5
56	8.52810	8.52848	8.52885	8.52922	8.52960	8.52997	8.53034	8.53071	8.53109	8.53146	4
57	8.53183	8.53220	8.53257	8.53294	8.53331	8.53368	8.53405	8.53442	8.53479	8.53515	3
58	8.53552	8.53589	8.53626	8.53663	8.53699	8.53736	8.53772	8.53809	8.53846	8.53882	2
59	8.53919	8.53955	8.53992	8.54028	8.54064	8.54101	8.54137	8.54173	8.54210	8.54246	1
60	8.54282	8.54318	8.54354	8.54390	8.54426	8.54462	8.54498	8.54534	8.54570	8.54606	0
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

88° — Log Cosine — 88°

1° — Log Tan — 1°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	8.24192	8.24264	8.24337	8.24409	8.24481	8.24553	8.24624	8.24696	8.24767	8.24839	60
1	8.24910	8.24981	8.25052	8.25123	8.25194	8.25265	8.25335	8.25406	8.25476	8.25546	59
2	8.25616	8.25686	8.25756	8.25826	8.25896	8.25965	8.26035	8.26104	8.26173	8.26243	58
3	8.26312	8.26380	8.26449	8.26518	8.26586	8.26655	8.26723	8.26792	8.26860	8.26928	57
4	8.26996	8.27063	8.27131	8.27199	8.27266	8.27334	8.27401	8.27468	8.27535	8.27602	56
5	8.27669	8.27736	8.27803	8.27869	8.27936	8.28002	8.28068	8.28134	8.28201	8.28266	55
6	8.28332	8.28398	8.28464	8.28529	8.28595	8.28660	8.28725	8.28791	8.28856	8.28921	54
7	8.28986	8.29050	8.29115	8.29180	8.29244	8.29309	8.29373	8.29437	8.29501	8.29565	53
8	8.29629	8.29693	8.29757	8.29820	8.29884	8.29947	8.30011	8.30074	8.30137	8.30200	52
9	8.30263	8.30326	8.30389	8.30452	8.30514	8.30577	8.30639	8.30702	8.30764	8.30826	51
10	8.30888	8.30950	8.31012	8.31074	8.31136	8.31198	8.31259	8.31321	8.31382	8.31443	50
11	8.31505	8.31566	8.31627	8.31688	8.31749	8.31809	8.31870	8.31931	8.31991	8.32052	49
12	8.32112	8.32173	8.32233	8.32293	8.32353	8.32413	8.32473	8.32533	8.32592	8.32652	48
13	8.32711	8.32771	8.32830	8.32890	8.32949	8.33008	8.33067	8.33126	8.33185	8.33244	47
14	8.33302	8.33361	8.33420	8.33478	8.33537	8.33595	8.33653	8.33712	8.33770	8.33828	46
15	8.33886	8.33944	8.34001	8.34059	8.34117	8.34174	8.34232	8.34289	8.34347	8.34404	45
16	8.34461	8.34518	8.34575	8.34632	8.34689	8.34746	8.34803	8.34859	8.34916	8.34972	44
17	8.35029	8.35085	8.35142	8.35198	8.35254	8.35310	8.35366	8.35422	8.35478	8.35534	43
18	8.35590	8.35645	8.35701	8.35756	8.35812	8.35867	8.35922	8.35978	8.36033	8.36088	42
19	8.36143	8.36198	8.36253	8.36308	8.36362	8.36417	8.36472	8.36526	8.36581	8.36635	41
20	8.36689	8.36744	8.36798	8.36852	8.36906	8.36960	8.37014	8.37068	8.37122	8.37175	40
21	8.37229	8.37283	8.37336	8.37390	8.37443	8.37497	8.37550	8.37603	8.37656	8.37709	39
22	8.37762	8.37815	8.37868	8.37921	8.37974	8.38026	8.38079	8.38132	8.38184	8.38236	38
23	8.38289	8.38341	8.38393	8.38446	8.38498	8.38550	8.38602	8.38654	8.38706	8.38757	37
24	8.38809	8.38861	8.38913	8.38964	8.39016	8.39067	8.39118	8.39170	8.39221	8.39272	36
25	8.39323	8.39374	8.39425	8.39476	8.39527	8.39578	8.39629	8.39680	8.39730	8.39781	35
26	8.39832	8.39882	8.39932	8.39983	8.40033	8.40083	8.40134	8.40184	8.40234	8.40284	34
27	8.40334	8.40384	8.40434	8.40483	8.40533	8.40583	8.40632	8.40682	8.40732	8.40781	33
28	8.40830	8.40880	8.40929	8.40978	8.41027	8.41077	8.41126	8.41175	8.41224	8.41272	32
29	8.41321	8.41370	8.41419	8.41468	8.41516	8.41565	8.41613	8.41662	8.41710	8.41758	31
30	8.41807	8.41855	8.41903	8.41951	8.41999	8.42048	8.42095	8.42143	8.42191	8.42239	30
31	8.42287	8.42335	8.42382	8.42430	8.42477	8.42525	8.42572	8.42620	8.42667	8.42715	29
32	8.42762	8.42809	8.42856	8.42903	8.42950	8.42997	8.43044	8.43091	8.43138	8.43185	28
33	8.43232	8.43278	8.43325	8.43371	8.43418	8.43464	8.43511	8.43557	8.43604	8.43650	27
34	8.43696	8.43742	8.43789	8.43835	8.43881	8.43927	8.43973	8.44019	8.44064	8.44110	26
35	8.44156	8.44202	8.44247	8.44293	8.44339	8.44384	8.44430	8.44475	8.44520	8.44566	25
36	8.44611	8.44656	8.44701	8.44747	8.44792	8.44837	8.44882	8.44927	8.44972	8.45016	24
37	8.45061	8.45106	8.45151	8.45195	8.45240	8.45285	8.45329	8.45374	8.45418	8.45463	23
38	8.45507	8.45551	8.45596	8.45640	8.45684	8.45728	8.45772	8.45816	8.45860	8.45904	22
39	8.45948	8.45992	8.46036	8.46080	8.46123	8.46167	8.46211	8.46254	8.46298	8.46341	21
40	8.46385	8.46428	8.46472	8.46515	8.46558	8.46602	8.46645	8.46688	8.46731	8.46774	20
41	8.46817	8.46860	8.46903	8.46946	8.46989	8.47032	8.47075	8.47117	8.47160	8.47203	19
42	8.47245	8.47288	8.47330	8.47373	8.47415	8.47458	8.47500	8.47543	8.47585	8.47627	18
43	8.47669	8.47712	8.47754	8.47796	8.47838	8.47880	8.47922	8.47964	8.48006	8.48047	17
44	8.48089	8.48131	8.48173	8.48214	8.48256	8.48298	8.48339	8.48381	8.48422	8.48464	16
45	8.48505	8.48546	8.48588	8.48629	8.48670	8.48711	8.48753	8.48794	8.48835	8.48876	15
46	8.48917	8.48958	8.48999	8.49040	8.49081	8.49121	8.49162	8.49203	8.49244	8.49284	14
47	8.49325	8.49366	8.49406	8.49447	8.49487	8.49528	8.49568	8.49608	8.49649	8.49689	13
48	8.49729	8.49769	8.49810	8.49850	8.49890	8.49930	8.49970	8.50010	8.50050	8.50090	12
49	8.50130	8.50170	8.50209	8.50249	8.50289	8.50329	8.50368	8.50408	8.50448	8.50487	11
50	8.50527	8.50566	8.50606	8.50645	8.50684	8.50724	8.50763	8.50802	8.50842	8.50881	10
51	8.50920	8.50959	8.50998	8.51037	8.51076	8.51115	8.51154	8.51193	8.51232	8.51271	9
52	8.51310	8.51349	8.51387	8.51426	8.51465	8.51503	8.51542	8.51581	8.51619	8.51658	8
53	8.51696	8.51735	8.51773	8.51811	8.51850	8.51888	8.51926	8.51964	8.52003	8.52041	7
54	8.52079	8.52117	8.52155	8.52193	8.52231	8.52269	8.52307	8.52345	8.52383	8.52421	6
55	8.52459	8.52496	8.52534	8.52572	8.52610	8.52647	8.52685	8.52722	8.52760	8.52797	5
56	8.52835	8.52872	8.52910	8.52947	8.52985	8.53022	8.53059	8.53096	8.53134	8.53171	4
57	8.53208	8.53245	8.53282	8.53319	8.53356	8.53393	8.53430	8.53467	8.53504	8.53541	3
58	8.53578	8.53615	8.53651	8.53688	8.53725	8.53762	8.53798	8.53835	8.53872	8.53908	2
59	8.53945	8.53981	8.54018	8.54054	8.54091	8.54127	8.54163	8.54200	8.54236	8.54272	1
60	8.54308	8.54345	8.54381	8.54417	8.54453	8.54489	8.54525	8.54561	8.54597	8.54633	0
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

88° — Log Ctn — 88°

2° — Log Sine — 2°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	8.54282	8.54318	8.54354	8.54390	8.54426	8.54462	8.54498	8.54534	8.54570	8.54606	60
1	8.54642	8.54678	8.54714	8.54750	8.54785	8.54821	8.54857	8.54893	8.54928	8.54964	59
2	8.54999	8.55035	8.55071	8.55106	8.55142	8.55177	8.55212	8.55248	8.55283	8.55319	58
3	8.55354	8.55389	8.55424	8.55460	8.55495	8.55530	8.55565	8.55600	8.55635	8.55670	57
4	8.55705	8.55740	8.55775	8.55810	8.55845	8.55880	8.55915	8.55950	8.55985	8.56019	56
5	8.56054	8.56089	8.56123	8.56158	8.56193	8.56227	8.56262	8.56296	8.56331	8.56365	55
6	8.56400	8.56434	8.56469	8.56503	8.56538	8.56572	8.56606	8.56640	8.56675	8.56709	54
7	8.56743	8.56777	8.56811	8.56846	8.56880	8.56914	8.56948	8.56982	8.57016	8.57050	53
8	8.57084	8.57117	8.57151	8.57185	8.57219	8.57253	8.57287	8.57320	8.57354	8.57388	52
9	8.57421	8.57455	8.57489	8.57522	8.57556	8.57589	8.57623	8.57656	8.57690	8.57723	51
10	8.57757	8.57790	8.57823	8.57857	8.57890	8.57923	8.57956	8.57990	8.58023	8.58056	50
11	8.58089	8.58122	8.58155	8.58189	8.58222	8.58255	8.58288	8.58321	8.58354	8.58386	49
12	8.58419	8.58452	8.58485	8.58518	8.58551	8.58583	8.58616	8.58649	8.58682	8.58714	48
13	8.58747	8.58780	8.58812	8.58845	8.58877	8.58910	8.58942	8.58975	8.59007	8.59040	47
14	8.59072	8.59104	8.59137	8.59169	8.59201	8.59234	8.59266	8.59298	8.59330	8.59363	46
15	8.59395	8.59427	8.59459	8.59491	8.59523	8.59555	8.59587	8.59619	8.59651	8.59683	45
16	8.59715	8.59747	8.59779	8.59811	8.59843	8.59874	8.59906	8.59938	8.59970	8.60001	44
17	8.60033	8.60065	8.60096	8.60128	8.60160	8.60191	8.60223	8.60254	8.60286	8.60317	43
18	8.60349	8.60380	8.60412	8.60443	8.60474	8.60506	8.60537	8.60568	8.60600	8.60631	42
19	8.60662	8.60693	8.60725	8.60756	8.60787	8.60818	8.60849	8.60880	8.60911	8.60942	41
20	8.60973	8.61004	8.61035	8.61066	8.61097	8.61128	8.61159	8.61190	8.61221	8.61252	40
21	8.61282	8.61313	8.61344	8.61375	8.61405	8.61436	8.61467	8.61497	8.61528	8.61559	39
22	8.61589	8.61620	8.61650	8.61681	8.61711	8.61742	8.61772	8.61803	8.61833	8.61863	38
23	8.61894	8.61924	8.61954	8.61985	8.62015	8.62045	8.62075	8.62106	8.62136	8.62166	37
24	8.62196	8.62226	8.62256	8.62286	8.62317	8.62347	8.62377	8.62407	8.62437	8.62467	36
25	8.62497	8.62526	8.62556	8.62586	8.62616	8.62646	8.62676	8.62706	8.62735	8.62765	35
26	8.62795	8.62825	8.62854	8.62884	8.62914	8.62943	8.62973	8.63002	8.63032	8.63062	34
27	8.63091	8.63121	8.63150	8.63180	8.63209	8.63238	8.63268	8.63297	8.63327	8.63356	33
28	8.63385	8.63415	8.63444	8.63473	8.63503	8.63532	8.63561	8.63590	8.63619	8.63649	32
29	8.63678	8.63707	8.63736	8.63765	8.63794	8.63823	8.63852	8.63881	8.63910	8.63939	31
30	8.63968	8.63997	8.64026	8.64055	8.64084	8.64112	8.64141	8.64170	8.64199	8.64228	30
31	8.64256	8.64285	8.64314	8.64342	8.64371	8.64400	8.64428	8.64457	8.64486	8.64514	29
32	8.64543	8.64571	8.64600	8.64628	8.64657	8.64685	8.64714	8.64742	8.64771	8.64799	28
33	8.64827	8.64856	8.64884	8.64912	8.64941	8.64969	8.64997	8.65026	8.65054	8.65082	27
34	8.65110	8.65138	8.65166	8.65195	8.65223	8.65251	8.65279	8.65307	8.65335	8.65363	26
35	8.65391	8.65419	8.65447	8.65475	8.65503	8.65531	8.65559	8.65587	8.65614	8.65642	25
36	8.65670	8.65698	8.65726	8.65754	8.65781	8.65809	8.65837	8.65864	8.65892	8.65920	24
37	8.65947	8.65975	8.66003	8.66030	8.66058	8.66085	8.66113	8.66141	8.66168	8.66196	23
38	8.66223	8.66250	8.66278	8.66305	8.66333	8.66360	8.66388	8.66415	8.66442	8.66470	22
39	8.66497	8.66524	8.66551	8.66579	8.66606	8.66633	8.66660	8.66687	8.66715	8.66742	21
40	8.66769	8.66796	8.66823	8.66850	8.66877	8.66904	8.66931	8.66958	8.66985	8.67012	20
41	8.67039	8.67066	8.67093	8.67120	8.67147	8.67174	8.67201	8.67228	8.67254	8.67281	19
42	8.67308	8.67335	8.67362	8.67388	8.67415	8.67442	8.67468	8.67495	8.67522	8.67548	18
43	8.67575	8.67602	8.67628	8.67655	8.67681	8.67708	8.67735	8.67761	8.67788	8.67814	17
44	8.67841	8.67867	8.67893	8.67920	8.67946	8.67973	8.67999	8.68025	8.68052	8.68078	16
45	8.68104	8.68131	8.68157	8.68183	8.68209	8.68236	8.68262	8.68288	8.68314	8.68340	15
46	8.68367	8.68393	8.68419	8.68445	8.68471	8.68497	8.68523	8.68549	8.68575	8.68601	14
47	8.68627	8.68653	8.68679	8.68705	8.68731	8.68757	8.68783	8.68809	8.68835	8.68860	13
48	8.68886	8.68912	8.68938	8.68964	8.68989	8.69015	8.69041	8.69067	8.69092	8.69118	12
49	8.69144	8.69169	8.69195	8.69221	8.69246	8.69272	8.69298	8.69323	8.69349	8.69374	11
50	8.69400	8.69425	8.69451	8.69476	8.69502	8.69527	8.69553	8.69578	8.69604	8.69629	10
51	8.69654	8.69680	8.69705	8.69730	8.69756	8.69781	8.69806	8.69832	8.69857	8.69882	9
52	8.69907	8.69933	8.69958	8.69983	8.70008	8.70033	8.70058	8.70084	8.70109	8.70134	8
53	8.70159	8.70184	8.70209	8.70234	8.70259	8.70284	8.70309	8.70334	8.70359	8.70384	7
54	8.70409	8.70434	8.70459	8.70484	8.70509	8.70534	8.70558	8.70583	8.70608	8.70633	6
55	8.70658	8.70682	8.70707	8.70732	8.70757	8.70781	8.70806	8.70831	8.70856	8.70880	5
56	8.70905	8.70930	8.70954	8.70979	8.71003	8.71028	8.71053	8.71077	8.71102	8.71126	4
57	8.71151	8.71175	8.71200	8.71224	8.71249	8.71273	8.71298	8.71322	8.71346	8.71371	3
58	8.71395	8.71420	8.71444	8.71468	8.71493	8.71517	8.71541	8.71566	8.71590	8.71614	2
59	8.71638	8.71663	8.71687	8.71711	8.71735	8.71759	8.71783	8.71808	8.71832	8.71856	1
60	8.71880	8.71904	8.71928	8.71952	8.71976	8.72000	8.72024	8.72048	8.72072	8.72096	0
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

87° — Log Cosine — 87°

2° — Log Tan — 2°

′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′
0	8.54308	8.54345	8.54381	8.54417	8.54453	8.54489	8.54525	8.54561	8.54597	8.54633	**60**
1	8.54669	8.54705	8.54741	8.54777	8.54813	8.54848	8.54884	8.54920	8.54956	8.54991	59
2	8.55027	8.55062	8.55098	8.55134	8.55169	8.55205	8.55240	8.55276	8.55311	8.55346	58
3	8.55382	8.55417	8.55452	8.55488	8.55523	8.55558	8.55593	8.55628	8.55663	8.55699	57
4	8.55734	8.55769	8.55804	8.55839	8.55874	8.55909	8.55943	8.55978	8.56013	8.56048	56
5	8.56083	8.56118	8.56152	8.56187	8.56222	8.56256	8.56291	8.56326	8.56360	8.56395	**55**
6	8.56429	8.56464	8.56498	8.56532	8.56567	8.56601	8.56636	8.56670	8.56704	8.56739	54
7	8.56773	8.56807	8.56841	8.56875	8.56909	8.56944	8.56978	8.57012	8.57046	8.57080	53
8	8.57114	8.57148	8.57182	8.57215	8.57249	8.57283	8.57317	8.57351	8.57385	8.57418	52
9	8.57452	8.57486	8.57519	8.57553	8.57587	8.57620	8.57654	8.57687	8.57721	8.57754	51
10	8.57788	8.57821	8.57854	8.57888	8.57921	8.57955	8.57988	8.58021	8.58054	8.58088	**50**
11	8.58121	8.58154	8.58187	8.58220	8.58253	8.58286	8.58319	8.58352	8.58385	8.58418	49
12	8.58451	8.58484	8.58517	8.58550	8.58583	8.58616	8.58649	8.58681	8.58714	8.58747	48
13	8.58779	8.58812	8.58845	8.58877	8.58910	8.58943	8.58975	8.59008	8.59040	8.59073	47
14	8.59105	8.59138	8.59170	8.59202	8.59235	8.59267	8.59299	8.59332	8.59364	8.59396	46
15	8.59428	8.59461	8.59493	8.59525	8.59557	8.59589	8.59621	8.59653	8.59685	8.59717	**45**
16	8.59749	8.59781	8.59813	8.59845	8.59877	8.59909	8.59941	8.59972	8.60004	8.60036	44
17	8.60068	8.60099	8.60131	8.60163	8.60194	8.60226	8.60258	8.60289	8.60321	8.60352	43
18	8.60384	8.60415	8.60447	8.60478	8.60510	8.60541	8.60572	8.60604	8.60635	8.60666	42
19	8.60698	8.60729	8.60760	8.60792	8.60823	8.60854	8.60885	8.60916	8.60947	8.60978	41
20	8.61009	8.61040	8.61071	8.61103	8.61133	8.61164	8.61195	8.61226	8.61257	8.61288	**40**
21	8.61319	8.61350	8.61381	8.61411	8.61442	8.61473	8.61504	8.61534	8.61565	8.61596	39
22	8.61626	8.61657	8.61687	8.61718	8.61748	8.61779	8.61809	8.61840	8.61870	8.61901	38
23	8.61931	8.61962	8.61992	8.62022	8.62053	8.62083	8.62113	8.62144	8.62174	8.62204	37
24	8.62234	8.62264	8.62295	8.62325	8.62355	8.62385	8.62415	8.62445	8.62475	8.62505	36
25	8.62535	8.62565	8.62595	8.62625	8.62655	8.62685	8.62715	8.62745	8.62774	8.62804	**35**
26	8.62834	8.62864	8.62894	8.62923	8.62953	8.62983	8.63012	8.63042	8.63072	8.63101	34
27	8.63131	8.63160	8.63190	8.63219	8.63249	8.63278	8.63308	8.63337	8.63367	8.63396	33
28	8.63426	8.63455	8.63484	8.63514	8.63543	8.63572	8.63602	8.63631	8.63660	8.63689	32
29	8.63718	8.63748	8.63777	8.63806	8.63835	8.63864	8.63893	8.63922	8.63951	8.63980	31
30	8.64009	8.64038	8.64067	8.64096	8.64125	8.64154	8.64183	8.64212	8.64241	8.64269	**30**
31	8.64298	8.64327	8.64356	8.64385	8.64413	8.64442	8.64471	8.64499	8.64528	8.64557	29
32	8.64585	8.64614	8.64642	8.64671	8.64700	8.64728	8.64757	8.64785	8.64814	8.64842	28
33	8.64870	8.64899	8.64927	8.64956	8.64984	8.65012	8.65041	8.65069	8.65097	8.65126	27
34	8.65154	8.65182	8.65210	8.65238	8.65267	8.65295	8.65323	8.65351	8.65379	8.65407	26
35	8.65435	8.65463	8.65491	8.65519	8.65547	8.65575	8.65603	8.65631	8.65659	8.65687	**25**
36	8.65715	8.65743	8.65771	8.65798	8.65826	8.65854	8.65882	8.65910	8.65937	8.65965	24
37	8.65993	8.66020	8.66048	8.66076	8.66103	8.66131	8.66159	8.66186	8.66214	8.66241	23
38	8.66269	8.66296	8.66324	8.66351	8.66379	8.66406	8.66434	8.66461	8.66489	8.66516	22
39	8.66543	8.66571	8.66598	8.66625	8.66653	8.66680	8.66707	8.66734	8.66762	8.66789	21
40	8.66816	8.66843	8.66870	8.66897	8.66925	8.66952	8.66979	8.67006	8.67033	8.67060	**20**
41	8.67087	8.67114	8.67141	8.67168	8.67195	8.67222	8.67249	8.67276	8.67303	8.67329	19
42	8.67356	8.67383	8.67410	8.67437	8.67464	8.67490	8.67517	8.67544	8.67571	8.67597	18
43	8.67624	8.67651	8.67677	8.67704	8.67731	8.67757	8.67784	8.67810	8.67837	8.67863	17
44	8.67890	8.67916	8.67943	8.67969	8.67996	8.68022	8.68049	8.68075	8.68102	8.68128	16
45	8.68154	8.68181	8.68207	8.68233	8.68260	8.68286	8.68312	8.68339	8.68365	8.68391	**15**
46	8.68417	8.68443	8.68470	8.68496	8.68522	8.68548	8.68574	8.68600	8.68626	8.68652	14
47	8.68678	8.68704	8.68731	8.68757	8.68782	8.68808	8.68834	8.68860	8.68886	8.68912	13
48	8.68938	8.68964	8.68990	8.69016	8.69042	8.69067	8.69093	8.69119	8.69145	8.69171	12
49	8.69196	8.69222	8.69248	8.69273	8.69299	8.69325	8.69350	8.69376	8.69402	8.69427	11
50	8.69453	8.69479	8.69504	8.69530	8.69555	8.69581	8.69606	8.69632	8.69657	8.69683	**10**
51	8.69708	8.69733	8.69759	8.69784	8.69810	8.69835	8.69860	8.69886	8.69911	8.69936	9
52	8.69962	8.69987	8.70012	8.70038	8.70063	8.70088	8.70113	8.70138	8.70164	8.70189	8
53	8.70214	8.70239	8.70264	8.70289	8.70314	8.70339	8.70365	8.70390	8.70415	8.70440	7
54	8.70465	8.70490	8.70515	8.70540	8.70565	8.70589	8.70614	8.70639	8.70664	8.70689	6
55	8.70714	8.70739	8.70764	8.70788	8.70813	8.70838	8.70863	8.70888	8.70912	8.70937	**5**
56	8.70962	8.70987	8.71011	8.71036	8.71061	8.71085	8.71110	8.71135	8.71159	8.71184	4
57	8.71208	8.71233	8.71257	8.71282	8.71307	8.71331	8.71356	8.71380	8.71405	8.71429	3
58	8.71453	8.71478	8.71502	8.71527	8.71551	8.71575	8.71600	8.71624	8.71649	8.71673	2
59	8.71697	8.71721	8.71746	8.71770	8.71794	8.71819	8.71843	8.71867	8.71891	8.71915	1
60	8.71940	8.71964	8.71988	8.72012	8.72036	8.72060	8.72084	8.72108	8.72133	8.72157	**0**
′	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	′

87° — Log Ctn — 87°

0° — Common Logarithms of Trigonometric Functions — 0°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	—		—		—	0.00 000	60
1	6.46 373	30103	6.46 373	30103	3.53 627	0.00 000	59
2	6.76 476	17609	6.76 476	17609	3.23 524	0.00 000	58
3	6.94 085	12494	6.94 085	12494	3.05 915	0.00 000	57
4	7.06 579	9691	7.06 579	9691	2.93 421	0.00 000	56
5	7.16 270	7918	7.16 270	7918	2.83 730	0.00 000	55
6	7.24 188	6694	7.24 188	6694	2.75 812	0.00 000	54
7	7.30 882	5800	7.30 882	5800	2.69 118	0.00 000	53
8	7.36 682	5115	7.36 682	5115	2.63 318	0.00 000	52
9	7.41 797	4576	7.41 797	4576	2.58 203	0.00 000	51
10	7.46 373	4139	7.46 373	4139	2.53 627	0.00 000	50
11	7.50 512	3779	7.50 512	3779	2.49 488	0.00 000	49
12	7.54 291	3476	7.54 291	3476	2.45 709	0.00 000	48
13	7.57 767	3218	7.57 767	3219	2.42 233	0.00 000	47
14	7.60 985	2997	7.60 986	2996	2.39 014	0.00 000	46
15	7.63 982	2802	7.63 982	2803	2.36 018	0.00 000	45
16	7.66 784	2633	7.66 785	2633	2.33 215	0.00 000	44
17	7.69 417	2483	7.69 418	2482	2.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	2.28 100	9.99 999	42
19	7.74 248	2227	7.74 248	2228	2.25 752	9.99 999	41
20	7.76 475	2119	7.76 476	2119	2.23 524	9.99 999	40
21	7.78 594	2021	7.78 595	2020	2.21 405	9.99 999	39
22	7.80 615	1930	7.80 615	1931	2.19 385	9.99 999	38
23	7.82 545	1848	7.82 546	1848	2.17 454	9.99 999	37
24	7.84 393	1773	7.84 394	1773	2.15 606	9.99 999	36
25	7.86 166	1704	7.86 167	1704	2.13 833	9.99 999	35
26	7.87 870	1639	7.87 871	1639	2.12 129	9.99 999	34
27	7.89 509	1579	7.89 510	1579	2.10 490	9.99 999	33
28	7.91 088	1524	7.91 089	1524	2.08 911	9.99 999	32
29	7.92 612	1472	7.92 613	1473	2.07 387	9.99 998	31
30	7.94 084	1424	7.94 086	1424	2.05 914	9.99 998	30
31	7.95 508	1379	7.95 510	1379	2.04 490	9.99 998	29
32	7.96 887	1336	7.96 889	1336	2.03 111	9.99 998	28
33	7.98 223	1297	7.98 225	1297	2.01 775	9.99 998	27
34	7.99 520	1259	7.99 522	1259	2.00 478	9.99 998	26
35	8.00 779	1223	8.00 781	1223	1.99 219	9.99 998	25
36	8.02 002	1190	8.02 004	1190	1.97 996	9.99 998	24
37	8.03 192	1158	8.03 194	1159	1.96 806	9.99 997	23
38	8.04 350	1128	8.04 353	1128	1.95 647	9.99 997	22
39	8.05 478	1100	8.05 481	1100	1.94 519	9.99 997	21
40	8.06 578	1072	8.06 581	1072	1.93 419	9.99 997	20
41	8.07 650	1046	8.07 653	1047	1.92 347	9.99 997	19
42	8.08 696	1022	8.08 700	1022	1.91 300	9.99 997	18
43	8.09 718	999	8.09 722	998	1.90 278	9.99 997	17
44	8.10 717	976	8.10 720	976	1.89 280	9.99 996	16
45	8.11 693	954	8.11 696	955	1.88 304	9.99 996	15
46	8.12 647	934	8.12 651	934	1.87 349	9.99 996	14
47	8.13 581	914	8.13 585	915	1.86 415	9.99 996	13
48	8.14 495	896	8.14 500	895	1.85 500	9.99 996	12
49	8.15 391	877	8.15 395	878	1.84 605	9.99 996	11
50	8.16 268	860	8.16 273	860	1.83 727	9.99 995	10
51	8.17 128	843	8.17 133	843	1.82 867	9.99 995	9
52	8.17 971	827	8.17 976	828	1.82 024	9.99 995	8
53	8.18 798	812	8.18 804	812	1.81 196	9.99 995	7
54	8.19 610	797	8.19 616	797	1.80 384	9.99 995	6
55	8.20 407	782	8.20 413	782	1.79 587	9.99 994	5
56	8.21 189	769	8.21 195	769	1.78 805	9.99 994	4
57	8.21 958	755	8.21 964	756	1.78 036	9.99 994	3
58	8.22 713	743	8.22 720	742	1.77 280	9.99 994	2
59	8.23 456	730	8.23 462	730	1.76 538	9.99 994	1
60	8.24 186		8.24 192		1.75 808	9.99 993	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

See pages 44–49 for the logarithms of sines and tangents of angles less than 3° and the logarithms of cosines and cotangents of angles greater than 87°.

89° — Common Logarithms of Trigonometric Functions — 89°

1° — Common Logarithms of Trigonometric Functions — 1°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	8.24 186	717	8.24 192	718	1.75 808	9.99 993	60
1	8.24 903	706	8.24 910	706	1.75 090	9.99 993	59
2	8.25 609	695	8.25 616	696	1.74 384	9.99 993	58
3	8.26 304	684	8.26 312	684	1.73 688	9.99 993	57
4	8.26 988	673	8.26 996	673	1.73 004	9.99 992	56
5	8.27 661	663	8.27 669	663	1.72 331	9.99 992	55
6	8.28 324	653	8.28 332	654	1.71 668	9.99 992	54
7	8.28 977	644	8.28 986	643	1.71 014	9.99 992	53
8	8.29 621	634	8.29 629	634	1.70 371	9.99 992	52
9	8.30 255	624	8.30 263	625	1.69 737	9.99 991	51
10	8.30 879	616	8.30 888	617	1.69 112	9.99 991	50
11	8.31 495	608	8.31 505	607	1.68 495	9.99 991	49
12	8.32 103	599	8.32 112	599	1.67 888	9.99 990	48
13	8.32 702	590	8.32 711	591	1.67 289	9.99 990	47
14	8.33 292	583	8.33 302	584	1.66 698	9.99 990	46
15	8.33 875	575	8.33 886	575	1.66 114	9.99 990	45
16	8.34 450	568	8.34 461	568	1.65 539	9.99 989	44
17	8.35 018	560	8.35 029	561	1.64 971	9.99 989	43
18	8.35 578	553	8.35 590	553	1.64 410	9.99 989	42
19	8.36 131	547	8.36 143	546	1.63 857	9.99 989	41
20	8.36 678	539	8.36 689	540	1.63 311	9.99 988	40
21	8.37 217	533	8.37 229	533	1.62 771	9.99 988	39
22	8.37 750	526	8.37 762	527	1.62 238	9.99 988	38
23	8.38 276	520	8.38 289	520	1.61 711	9.99 987	37
24	8.38 796	514	8.38 809	514	1.61 191	9.99 987	36
25	8.39 310	508	8.39 323	509	1.60 677	9.99 987	35
26	8.39 818	502	8.39 832	502	1.60 168	9.99 986	34
27	8.40 320	496	8.40 334	496	1.59 666	9.99 986	33
28	8.40 816	491	8.40 830	491	1.59 170	9.99 986	32
29	8.41 307	485	8.41 321	486	1.58 679	9.99 985	31
30	8.41 792	480	8.41 807	480	1.58 193	9.99 985	30
31	8.42 272	474	8.42 287	475	1.57 713	9.99 985	29
32	8.42 746	470	8.42 762	470	1.57 238	9.99 984	28
33	8.43 216	464	8.43 232	464	1.56 768	9.99 984	27
34	8.43 680	459	8.43 696	460	1.56 304	9.99 984	26
35	8.44 139	455	8.44 156	455	1.55 844	9.99 983	25
36	8.44 594	450	8.44 611	450	1.55 389	9.99 983	24
37	8.45 044	445	8.45 061	446	1.54 939	9.99 983	23
38	8.45 489	441	8.45 507	441	1.54 493	9.99 982	22
39	8.45 930	436	8.45 948	437	1.54 052	9.99 982	21
40	8.46 366	433	8.46 385	432	1.53 615	9.99 982	20
41	8.46 799	427	8.46 817	428	1.53 183	9.99 981	19
42	8.47 226	424	8.47 245	424	1.52 755	9.99 981	18
43	8.47 650	419	8.47 669	420	1.52 331	9.99 981	17
44	8.48 069	416	8.48 089	416	1.51 911	9.99 980	16
45	8.48 485	411	8.48 505	412	1.51 495	9.99 980	15
46	8.48 896	408	8.48 917	408	1.51 083	9.99 979	14
47	8.49 304	404	8.49 325	404	1.50 675	9.99 979	13
48	8.49 708	400	8.49 729	401	1.50 271	9.99 979	12
49	8.50 108	396	8.50 130	397	1.49 870	9.99 978	11
50	8.50 504	393	8.50 527	393	1.49 473	9.99 978	10
51	8.50 897	390	8.50 920	390	1.49 080	9.99 977	9
52	8.51 287	386	8.51 310	386	1.48 690	9.99 977	8
53	8.51 673	382	8.51 696	383	1.48 304	9.99 977	7
54	8.52 055	379	8.52 079	380	1.47 921	9.99 976	6
55	8.52 434	376	8.52 459	376	1.47 541	9.99 976	5
56	8.52 810	373	8.52 835	373	1.47 165	9.99 975	4
57	8.53 183	369	8.53 208	370	1.46 792	9.99 975	3
58	8.53 552	367	8.53 578	367	1.46 422	9.99 974	2
59	8.53 919	363	8.53 945	363	1.46 055	9.99 974	1
60	8.54 282		8.54 308		1.45 692	9.99 974	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

See pages 44–49 for the logarithms of sines and tangents of angles less than 3° and the logarithms of cosines and cotangents of angles greater than 87°.

88° — Common Logarithms of Trigonometric Functions — 88°

2° — Common Logarithms of Trigonometric Functions — 2°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	8.54 282	360	8.54 308	361	1.45 692	9.99 974	60
1	8.54 642	357	8.54 669	358	1.45 331	9.99 973	59
2	8.54 999	355	8.55 027	355	1.44 973	9.99 973	58
3	8.55 354	351	8.55 382	352	1.44 618	9.99 972	57
4	8.55 705	349	8.55 734	349	1.44 266	9.99 972	56
5	8.56 054	346	8.56 083	346	1.43 917	9.99 971	55
6	8.56 400	343	8.56 429	344	1.43 571	9.99 971	54
7	8.56 743	341	8.56 773	341	1.43 227	9.99 970	53
8	8.57 084	337	8.57 114	338	1.42 886	9.99 970	52
9	8.57 421	336	8.57 452	336	1.42 548	9.99 969	51
10	8.57 757	332	8.57 788	333	1.42 212	9.99 969	50
11	8.58 089	330	8.58 121	330	1.41 879	9.99 968	49
12	8.58 419	328	8.58 451	328	1.41 549	9.99 968	48
13	8.58 747	325	8.58 779	326	1.41 221	9.99 967	47
14	8.59 072	323	8.59 105	323	1.40 895	9.99 967	46
15	8.59 395	320	8.59 428	321	1.40 572	9.99 967	45
16	8.59 715	318	8.59 749	319	1.40 251	9.99 966	44
17	8.60 033	316	8.60 068	316	1.39 932	9.99 966	43
18	8.60 349	313	8.60 384	314	1.39 616	9.99 965	42
19	8.60 662	311	8.60 698	311	1.39 302	9.99 964	41
20	8.60 973	309	8.61 009	310	1.38 991	9.99 964	40
21	8.61 282	307	8.61 319	307	1.38 681	9.99 963	39
22	8.61 589	305	8.61 626	305	1.38 374	9.99 963	38
23	8.61 894	302	8.61 931	303	1.38 069	9.99 962	37
24	8.62 196	301	8.62 234	301	1.37 766	9.99 962	36
25	8.62 497	298	8.62 535	299	1.37 465	9.99 961	35
26	8.62 795	296	8.62 834	297	1.37 166	9.99 961	34
27	8.63 091	294	8.63 131	295	1.36 869	9.99 960	33
28	8.63 385	293	8.63 426	292	1.36 574	9.99 960	32
29	8.63 678	290	8.63 718	291	1.36 282	9.99 959	31
30	8.63 968	288	8.64 009	289	1.35 991	9.99 959	30
31	8.64 256	287	8.64 298	287	1.35 702	9.99 958	29
32	8.64 543	284	8.64 585	285	1.35 415	9.99 958	28
33	8.64 827	283	8.64 870	284	1.35 130	9.99 957	27
34	8.65 110	281	8.65 154	281	1.34 846	9.99 956	26
35	8.65 391	279	8.65 435	280	1.34 565	9.99 956	25
36	8.65 670	277	8.65 715	278	1.34 285	9.99 955	24
37	8.65 947	276	8.65 993	276	1.34 007	9.99 955	23
38	8.66 223	274	8.66 269	274	1.33 731	9.99 954	22
39	8.66 497	272	8.66 543	273	1.33 457	9.99 954	21
40	8.66 769	270	8.66 816	271	1.33 184	9.99 953	20
41	8.67 039	269	8.67 087	269	1.32 913	9.99 952	19
42	8.67 308	267	8.67 356	268	1.32 644	9.99 952	18
43	8.67 575	266	8.67 624	266	1.32 376	9.99 951	17
44	8.67 841	263	8.67 890	264	1.32 110	9.99 951	16
45	8.68 104	263	8.68 154	263	1.31 846	9.99 950	15
46	8.68 367	260	8.68 417	261	1.31 583	9.99 949	14
47	8.68 627	259	8.68 678	260	1.31 322	9.99 949	13
48	8.68 886	258	8.68 938	258	1.31 062	9.99 948	12
49	8.69 144	256	8.69 196	257	1.30 804	9.99 948	11
50	8.69 400	254	8.69 453	255	1.30 547	9.99 947	10
51	8.69 654	253	8.69 708	254	1.30 292	9.99 946	9
52	8.69 907	252	8.69 962	252	1.30 038	9.99 946	8
53	8.70 159	250	8.70 214	251	1.29 786	9.99 945	7
54	8.70 409	249	8.70 465	249	1.29 535	9.99 944	6
55	8.70 658	247	8.70 714	248	1.29 286	9.99 944	5
56	8.70 905	246	8.70 962	246	1.29 038	9.99 943	4
57	8.71 151	244	8.71 208	245	1.28 792	9.99 942	3
58	8.71 395	243	8.71 453	244	1.28 547	9.99 942	2
59	8.71 638	242	8.71 697	243	1.28 303	9.99 941	1
60	8.71 880		8.71 940		1.28 060	9.99 940	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

See pages 44–49 for the logarithms of sines and tangents of angles less than 3° and the logarithms of cosines and cotangents of angles greater than 87°.

87° — Common Logarithms of Trigonometric Functions — 87°

3° — Common Logarithms of Trigonometric Functions — 3°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	1.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	1.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925	38
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	30
31	8.78 774	205	8.78 855	206	1.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	1.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	20
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	1.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895	1
60	8.84 358		8.84 464		1.15 536	9.99 894	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	240	235	230	225	220	215	210	205
1	24.0	23.5	23.0	22.5	22.0	21.5	21.0	20.5
2	48.0	47.0	46.0	45.0	44.0	43.0	42.0	41.0
3	72.0	70.5	69.0	67.5	66.0	64.5	63.0	61.5
4	96.0	94.0	92.0	90.0	88.0	86.0	84.0	82.0
5	120.0	117.5	115.0	112.5	110.0	107.5	105.0	102.5
6	144.0	141.0	138.0	135.0	132.0	129.0	126.0	123.0
7	168.0	164.5	161.0	157.5	154.0	150.5	147.0	143.5
8	192.0	188.0	184.0	180.0	176.0	172.0	168.0	164.0
9	216.0	211.5	207.0	202.5	198.0	193.5	189.0	184.5

	195	192	188	184	182	180
1	19.5	19.2	18.8	18.4	18.2	18.0
2	39.0	38.4	37.6	36.8	36.4	36.0
3	58.5	57.6	56.4	55.2	54.6	54.0
4	78.0	76.8	75.2	73.6	72.8	72.0
5	97.5	96.0	94.0	92.0	91.0	90.0
6	117.0	115.2	112.8	110.4	109.2	108.0
7	136.5	134.4	131.6	128.8	127.4	126.0
8	156.0	153.6	150.4	147.2	145.6	144.0
9	175.5	172.8	169.2	165.6	163.8	162.0

86° — Common Logarithms of Trigonometric Functions — 86°

4° — Common Logarithms of Trigonometric Functions — 4°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	1.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	1.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	1.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	1.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	1.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	178	176	174	172	170	168	166	164
1	17.8	17.6	17.4	17.2	17.0	16.8	16.6	16.4
2	35.6	35.2	34.8	34.4	34.0	33.6	33.2	32.8
3	53.4	52.8	52.2	51.6	51.0	50.4	49.8	49.2
4	71.2	70.4	69.6	68.8	68.0	67.2	66.4	65.6
5	89.0	88.0	87.0	86.0	85.0	84.0	83.0	82.0
6	106.8	105.6	104.4	103.2	102.0	100.8	99.6	98.4
7	124.6	123.2	121.8	120.4	119.0	117.6	116.2	114.8
8	142.4	140.8	139.2	137.6	136.0	134.4	132.8	131.2
9	160.2	158.4	156.6	154.8	153.0	151.2	149.4	147.6

	162	160	158	156	154	152
1	16.2	16.0	15.8	15.6	15.4	15.2
2	32.4	32.0	31.6	31.2	30.8	30.4
3	48.6	48.0	47.4	46.8	46.2	45.6
4	64.8	64.0	63.2	62.4	61.6	60.8
5	81.0	80.0	79.0	78.0	77.0	76.0
6	97.2	96.0	94.8	93.6	92.4	91.2
7	113.4	112.0	110.6	109.2	107.8	106.4
8	129.6	128.0	126.4	124.8	123.2	121.6
9	145.8	144.0	142.2	140.4	138.6	136.8

85° — Common Logarithms of Trigonometric Functions — 85°

5° — Common Logarithms of Trigonometric Functions — 5°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	8.94 030	144	8.94 195	145	1.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	1.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	1.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	150	148	146	144	142	140	138	136	134	132	130	128	126	124
1	15.0	14.8	14.6	14.4	14.2	14.0	13.8	13.6	13.4	13.2	13.0	12.8	12.6	12.4
2	30.0	29.6	29.2	28.8	28.4	28.0	27.6	27.2	26.8	26.4	26.0	25.6	25.2	24.8
3	45.0	44 4	43.8	43.2	42.6	42.0	41.4	40.8	40.2	39.6	39.0	38.4	37.8	37.2
4	60.0	59.2	58.4	57.6	56.8	56.0	55.2	54.4	53.6	52.8	52.0	51.2	50.4	49.6
5	75.0	74.0	73.0	72.0	71.0	70.0	69.0	68.0	67.0	66.0	65.0	64.0	63.0	62.0
6	90.0	88.8	87.6	86.4	85.2	84.0	82.8	81.6	80.4	79.2	78.0	76.8	75.6	74.4
7	105.0	103.6	102.2	100.8	99.4	98.0	96.6	95.2	93.8	92.4	91.0	89.6	88.2	86.8
8	120.0	118.4	116.8	115.2	113.6	112.0	110.4	108.8	107.2	105.6	104.0	102.4	100.8	99.2
9	135.0	133.2	131.4	129.6	127.8	126.0	124.2	122.4	120.6	118.8	117.0	115.2	113.4	111 6

84° — Common Logarithms of Trigonometric Functions — 84°

6° — Common Logarithms of Trigonometric Functions — 6°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	0.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	0.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	0.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	122	**120**
1	12.2	12.0
2	24.4	24.0
3	36.6	36.0
4	48.8	48.0
5	61.0	60.0
6	73.2	72.0
7	85.4	84.0
8	97.6	96.0
9	109.8	108.0
	118	**116**
1	11.8	11.6
2	23.6	23.2
3	35.4	34.8
4	47.2	46.4
5	59.0	58.0
6	70.8	69.6
7	82.6	81.2
8	94.4	92.8
9	106.2	104.4
	114	**112**
1	11.4	11.2
2	22.8	22.4
3	34.2	33.6
4	45.6	44.8
5	57.0	56.0
6	68.4	67.2
7	79.8	78.4
8	91.2	89.6
9	102.6	100.8
	110	**109**
1	11.0	10.9
2	22.0	21.8
3	33.0	32.7
4	44.0	43.6
5	55.0	54.5
6	66.0	65.4
7	77.0	76.3
8	88.0	87.2
9	99.0	98.1
	108	**107**
1	10.8	10.7
2	21.6	21.4
3	32.4	32.1
4	43.2	42.8
5	54.0	53.5
6	64.8	64.2
7	75.6	74.9
8	86.4	85.6
9	97.2	96.3
	106	**105**
1	10.6	10.5
2	21.2	21.0
3	31.8	31.5
4	42.4	42.0
5	53.0	52.5
6	63.6	63.0
7	74.2	73.5
8	84.8	84.0
9	95.4	94.5
	104	**103**
1	10.4	10.3
2	20.8	20.6
3	31.2	30.9
4	41.6	41.2
5	52.0	51.5
6	62.4	61.8
7	72.8	72.1
8	83.2	82.4
9	93.6	92.7

Prop. Parts

83° — Common Logarithms of Trigonometric Functions — 83°

7° — Common Logarithms of Trigonometric Functions — 7°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	**60**
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	**55**
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	**50**
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	**45**
16	9.10 205	99	9.10 555	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	**40**
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	**35**
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	**30**
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	**25**
36	9.12 142	94	9.12 525	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	**20**
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	**15**
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	**10**
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	**5**
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	103	102	101	99	98	97	96	95	94	93	92	91	90
1	10.3	10.2	10.1	9.9	9.8	9.7	9.6	9.5	9.4	9.3	9.2	9.1	9.0
2	20.6	20.4	20.2	19.8	19.6	19.4	19.2	19.0	18.8	18.6	18.4	18.2	18.0
3	30.9	30.6	30.3	29.7	29.4	29.1	28.8	28.5	28.2	27.9	27.6	27.3	27.0
4	41.2	40.8	40.4	39.6	39.2	38.8	38.4	38.0	37.6	37.2	36.8	36.4	36.0
5	51.5	51.0	50.5	49.5	49.0	48.5	48.0	47.5	47.0	46.5	46.0	45.5	45.0
6	61.8	61.2	60.6	59.4	58.8	58.2	57.6	57.0	56.4	55.8	55.2	54.6	54.0
7	72.1	71.4	70.7	69.3	68.6	67.9	67.2	66.5	65.8	65.1	64.4	63.7	63.0
8	82.4	81.6	80.8	79.2	78.4	77.6	76.8	76.0	75.2	74.4	73.6	72.8	72.0
9	92.7	91.8	90.9	89.1	88.2	87.3	86.4	85.5	84.6	83.7	82.8	81.9	81.0

Prop. Parts

82° — Common Logarithms of Trigonometric Functions — 82°

8° — Common Logarithms of Trigonometric Functions — 8°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

	92	91	90	89	88	87	86	85	84	83	82	81	80
1	9.2	9.1	9.0	8.9	8.8	8.7	8.6	8.5	8.4	8.3	8.2	8.1	8.0
2	18.4	18.2	18.0	17.8	17.6	17.4	17.2	17.0	16.8	16.6	16.4	16.2	16.0
3	27.6	27.3	27.0	26.7	26.4	26.1	25.8	25.5	25.2	24.9	24.6	24.3	24.0
4	36.8	36.4	36.0	35.6	35.2	34.8	34.4	34.0	33.6	33.2	32.8	32.4	32.0
5	46.0	45.5	45.0	44.5	44.0	43.5	43.0	42.5	42.0	41.5	41.0	40.5	40.0
6	55.2	54.6	54.0	53.4	52.8	52.2	51.6	51.0	50.4	49.8	49.2	48.6	48.0
7	64.4	63.7	63.0	62.3	61.6	60.9	60.2	59.5	58.8	58.1	57.4	56.7	56.0
8	73.6	72.8	72.0	71.2	70.4	69.6	68.8	68.0	67.2	66.4	65.6	64.8	64.0
9	82.8	81.9	81.0	80.1	79.2	78.3	77.4	76.5	75.6	74.7	73.8	72.9	72.0

81° — Common Logarithms of Trigonometric Functions — 81°

9° — Common Logarithms of Trigonometric Functions — 9°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	′
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	25
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	0.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Parts

′	82	81
1	8.2	8.1
2	16.4	16.2
3	24.6	24.3
4	32.8	32.4
5	41.0	40.5
6	49.2	48.6
7	57.4	56.7
8	65.6	64.8
9	73.8	72.9

	80	79
1	8.0	7.9
2	16.0	15.8
3	24.0	23.7
4	32.0	31.6
5	40.0	39.5
6	48.0	47.4
7	56.0	55.3
8	64.0	63.2
9	72.0	71.1

	78	77
1	7.8	7.7
2	15.6	15.4
3	23.4	23.1
4	31.2	30.8
5	39.0	38.5
6	46.8	46.2
7	54.6	53.9
8	62.4	61.6
9	70.2	69.3

	76	75
1	7.6	7.5
2	15.2	15.0
3	22.8	22.5
4	30.4	30.0
5	38.0	37.5
6	45.6	45.0
7	53.2	52.5
8	60.8	60.0
9	68.4	67.5

	74	73
1	7.4	7.3
2	14.8	14.6
3	22.2	21.9
4	29.6	29.2
5	37.0	36.5
6	44.4	43.8
7	51.8	51.1
8	59.2	58.4
9	66.6	65.7

	72	71
1	7.2	7.1
2	14.4	14.2
3	21.6	21.3
4	28.8	28.4
5	36.0	35.5
6	43.2	42.6
7	50.4	49.7
8	57.6	56.8
9	64.8	63.9

Prop. Parts

80° — Common Logarithms of Trigonometric Functions — 80°

10° — Common Logarithms of Trigonometric Functions — 10°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.23 967		9.24 632		0.75 368	9.99 335		60
1	9.24 039	72	9.24 706	74	0.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	73	0.75 221	9.99 331	2	58
3	9.24 181	71	9.24 853	74	0.75 147	9.99 328	3	57
4	9.24 253	72	9.24 926	73	0.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	74	0.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	2	54
7	9.24 466	71	9.25 146	73	0.74 854	9.99 319	3	53
8	9.24 536	70	9.25 219	73	0.74 781	9.99 317	2	52
9	9.24 607	71	9.25 292	73	0.74 708	9.99 315	2	51
10	9.24 677	70	9.25 365	73	0.74 635	9.99 313	2	50
11	9.24 748	71	9.25 437	72	0.74 563	9.99 310	3	49
12	9.24 818	70	9.25 510	73	0.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	72	0.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	73	0.74 345	9.99 304	2	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	3	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44
17	9.25 168	70	9.25 871	72	0.74 129	9.99 297	2	43
18	9.25 237	69	9.25 943	72	0.74 057	9.99 294	3	42
19	9.25 307	70	9.26 015	72	0.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	71	0.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	72	0.73 842	9.99 288	2	39
22	9.25 514	69	9.26 229	71	0.73 771	9.99 285	3	38
23	9.25 583	69	9.26 301	72	0.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	2	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	3	35
26	9.25 790	69	9.26 514	71	0.73 486	9.99 276	2	34
27	9.25 858	68	9.26 585	71	0.73 415	9.99 274	2	33
28	9.25 927	69	9.26 655	70	0.73 345	9.99 271	3	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	71	0.73 203	9.99 267	2	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	3	29
32	9.26 199	68	9.26 937	70	0.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	71	0.72 992	9.99 260	2	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	3	26
35	9.26 403	68	9.27 148	70	0.72 852	9.99 255	2	25
36	9.26 470	67	9.27 218	70	0.72 782	9.99 252	3	24
37	9.26 538	68	9.27 288	70	0.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	69	0.72 643	9.99 248	2	22
39	9.26 672	67	9.27 427	70	0.72 573	9.99 245	3	21
40	9.26 739	67	9.27 496	69	0.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	70	0.72 434	9.99 241	2	19
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	3	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	2	17
44	9.27 007	67	9.27 773	69	0.72 227	9.99 233	3	16
45	9.27 073	66	9.27 842	69	0.72 158	9.99 231	2	15
46	9.27 140	67	9.27 911	69	0.72 089	9.99 229	2	14
47	9.27 206	66	9.27 980	69	0.72 020	9.99 226	3	13
48	9.27 273	67	9.28 049	69	0.71 951	9.99 224	2	12
49	9.27 339	66	9.28 117	68	0.71 883	9.99 221	3	11
50	9.27 405	66	9.28 186	69	0.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	68	0.71 746	9.99 217	2	9
52	9.27 537	66	9.28 323	69	0.71 677	9.99 214	3	8
53	9.27 602	65	9.28 391	68	0.71 609	9.99 212	2	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	3	6
55	9.27 734	66	9.28 527	68	0.71 473	9.99 207	2	5
56	9.27 799	65	9.28 595	68	0.71 405	9.99 204	3	4
57	9.27 864	65	9.28 662	67	0.71 338	9.99 202	2	3
58	9.27 930	66	9.28 730	68	0.71 270	9.99 200	2	2
59	9.27 995	65	9.28 798	68	0.71 202	9.99 197	3	1
60	9.28 060	65	9.28 865	67	0.71 135	9.99 195	2	0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	74	73	72	71	70	69	68	67	66	65	3
1	7.4	7.3	7.2	7.1	7.0	6.9	6.8	6.7	6.6	6.5	0.3
2	14.8	14.6	14.4	14.2	14.0	13.8	13.6	13.4	13.2	13.0	0.6
3	22.2	21.9	21.6	21.3	21.0	20.7	20.4	20.1	19.8	19.5	0.9
4	29.6	29.2	28.8	28.4	28.0	27.6	27.2	26.8	26.4	26.0	1.2
5	37.0	36.5	36.0	35.5	35.0	34.5	34.0	33.5	33.0	32.5	1.5
6	44.4	43.8	43.2	42.6	42.0	41.4	40.8	40.2	39.6	39.0	1.8
7	51.8	51.1	50.4	49.7	49.0	48.3	47.6	46.9	46.2	45.5	2.1
8	59.2	58.4	57.6	56.8	56.0	55.2	54.4	53.6	52.8	52.0	2.4
9	66.6	65.7	64 8	63.9	63.0	62.1	61.2	60.3	59.4	58.5	2.7

79° — Common Logarithms of Trigonometric Functions — 79°

11° — Common Logarithms of Trigonometric Functions — 11°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	68	67	66	65	64	63	62	61	60	59	3
1	6.8	6.7	6.6	6.5	6.4	6.3	6.2	6.1	6.0	5 9	0.3
2	13.6	13.4	13.2	13.0	12.8	12.6	12.4	12.2	12.0	11 8	0.6
3	20.4	20.1	19.8	19.5	19.2	18.9	18.6	18.3	18.0	17.7	0.9
4	27.2	26.8	26.4	26.0	25.6	25.2	24 8	24.4	24 0	23.6	1.2
5	34.0	33.5	33.0	32.5	32.0	31.5	31.0	30.5	30.0	29.5	1.5
6	40.8	40.2	39.6	39.0	38.4	37.8	37.2	36.6	36.0	35.4	1.8
7	47.6	46.9	46.2	45.5	44.8	44.1	43.4	42.7	42.0	41.3	2.1
8	54.4	53.6	52.8	52.0	51.2	50.4	49.6	48.8	48.0	47.2	2.4
9	61.2	60.3	59.4	58.5	57 6	56.7	55.8	54.9	54.0	53.1	2.7

78° — Common Logarithms of Trigonometric Functions — 78°

12° — Common Logarithms of Trigonometric Functions — 12°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	60
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	2	57
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	3	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	2	50
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	3	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	40
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	3	37
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	3	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	30
31	9.33 591	56	9.34 635	60	0.65 365	9.98 955	2	29
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755	59	0.65 245	9.98 950	3	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	20
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	3	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	10
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	3	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	63	62	61	60	59	58	57	56	55	3
1	6.3	6.2	6.1	6.0	5.9	5.8	5.7	5.6	5.5	0.3
2	12.6	12.4	12.2	12.0	11.8	11.6	11.4	11.2	11.0	0.6
3	18.9	18.6	18.3	18.0	17.7	17.4	17.1	16.8	16.5	0.9
4	25.2	24.8	24.4	24.0	23.6	23.2	22.8	22.4	22.0	1.2
5	31.5	31.0	30.5	30.0	29.5	29.0	28.5	28.0	27.5	1.5
6	37.8	37.2	36.6	36.0	35.4	34.8	34.2	33.6	33.0	1.8
7	44.1	43.4	42.7	42.0	41.3	40.6	39.9	39.2	38.5	2.1
8	50.4	49.6	48 8	48.0	47.2	46.4	45.6	44.8	44.0	2.4
9	56.7	55.8	54.9	54.0	53.1	52.2	51.3	50.4	49.5	2.7

77° — Common Logarithms of Trigonometric Functions — 77°

13° — Common Logarithms of Trigonometric Functions — 13°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	3	60
1	9.35 263	55	9.36 394	58	0.63 606	9.98 869	2	59
2	9.35 318	55	9.36 452	57	0.63 548	9.98 867	3	58
3	9.35 373	54	9.36 509	57	0.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	58	0.63 434	9.98 861	3	56
5	9.35 481	55	9.36 624	57	0.63 376	9.98 858	3	55
6	9.35 536	54	9.36 681	57	0.63 319	9.98 855	3	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	3	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	3	46
15	9.36 022	53	9.37 193	57	0.62 807	9.98 828	3	45
16	9.36 075	54	9.37 250	56	0.62 750	9.98 825	3	44
17	9.36 129	53	9.37 306	57	0.62 694	9.98 822	3	43
18	9.36 182	54	9.37 363	56	0.62 637	9.98 819	3	42
19	9.36 236	53	9.37 419	57	0.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39
22	9.36 395	54	9.37 588	56	0.62 412	9.98 807	3	38
23	9.36 449	53	9.37 644	56	0.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	3	35
26	9.36 608	52	9.37 812	56	0.62 188	9.98 795	3	34
27	9.36 660	53	9.37 868	56	0.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	55	0.62 020	9.98 786	3	31
30	9.36 819	52	9.38 035	56	0.61 965	9.98 783	3	30
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	3	29
32	9.36 924	52	9.38 147	55	0.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27
34	9.37 028	53	9.38 257	56	0.61 743	9.98 771	3	26
35	9.37 081	52	9.38 313	55	0.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	3	24
37	9.37 185	52	9.38 423	56	0.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	4	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	3	18
43	9.37 497	52	9.38 754	54	0.61 246	9.98 743	3	17
44	9.37 549	51	9.38 808	55	0.61 192	9.98 740	3	16
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	3	15
46	9.37 652	51	9.38 918	54	0.61 082	9.98 734	3	14
47	9.37 703	52	9.38 972	55	0.61 028	9.98 731	3	13
48	9.37 755	51	9.39 027	55	0.60 973	9.98 728	3	12
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725	3	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	4	9
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	3	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	4	1
60	9.38 368		9.39 677		0.60 323	9.98 690		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	58	57	56	55	54	53	52	51	4	3
1	5.8	5.7	5.6	5.5	5.4	5.3	5.2	5.1	0.4	0.3
2	11.6	11.4	11.2	11.0	10.8	10.6	10.4	10.2	0.8	0.6
3	17.4	17.1	16.8	16.5	16.2	15.9	15.6	15.3	1.2	0.9
4	23.2	22.8	22.4	22.0	21.6	21.2	20.8	20.4	1.6	1.2
5	29.0	28.5	28.0	27.5	27.0	26.5	26.0	25.5	2.0	1.5
6	34.8	34.2	33.6	33.0	32.4	31.8	31.2	30.6	2.4	1.8
7	40.6	39.9	39.2	38.5	37.8	37.1	36.4	35.7	2.8	2.1
8	46.4	45.6	44.8	44.0	43.2	42.4	41.6	40.8	3.2	2.4
9	52.2	51.3	50.4	49.5	48.6	47.7	46.8	45.9	3.6	2.7

76° — Common Logarithms of Trigonometric Functions — 76°

14° — Common Logarithms of Trigonometric Functions — 14°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	50	9.39 785	53	0.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	0.60 055	9.98 675	4	55
6	9.38 670	51	9.39 999	53	0.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	53	0.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425	53	0.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	40
21	9.39 418	49	9.40 795	52	0.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	4	36
25	9.39 615	49	9.41 005	52	0.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	0·58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	0.58 578	9.98 584	3	27
34	9.40 055	48	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	0.58 422	9.98 574	3	24
37	9.40 200	49	9.41 629	52	0.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	0.58 267	9.98 565	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	20
41	9.40 394	48	9.41 836	51	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	0.58 113	9.98 555	4	18
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	0.57 959	9.98 545	4	15
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805	9.98 535	4	12
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525	4	9
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	0.57 550	9.98 518	3	7
54	9.41 016	47	9.42 501	51	0.57 499	9.98 515	4	6
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	0.57 347	9.98 505	4	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	3	2
59	9.41 252	48	9.42 755	50	0.57 245	9.98 498	4	1
60	9.41 300		9.42 805		0.57 195	9.98 494		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	54	53	52	51	50	49	48	47	4	3
1	5.4	5.3	5.2	5.1	5.0	4.9	4.8	4.7	0.4	0.3
2	10.8	10.6	10.4	10.2	10.0	9.8	9.6	9.4	0.8	0.6
3	16.2	15.9	15.6	15.3	15.0	14.7	14.4	14.1	1.2	0.9
4	21.6	21.2	20.8	20.4	20.0	19.6	19.2	18.8	1.6	1.2
5	27.0	26.5	26.0	25.5	25.0	24.5	24.0	23.5	2.0	1.5
6	32.4	31.8	31.2	30.6	30.0	29.4	28.8	28.2	2.4	1.8
7	37.8	37.1	36.4	35.7	35.0	34.3	33.6	32.9	2.8	2.1
8	43.2	42.4	41.6	40.8	40.0	39.2	38.4	37.6	3.2	2.4
9	48.6	47.7	46.8	45.9	45.0	44.1	43.2	42.3	3.6	2.7

75° — Common Logarithms of Trigonometric Functions — 75°

15° — Common Logarithms of Trigonometric Functions — 15°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.41 300	47	9.42 805	51	0.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	0.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 750		0.54 250	9.98 284		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	51	50	49	48	47	46	45	44	4	3
1	5.1	5.0	4.9	4.8	4.7	4.6	4.5	4.4	0.4	0.3
2	10.2	10.0	9.8	9.6	9.4	9.2	9.0	8.8	0.8	0.6
3	15.3	15.0	14.7	14.4	14.1	13.8	13.5	13.2	1.2	0.9
4	20.4	20.0	19.6	19.2	18.8	18.4	18.0	17.6	1.6	1.2
5	25.5	25.0	24.5	24.0	23.5	23.0	22.5	22.0	2.0	1.5
6	30.6	30.0	29.4	28.8	28.2	27.6	27.0	26.4	2.4	1.8
7	35.7	35.0	34.3	33.6	32.9	32.2	31.5	30.8	2.8	2.1
8	40.8	40.0	39.2	38.4	37.6	36.8	36.0	35.2	3.2	2.4
9	45.9	45.0	44.1	43.2	42.3	41.4	40.5	39.6	3.6	2.7

Prop. Parts

74° — Common Logarithms of Trigonometric Functions — 74°

16° — Common Logarithms of Trigonometric Functions — 16°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	60
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	0.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	0.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	0.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	0.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	0.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	0.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	4	50
11	9.44 516	43	9.46 271	48	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	0.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	0.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	0.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	0.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	0.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	40
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	0.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	0.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	0.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	0.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	0.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	30
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	0.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	20
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	48	47
1	4.8	4.7
2	9.6	9.4
3	14.4	14.1
4	19.2	18.8
5	24.0	23.5
6	28.8	28.2
7	33.6	32.9
8	38.4	37.6
9	43.2	42.3

	46	45
1	4.6	4.5
2	9.2	9.0
3	13.8	13.5
4	18.4	18.0
5	23.0	22.5
6	27.6	27.0
7	32.2	31.5
8	36.8	36.0
9	41.4	40.5

	44	43
1	4.4	4.3
2	8.8	8.6
3	13.2	12.9
4	17.6	17.2
5	22.0	21.5
6	26.4	25.8
7	30.8	30.1
8	35.2	34.4
9	39.6	38.7

	42	41
1	4.2	4.1
2	8.4	8.2
3	12.6	12.3
4	16.8	16.4
5	21.0	20.5
6	25.2	24.6
7	29.4	28.7
8	33.6	32.8
9	37.8	36.9

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

Prop. Parts

73° — Common Logarithms of Trigonometric Functions — 73°

17° — Common Logarithms of Trigonometric Functions — 17°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	45	44	43	42	41	40	39	5	4	3
1	4.5	4.4	4.3	4.2	4.1	4.0	3.9	0.5	0.4	0.3
2	9.0	8.8	8.6	8.4	8.2	8 0	7.8	1.0	0.8	0.6
3	13.5	13.2	12.9	12.6	12.3	12.0	11.7	1.5	1.2	0.9
4	18.0	17.6	17.2	16.8	16.4	16.0	15.6	2.0	1.6	1.2
5	22.5	22.0	21.5	21.0	20.5	20.0	19.5	2.5	2.0	1.5
6	27.0	26.4	25.8	25.2	24.6	24.0	23.4	3.0	2.4	1.8
7	31.5	30 8	30.1	29.4	28.7	28.0	27.3	3.5	2.8	2.1
8	36.0	35.2	34.4	33.6	32.8	32.0	31.2	4.0	3.2	2.4
9	40.5	39.6	38.7	37.8	36.9	36.0	35.1	4.5	3.6	2.7

72° — Common Logarithms of Trigonometric Functions — 72°

18° — Common Logarithms of Trigonometric Functions — 18°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	0.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	0.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	43	42
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8

	41	39
1	4.1	3.9
2	8.2	7.8
3	12.3	11.7
4	16.4	15.6
5	20.5	19.5
6	24.6	23.4
7	28.7	27.3
8	32.8	31.2
9	36.9	35.1

	38	37
1	3.8	3.7
2	7.6	7.4
3	11.4	11.1
4	15.2	14.8
5	19.0	18.5
6	22.8	22.2
7	26.6	25.9
8	30.4	29.6
9	34.2	33.3

	36
1	3.6
2	7.2
3	10.8
4	14.4
5	18.0
6	21.6
7	25.2
8	28.8
9	32.4

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

71° — Common Logarithms of Trigonometric Functions — 71°

19° — Common Logarithms of Trigonometric Functions — 19°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	**60**
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	**55**
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	**50**
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	**45**
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	**40**
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	**35**
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	**30**
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	4	**25**
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	**20**
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	**15**
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	**10**
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	**5**
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	41	40
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0

	39	37
1	3.9	3.7
2	7.8	7.4
3	11.7	11.1
4	15.6	14.8
5	19.5	18.5
6	23.4	22.2
7	27.3	25.9
8	31.2	29.6
9	35.1	33.3

	36	35
1	3.6	3.5
2	7.2	7.0
3	10.8	10.5
4	14.4	14.0
5	18.0	17.5
6	21.6	21.0
7	25.2	24.5
8	28.8	28.0
9	32.4	31.5

	34
1	3.4
2	6.8
3	10.2
4	13.6
5	17.0
6	20.4
7	23.8
8	27.2
9	30.6

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

Prop. Parts

70° — Common Logarithms of Trigonometric Functions — 70°

20° — Common Logarithms of Trigonometric Functions — 20°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	0.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	0.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	0.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	0.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	0.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	0.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	0.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	0.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	0.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 236	39	0.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	0.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	0.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	0.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	0.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	0.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	0.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	0.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	0.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	0.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	40	39	38	37	35	34	33	5	4
1	4.0	3.9	3.8	3.7	3.5	3.4	3.3	0.5	0.4
2	8.0	7.8	7.6	7.4	7.0	6.8	6.6	1.0	0.8
3	12.0	11.7	11.4	11.1	10.5	10.2	9.9	1.5	1.2
4	16.0	15.6	15.2	14.8	14.0	13.6	13.2	2.0	1.6
5	20.0	19.5	19.0	18.5	17.5	17.0	16.5	2.5	2.0
6	24.0	23.4	22.8	22.2	21.0	20.4	19.8	3.0	2.4
7	28.0	27.3	26.6	25.9	24.5	23.8	23.1	3.5	2.8
8	32.0	31.2	30.4	29.6	28.0	27.2	26.4	4.0	3.2
9	36.0	35.1	34.2	33.3	31.5	30.6	29.7	4.5	3.6

69° — Common Logarithms of Trigonometric Functions — 69°

21° — Common Logarithms of Trigonometric Functions — 21°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	**60**
1	9.55 466	33	9.58 455	38	0.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	**55**
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	**50**
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	**45**
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	**40**
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	**35**
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	**30**
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	0.40 275	9.96 843	5	**25**
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	**20**
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	**15**
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	**10**
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	**5**
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	38	37
1	3.8	3.7
2	7.6	7.4
3	11.4	11.1
4	15.2	14.8
5	19.0	18.5
6	22.8	22.2
7	26.6	25.9
8	30.4	29.6
9	34.2	33.3

	36	33
1	3.6	3.3
2	7.2	6.6
3	10.8	9.9
4	14.4	13.2
5	18.0	16.5
6	21.6	19.8
7	25.2	23.1
8	28.8	26.4
9	32.4	29.7

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

68° — Common Logarithms of Trigonometric Functions — 68°

22° — Common Logarithms of Trigonometric Functions — 22°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	32
1	3.5	3.2
2	7.0	6.4
3	10.5	9.6
4	14.0	12.8
5	17.5	16.0
6	21.0	19.2
7	24.5	22.4
8	28.0	25.6
9	31.5	28.8

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29
1	2.9
2	5.8
3	8.7
4	11.6
5	14.5
6	17.4
7	20.3
8	23.2
9	26.1

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

67° — Common Logarithms of Trigonometric Functions — 67°

23° — Common Logarithms of Trigonometric Functions — 23°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.59 188	30	9.62 785	35	0.37 215	9.96 403	6	60
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865	9.96 349	6	50
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	0.35 825	9.96 185	6	20
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	0.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	36	35
1	3.6	3.5
2	7.2	7.0
3	10.8	10.5
4	14.4	14.0
5	18.0	17.5
6	21.6	21.0
7	25.2	24.5
8	28.8	28.0
9	32.4	31.5

	34	30
1	3.4	3.0
2	6.8	6.0
3	10.2	9.0
4	13.6	12.0
5	17.0	15.0
6	20.4	18.0
7	23.8	21.0
8	27.2	24.0
9	30.6	27.0

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

66° — Common Logarithms of Trigonometric Functions — 66°

24° — Common Logarithms of Trigonometric Functions — 24°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	60
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	59
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	6	58
3	9.61 016	29	9.64 960	34	0.35 040	9.96 056	6	57
4	9.61 045	28	9.64 994	34	0.35 006	9.96 050	5	56
5	9.61 073	28	9.65 028	34	0.34 972	9.96 045	6	55
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	5	54
7	9.61 129	29	9.65 096	34	0.34 904	9.96 034	6	53
8	9.61 158	28	9.65 130	34	0.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	33	0.34 836	9.96 022	5	51
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	50
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	0.34 735	9.96 005	5	48
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	6	47
14	9.61 326	28	9.65 333	33	0.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	34	0.34 634	9.95 988	6	45
16	9.61 382	29	9.65 400	34	0.34 600	9.95 982	5	44
17	9.61 411	27	9.65 434	33	0.34 566	9.95 977	6	43
18	9.61 438	28	9.65 467	34	0.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	41
20	9.61 494	28	9.65 535	33	0.34 465	9.95 960	6	40
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	33	0.34 364	9.95 942	5	37
24	9.61 606	28	9.65 669	34	0.34 331	9.95 937	6	36
25	9.61 634	28	9.65 703	33	0.34 297	9.95 931	6	35
26	9.61 662	27	9.65 736	34	0.34 264	9.95 925	5	34
27	9.61 689	28	9.65 770	33	0.34 230	9.95 920	6	33
28	9.61 717	28	9.65 803	34	0.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	33	0.34 163	9.95 908	6	31
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	30
31	9.61 800	28	9.65 904	33	0.34 096	9.95 897	6	29
32	9.61 828	28	9.65 937	34	0.34 063	9.95 891	6	28
33	9.61 856	27	9.65 971	33	0.34 029	9.95 885	6	27
34	9.61 883	28	9.66 004	34	0.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	33	0.33 962	9.95 873	5	25
36	9.61 939	27	9.66 071	33	0.33 929	9.95 868	6	24
37	9.61 966	28	9.66 104	34	0.33 896	9.95 862	6	23
38	9.61 994	27	9.66 138	33	0.33 862	9.95 856	6	22
39	9.62 021	28	9.66 171	33	0.33 829	9.95 850	6	21
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	20
41	9.62 076	28	9.66 238	33	0.33 762	9.95 839	6	19
42	9.62 104	27	9.66 271	33	0.33 729	9.95 833	6	18
43	9.62 131	28	9.66 304	33	0.33 696	9.95 827	6	17
44	9.62 159	27	9.66 337	34	0.33 663	9.95 821	6	16
45	9.62 186	28	9.66 371	33	0.33 629	9.95 815	5	15
46	9.62 214	27	9.66 404	33	0.33 596	9.95 810	6	14
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	13
48	9.62 268	28	9.66 470	33	0.33 530	9.95 798	6	12
49	9.62 296	27	9.66 503	34	0.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	10
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	9
52	9.62 377	28	9.66 603	33	0.33 397	9.95 775	6	8
53	9.62 405	27	9.66 636	33	0.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	0.33 265	9.95 751	6	4
57	9.62 513	28	9.66 768	33	0.33 232	9.95 745	6	3
58	9.62 541	27	9.66 801	33	0.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	1
60	9.62 595		9.66 867		0.33 133	9.95 728		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	34	33
1	3.4	3.3
2	6.8	6.6
3	10.2	9.9
4	13.6	13.2
5	17.0	16.5
6	20.4	19.8
7	23.8	23.1
8	27.2	26.4
9	30.6	29.7

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27
1	2.7
2	5.4
3	8.1
4	10.8
5	13.5
6	16.2
7	18.9
8	21.6
9	24.3

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

65° — Common Logarithms of Trigonometric Functions — 65°

25° — Common Logarithms of Trigonometric Functions — 25°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.62 595	27	9.66 867	33	0.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	5	50
11	9.62 892	26	9.67 229	33	0.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	0.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	0.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	0.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	9.63 425	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	0.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	10
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

64° — Common Logarithms of Trigonometric Functions — 64°

26° — Common Logarithms of Trigonometric Functions — 26°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	60
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	50
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	0.30 575	9,95 248	6	41
20	9.64 698	26	9.69˙457	31	0.30 543	9.95 242	6	40
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	0·30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	0.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	30
31	9.64 978	25	9.69 805	32	0.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	0.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	20
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	0.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	10
51	9.65 481	25	9.70 435	31	0.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	0.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	0.29 315	9.94 995	7	1
60	9.65 705		9.70 717		0.29 283	9.94 988		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	24
1	2.4
2	4.8
3	7.2
4	9.6
5	12.0
6	14.4
7	16.8
8	19.2
9	21.6

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5 4

Prop. Parts

63° — Common Logarithms of Trigonometric Functions — 63°

27° — Common Logarithms of Trigonometric Functions — 27°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.65 705	24	9.70 717	31	0.29 283	9.94 988	6	60
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	0.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	6	54
7	9.65 878	24	9.70 935	31	0.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	50
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	0.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	0.28 847	9.94 898	7	46
15	9.66 075	24	9.71 184	31	0.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215	31	0.28 785	9.94 885	7	44
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	0.28 692	9.94 865	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	40
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	0.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	0.28 569	9.94 839	7	37
24	9.66 295	24	9.71 462	31	0.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	0.28 445	9.94 813	7	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	30
31	9.66 465	24	9.71 679	30	0.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	0.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	0.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925	30	0.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	0.28 045	9.94 727	7	20
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18
43	9.66 755	24	9.72 048	30	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	0.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	0.27 830	9.94 680	6	13
48	9.66 875	24	9.72 201	30	0.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	10
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	0.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	0.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415	30	0.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	0.27 555	9.94 620	6	4
57	9.67 090	23	9.72 476	30	0.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	0.27 463	9.94 600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	30	25
1	3.0	2.5
2	6.0	5.0
3	9.0	7.5
4	12.0	10.0
5	15.0	12.5
6	18.0	15.0
7	21.0	17.5
8	24.0	20.0
9	27.0	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

Prop. Parts

62° — Common Logarithms of Trigonometric Functions — 62°

28° — Common Logarithms of Trigonometric Functions — 28°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
1	9.67 185	23	9.72 598	30	0.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	0.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	0.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	0.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	0.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	0.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	0.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	0.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	0.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	0.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	0.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	0.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	0.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	0.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	0.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	0.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	0.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	30	0.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205	30	0.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	0.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	0.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	0.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	0.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	0.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	0.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	0.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	0.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	0.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	0.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	0.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	0.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1
60	9.68 557		9.74 375		0.25 625	9.94 182		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29	24
1	2.9	2.4
2	5.8	4.8
3	8.7	7.2
4	11.6	9.6
5	14.5	12.0
6	17.4	14.4
7	20.3	16.8
8	23.2	19.2
9	26.1	21.6

	23	22
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

Prop. Parts

61° — Common Logarithms of Trigonometric Functions — 61°

29° — Common Logarithms of Trigonometric Functions — 29°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	**60**
1	9.68 580	23	9.74 405	30	0.25 595	9.94 175	7	59
2	9.68 603	22	9.74 435	30	0.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465	29	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	**55**
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	**50**
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105	7	49
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47
14	9.68 875	22	9.74 791	30	0.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	**45**
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43
18	9.68 965	22	9.74 910	29	0.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	0.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	**40**
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39
22	9.69 055	22	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	**35**
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	0.24 795	9.93 984	7	32
29	9.69 212	22	9.75 235	29	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	**30**
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	**25**
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	0.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	**20**
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	0.24 295	9.93 862	7	**15**
46	9.69 589	22	9.75 735	29	0.24 265	9.93 855	8	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	**10**
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	**5**
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115	29	0.23 885	9.93 760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	30	29
1	3.0	2.9
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	23	22
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

Prop. Parts

60° — Common Logarithms of Trigonometric Functions — 60°

30° — Common Logarithms of Trigonometric Functions — 30°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	20
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	30	29
1	3.0	2.9
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	28
1	2.8
2	5.6
3	8.4
4	11.2
5	14.0
6	16.8
7	19.6
8	22.4
9	25.2

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

Prop. Parts

59° — Common Logarithms of Trigonometric Functions — 59°

31° — Common Logarithms of Trigonometric Functions — 31°

'	L Sin	d	L Tan	cd	L Ctn	L Cos	d	'
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	60
1	9.71 205	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	20
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		0
'	L Cos	d	L Ctn	cd	L Tan	L Sin	d	'

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	21	20
1	2.1	2.0
2	4.2	4.0
3	6.3	6.0
4	8.4	8.0
5	10.5	10.0
6	12.6	12.0
7	14.7	14.0
8	16.8	16.0
9	18.9	18.0

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

Prop. Parts

58° — Common Logarithms of Trigonometric Functions — 58°

32° — Common Logarithms of Trigonometric Functions — 32°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	21
1	2.7	2.1
2	5.4	4.2
3	8.1	6.3
4	10.8	8.4
5	13.5	10.5
6	16.2	12.6
7	18.9	14.7
8	21.6	16.8
9	24.3	18.9

	20	19
1	2.0	1.9
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Parts

57° — Common Logarithms of Trigonometric Functions — 57°

33° — Common Logarithms of Trigonometric Functions — 33°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	**60**
1	9.73 630	20	9.81 279	28	0.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	**55**
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	**50**
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	**45**
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	**40**
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	**35**
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	**30**
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	0.17 785	9.92 069	9	**25**
36	9.74 303	19	9.82 243	27	0.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	0.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	**20**
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	0.17 511	9.91 985	9	**15**
46	9.74 493	19	9.82 517	27	0.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	**10**
51	9.74 587	19	9.82 653	28	0.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	0.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	0.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	0.17 238	9.91 900	9	**5**
56	9.74 681	19	9.82 790	27	0.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	0.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	0.17 129	9.91 866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	20	19
1	2.0	1.9
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

Prop. Parts

56° — Common Logarithms of Trigonometric Functions — 56°

34° — Common Logarithms of Trigonometric Functions — 34°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	**60**
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	**55**
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	**50**
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	**45**
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	**40**
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	**35**
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	**30**
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	**25**
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	**20**
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	**15**
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	**10**
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	**5**
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	26
1	2.6
2	5.2
3	7.8
4	10.4
5	13.0
6	15.6
7	18.2
8	20.8
9	23.4

	19	18
1	1.9	1.8
2	3.8	3.6
3	5.7	5.4
4	7.6	7.2
5	9.5	9.0
6	11.4	10.8
7	13.3	12.6
8	15.2	14.4
9	17.1	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

Prop. Parts

55° — Common Logarithms of Trigonometric Functions — 55°

35° — Common Logarithms of Trigonometric Functions — 35°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	**60**
1	9.75 877	18	9.84 550	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	**55**
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	**50**
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075	9.91 203	9	**45**
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	**40**
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	**35**
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	**30**
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	**25**
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	**20**
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	**15**
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	0.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	**10**
51	9.76 765	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	0.14 007	9.90 842	10	**5**
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

Prop. Parts

54° — Common Logarithms of Trigonometric Functions — 54°

36° — Common Logarithms of Trigonometric Functions — 36°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	0.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	0.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	0.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	0.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	0.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	0.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	0.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	0.12 315	9.90 244	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	27	26	18	17	16	10	9
1	2.7	2.6	1.8	1.7	1.6	1.0	0.9
2	5.4	5.2	3.6	3.4	3.2	2.0	1.8
3	8.1	7.8	5.4	5.1	4.8	3.0	2.7
4	10.8	10.4	7.2	6.8	6.4	4.0	3.6
5	13.5	13.0	9.0	8.5	8.0	5.0	4.5
6	16.2	15.6	10.8	10.2	9.6	6.0	5.4
7	18.9	18.2	12.6	11.9	11.2	7.0	6.3
8	21.6	20.8	14.4	13.6	12.8	8.0	7.2
9	24.3	23.4	16.2	15.3	14.4	9.0	8.1

53° — Common Logarithms of Trigonometric Functions — 53°

37° — Common Logarithms of Trigonometric Functions — 37°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	**60**
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	**55**
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	**50**
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895	9.90 091	9	**45**
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	**40**
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	**35**
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	**30**
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	**25**
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	**20**
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	**15**
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	**10**
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	**5**
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	17	16
1	1.7	1.6
2	3.4	3.2
3	5.1	4.8
4	6.8	6.4
5	8.5	8.0
6	10.2	9.6
7	11.9	11.2
8	13.6	12.8
9	15.3	14.4

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

Prop. Parts

52° — Common Logarithms of Trigonometric Functions — 52°

38° — Common Logarithms of Trigonometric Functions — 38°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25	17	16	15	11	10	9
1	2.6	2.5	1.7	1.6	1.5	1.1	1.0	0.9
2	5.2	5.0	3.4	3.2	3.0	2.2	2.0	1.8
3	7.8	7.5	5.1	4.8	4.5	3.3	3.0	2.7
4	10.4	10.0	6.8	6.4	6.0	4.4	4.0	3.6
5	13.0	12.5	8.5	8.0	7.5	5.5	5.0	4.5
6	15.6	15.0	10.2	9.6	9.0	6.6	6.0	5.4
7	18.2	17.5	11.9	11.2	10.5	7.7	7.0	6.3
8	20.8	20.0	13.6	12.8	12.0	8.8	8.0	7.2
9	23.4	22.5	15.3	14.4	13.5	9.9	9.0	8.1

51° — Common Logarithms of Trigonometric Functions — 51°

39° — Common Logarithms of Trigonometric Functions — 39°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	**60**
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	**55**
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.09 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	**50**
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	**45**
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	**40**
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	**35**
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	**30**
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	**25**
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	**20**
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	**15**
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	**10**
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	**5**
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

Prop. Parts

50° — Common Logarithms of Trigonometric Functions — 50°

40° — Common Logarithms of Trigonometric Functions — 40°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	**60**
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	**55**
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	**50**
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	**45**
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	**40**
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	**35**
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	**30**
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	**25**
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	**20**
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	**15**
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	**10**
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	**5**
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25	15	14	11	10
1	2.6	2.5	1.5	1.4	1.1	1.0
2	5.2	5.0	3.0	2.8	2.2	2.0
3	7.8	7.5	4.5	4.2	3.3	3.0
4	10.4	10.0	6.0	5.6	4.4	4.0
5	13.0	12.5	7.5	7.0	5.5	5.0
6	15.6	15.0	9.0	8.4	6.6	6.0
7	18.2	17.5	10.5	9.8	7.7	7.0
8	20.8	20.0	12.0	11.2	8.8	8.0
9	23.4	22.5	13.5	12.6	9.9	9.0

49° — Common Logarithms of Trigonometric Functions — 49°

41° — Common Logarithms of Trigonometric Functions — 41°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	0.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	0.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	20
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	0.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

Prop. Parts

48° — Common Logarithms of Trigonometric Functions — 48°

42° — Common Logarithms of Trigonometric Functions — 42°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	0.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	0.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	0.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	0.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	0.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	0.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	0.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	0.03 060	9.86 425	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

Prop. Parts

47° — Common Logarithms of Trigonometric Functions — 47°

43° — Common Logarithms of Trigonometric Functions — 43°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

Prop. Parts

46° — Common Logarithms of Trigonometric Functions — 46°

44° — Common Logarithms of Trigonometric Functions — 44°

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	**60**
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	0.01 415	9.85 645	13	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	**55**
6	9.84 255	14	9.98 635	26	0.01 365	9.85 620	12	54
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	13	52
9	9.84 295	13	9.98 711	26	0.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	**50**
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	13	**45**
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	0.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	**40**
21	9.84 450	13	9.99 015	25	0.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	0.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	**35**
26	9.84 515	13	9.99 141	25	0.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	**30**
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28
33	9.84 605	13	9.99 318	25	0.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	**25**
36	9.84 643	13	9.99 394	25	0.00 606	9.85 250	13	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225	13	22
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495	25	0.00 505	9.85 200	13	**20**
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	0.00 455	9.85 175	13	18
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	0.00 404	9.85 150	13	16
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	12	**15**
46	9.84 771	13	9.99 646	26	0.00 354	9.85 125	13	14
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	**10**
51	9.84 835	12	9.99 773	25	0.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	13	**5**
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975	25	0.00 025	9.84 961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		**0**
′	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

Prop. Parts

45° — Common Logarithms of Trigonometric Functions — 45°

TABLE 4

1 — Powers, Roots, Reciprocals — 50

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	$1000/N$
1	1	1.00 000	3.16 228	1	1.00 000	2.15 443	4.64 159	1000.00
2	4	1.41 421	4.47 214	8	1.25 992	2.71 442	5.84 804	500.00 0
3	9	1.73 205	5.47 723	27	1.44 225	3.10 723	6.69 433	333.33 3
4	16	2.00 000	6.32 456	64	1.58 740	3.41 995	7.36 806	250.00 0
5	25	2.23 607	7.07 107	125	1.70 998	3.68 403	7.93 701	200.00 0
6	36	2.44 949	7.74 597	216	1.81 712	3.91 487	8.43 433	166.66 7
7	49	2.64 575	8.36 660	343	1.91 293	4.12 129	8.87 904	142.85 7
8	64	2.82 843	8.94 427	512	2.00 000	4.30 887	9.28 318	125.00 0
9	81	3.00 000	9.48 683	729	2.08 008	4.48 140	9.65 489	111.11 1
10	100	3.16 228	10.00 00	1 000	2.15 443	4.64 159	10.00 00	100.00 0
11	121	3.31 662	10.48 81	1 331	2.22 398	4.79 142	10.32 28	90.90 91
12	144	3.46 410	10.95 45	1 728	2.28 943	4.93 242	10.62 66	83.33 33
13	169	3.60 555	11.40 18	2 197	2.35 133	5.06 580	10.91 39	76.92 31
14	196	3.74 166	11.83 22	2 744	2.41 014	5.19 249	11.18 69	71.42 86
15	225	3.87 298	12.24 74	3 375	2.46 621	5.31 329	11.44 71	66.66 67
16	256	4.00 000	12.64 91	4 096	2.51 984	5.42 884	11.69 61	62.50 00
17	289	4.12 311	13.03 84	4 913	2.57 128	5.53 966	11.93 48	58.82 35
18	324	4.24 264	13.41 64	5 832	2.62 074	5.64 622	12.16 44	55.55 56
19	361	4.35 890	13.78 40	6 859	2.66 840	5.74 890	12.38 56	52.63 16
20	400	4.47 214	14.14 21	8 000	2.71 442	5.84 804	12.59 92	50.00 00
21	441	4.58 258	14.49 14	9 261	2.75 892	5.94 392	12.80 58	47.61 90
22	484	4.69 042	14.83 24	10 648	2.80 204	6.03 681	13.00 59	45.45 45
23	529	4.79 583	15.16 58	12 167	2.84 387	6.12 693	13.20 01	43.47 83
24	576	4.89 898	15.49 19	13 824	2.88 450	6.21 446	13.38 87	41.66 67
25	625	5.00 000	15.81 14	15 625	2.92 402	6.29 961	13.57 21	40.00 00
26	676	5.09 902	16.12 45	17 576	2.96 250	6.38 250	13.75 07	38.46 15
27	729	5.19 615	16.43 17	19 683	3.00 000	6.46 330	13.92 48	37.03 70
28	784	5.29 150	16.73 32	21 952	3.03 659	6.54 213	14.09 46	35.71 43
29	841	5.38 516	17.02 94	24 389	3.07 232	6.61 911	14.26 04	34.48 28
30	900	5.47 723	17.32 05	27 000	3.10 723	6.69 433	14.42 25	33.33 33
31	961	5.56 776	17.60 68	29 791	3.14 138	6.76 790	14.58 10	32.25 81
32	1 024	5.65 685	17.88 85	32 768	3.17 480	6.83 990	14.73 61	31.25 00
33	1 089	5.74 456	18.16 59	35 937	3.20 753	6.91 042	14.88 81	30.30 30
34	1 156	5.83 095	18.43 91	39 304	3.23 961	6.97 953	15.03 69	29.41 18
35	1 225	5.91 608	18.70 83	42 875	3.27 107	7.04 730	15.18 29	28.57 14
36	1 296	6.00 000	18.97 37	46 656	3.30 193	7.11 379	15.32 62	27.77 78
37	1 369	6.08 276	19.23 54	50 653	3.33 222	7.17 905	15.46 68	27.02 70
38	1 444	6.16 441	19.49 36	54 872	3.36 198	7.24 316	15.60 49	26.31 58
39	1 521	6.24 500	19.74 84	59 319	3.39 121	7.30 614	15.74 06	25.64 10
40	1 600	6.32 456	20.00 00	64 000	3.41 995	7.36 806	15.87 40	25.00 00
41	1 681	6.40 312	20.24 85	68 921	3.44 822	7.42 896	16.00 52	24.39 02
42	1 764	6.48 074	20.49 39	74 088	3.47 603	7.48 887	16.13 43	23.80 95
43	1 849	6.55 744	20.73 64	79 507	3.50 340	7.54 784	16.26 13	23.25 58
44	1 936	6.63 325	20.97 62	85 184	3.53 035	7.60 590	16.38 64	22.72 73
45	2 025	6.70 820	21.21 32	91 125	3.55 689	7.66 309	16.50 96	22.22 22
46	2 116	6.78 233	21.44 76	97 336	3.58 305	7.71 944	16.63 10	21.73 91
47	2 209	6.85 565	21.67 95	103 823	3.60 883	7.77 498	16.75 07	21.27 66
48	2 304	6.92 820	21.90 89	110 592	3.63 424	7.82 974	16.86 87	20.83 33
49	2 401	7.00 000	22.13 59	117 649	3.65 931	7.88 374	16.98 50	20.40 82
50	2 500	7.07 107	22.36 07	125 000	3.68 403	7.93 701	17.09 98	20.00 00
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	$1000/N$

1 — Powers, Roots, Reciprocals — 50

50 — Powers, Roots, Reciprocals — 100

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	$1000/N$
50	2 500	7.07 107	22.36 07	125 000	3.68 403	7.93 701	17.09 98	20.00 00
51	2 601	7.14 143	22.58 32	132 651	3.70 843	7.98 957	17.21 30	19.60 78
52	2 704	7.21 110	22.80 35	140 608	3.73 251	8.04 145	17.32 48	19.23 08
53	2 809	7.28 011	23.02 17	148 877	3.75 629	8.09 267	17.43 51	18.86 79
54	2 916	7.34 847	23.23 79	157 464	3.77 976	8.14 325	17.54 41	18.51 85
55	3 025	7.41 620	23.45 21	166 375	3.80 295	8.19 321	17.65 17	18.18 18
56	3 136	7.48 331	23.66 43	175 616	3.82 586	8.24 257	17.75 81	17.85 71
57	3 249	7.54 983	23.87 47	185 193	3.84 850	8.29 134	17.86 32	17.54 39
58	3 364	7.61 577	24.08 32	195 112	3.87 088	8.33 955	17.96 70	17.24 14
59	3 481	7.68 115	24.28 99	205 379	3.89 300	8.38 721	18.06 97	16.94 92
60	3 600	7.74 597	24.49 49	216 000	3.91 487	8.43 433	18.17 12	16.66 67
61	3 721	7.81 025	24.69 82	226 981	3.93 650	8.48 093	18.27 16	16.39 34
62	3 844	7.87 401	24.89 98	238 328	3.95 789	8.52 702	18.37 09	16.12 90
63	3 969	7.93 725	25.09 98	250 047	3.97 906	8.57 262	18.46 91	15.87 30
64	4 096	8.00 000	25.29 82	262 144	4.00 000	8.61 774	18.56 64	15.62 50
65	4 225	8.06 226	25.49 51	274 625	4.02 073	8.66 239	18.66 26	15.38 46
66	4 356	8.12 404	25.69 05	287 496	4.04 124	8.70 659	18.75 78	15.15 15
67	4 489	8.18 535	25.88 44	300 763	4.06 155	8.75 034	18.85 20	14.92 54
68	4 624	8.24 621	26.07 68	314 432	4.08 166	8.79 366	18.94 54	14.70 59
69	4 761	8.30 662	26.26 79	328 509	4.10 157	8.83 656	19.03 78	14.49 28
70	4 900	8.36 660	26.45 75	343 000	4.12 129	8.87 904	19.12 93	14.28 57
71	5 041	8.42 615	26.64 58	357 911	4.14 082	8.92 112	19.22 00	14.08 45
72	5 184	8.48 528	26.83 28	373 248	4.16 017	8.96 281	19.30 98	13.88 89
73	5 329	8.54 400	27.01 85	389 017	4.17 934	9.00 411	19.39 88	13.69 86
74	5 476	8.60 233	27.20 29	405 224	4.19 834	9.04 504	19.48 70	13.51 35
75	5 625	8.66 025	27.38 61	421 875	4.21 716	9.08 560	19.57 43	13.33 33
76	5 776	8.71 780	27.56 81	438 976	4.23 582	9.12 581	19.66 10	13.15 79
77	5 929	8.77 496	27.74 89	456 533	4.25 432	9.16 566	19.74 68	12.98 70
78	6 084	8.83 176	27.92 85	474 552	4.27 266	9.20 516	19.83 19	12.82 05
79	6 241	8.88 819	28.10 69	493 039	4.29 084	9.24 434	19.91 63	12.65 82
80	6 400	8.94 427	28.28 43	512 000	4.30 887	9.28 318	20.00 00	12.50 00
81	6 561	9.00 000	28.46 05	531 441	4.32 675	9.32 170	20.08 30	12.34 57
82	6 724	9.05 539	28.63 56	551 368	4.34 448	9.35 990	20.16 53	12.19 51
83	6 889	9.11 043	28.80 97	571 787	4.36 207	9.39 780	20.24 69	12.04 82
84	7 056	9.16 515	28.98 28	592 704	4.37 952	9.43 539	20.32 79	11.90 48
85	7 225	9.21 954	29.15 48	614 125	4.39 683	9.47 268	20.40 83	11.76 47
86	7 396	9.27 362	29.32 58	636 056	4.41 400	9.50 969	20.48 80	11.62 79
87	7 569	9.32 738	29.49 58	658 503	4.43 105	9.54 640	20.56 71	11.49 43
88	7 744	9.38 083	29.66 48	681 472	4.44 796	9.58 284	20.64 56	11.36 36
89	7 921	9.43 398	29.83 29	704 969	4.46 475	9.61 900	20.72 35	11.23 60
90	8 100	9.48 683	30.00 00	729 000	4.48 140	9.65 489	20.80 08	11.11 11
91	8 281	9.53 939	30.16 62	753 571	4.49 794	9.69 052	20.87 76	10.98 90
92	8 464	9.59 166	30.33 15	778 688	4.51 436	9.72 589	20.95 38	10.86 96
93	8 649	9.64 365	30.49 59	804 357	4.53 065	9.76 100	21.02 94	10.75 27
94	8 836	9.69 536	30.65 94	830 584	4.54 684	9.79 586	21.10 45	10.63 83
95	9 025	9.74 679	30.82 21	857 375	4.56 290	9.83 048	21.17 91	10.52 63
96	9 216	9.79 796	30.98 39	884 736	4.57 886	9.86 485	21.25 32	10.41 67
97	9 409	9.84 886	31.14 48	912 673	4.59 470	9.89 898	21.32 67	10.30 93
98	9 604	9.89 949	31.30 50	941 192	4.61 044	9.93 288	21.39 97	10.20 41
99	9 801	9.94 987	31.46 43	970 299	4.62 607	9.96 655	21.47 23	10.10 10
100	10 000	10.00 000	31.62 28	1 000 000	4.64 159	10.00 000	21.54 43	10.00 00
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$	$1000/N$

ANSWERS TO ODD-NUMBERED PROBLEMS

BOOK I

Exercises 1, Page 4

5. 3π; $1/\sqrt{2}$; $\sqrt{18}$; $\pi + 7$; $\sqrt{5}$

Exercises 2, Page 6

1. -3π; $-\sqrt{2}$; -1; 0; $\sqrt{3}$; 2; π; $5\frac{1}{2}$

3. (*a*) 9, 3; (*b*) 6, 2, 0, 6; (*c*) 1; (*d*) 2, 0, 0, 0, 4; (*e*) 5

5. (*a*) 15.641; 15.638; 15.645; 15.635
(*b*) 15.635 — 15.645, inclusive

Exercises 3, Page 7

3. 221/216 **5.** $\frac{13}{12}$ **7.** 12
9. 5 **11.** $3\frac{11}{18}$ **13.** $-255/32$
15. 314.16 **17.** 12.65 **19.** 7.48

Exercises 4, Page 12

1. $x = u + v - y$; $y = u + v - x$; $u = x + y - v$; $v = x + y - u$

3. $x = n\sqrt{z/y}$; $y = \dfrac{n^2z}{x^2}$; $z = \dfrac{x^2y}{n^2}$

5. $r = \frac{1}{2}\sqrt{A/\pi}$

7. $d = \dfrac{d_1v - d_2v}{2}$; $d_1 = \dfrac{2d}{v} + d_2$; $d_2 = d_1 - \dfrac{2d}{v}$

9. $R = E^2/P$; $E = \sqrt{RP}$

11. $L = \dfrac{32.2t^2}{(6.28)^2}$

13. $x = (d^2 + 4 - y)^3$; $y = d^2 + 4 - \sqrt[3]{x}$; $d = \sqrt{y - 4 + \sqrt[3]{x}}$

15. $M = \dfrac{EI}{R}$; $E = \dfrac{MR}{I}$; $I = \dfrac{MR}{E}$; $R = \dfrac{EI}{M}$

17. $l = \dfrac{\pi^2EI}{P}$; $E = \dfrac{Pl}{\pi^2I}$; $I = \dfrac{Pl}{\pi^2E}$

19. $S_t = \dfrac{mdS_c}{p - d}$; $S_c = \dfrac{S_t(p - d)}{md}$; $d = \dfrac{S_tp}{mS_c + S_t}$

21. $R_1 = R_t - R_2\left(\dfrac{N_1}{N_2}\right)^2$; $R_2 = (R_t - R_1)\left(\dfrac{N_2}{N_1}\right)^2$; $N_1 = \sqrt{\dfrac{N_2(R_t - R_1)}{R_2}}$;
$N_2 = N_1\sqrt{\dfrac{R_2}{R_t - R_1}}$

23. $P = \dfrac{AS}{l + q(1/d)^2}$; $s = \dfrac{P}{A}\left[l + q\left(\dfrac{1}{d}\right)^2\right]$; $d = l\sqrt{\dfrac{Pq}{AS - pl}}$

25. $p = \dfrac{S}{(1+i)^n}$; $i = \sqrt[n]{\dfrac{S}{P}} - 1$

Exercises 5, Page 14

1. $a^2 = c^2 - b^2$
3. $16\sqrt{3}$ sq in.
5. $A_t/A_s = 4\sqrt{3}/9$
7. 67.2 ft
9. 15.1987 ft
11. 22.6 sq ft
13. 41.57 cu ft
15. 1.96 cu yd
17. 3280 cu ft

Exercises 6, Page 17

1. $x^2 - 4y^2$
3. $9x^2 - 49y^2$
5. $x^2 - 16/y^2$
7. $a^2 - b^2 - 2bc - c^2$
9. $x^2 - 2xy + y^2 - z^2$

Exercises 7, Page 18

1. $(x-2)(x+2)$
3. $(4b - 2a)(4b + 2a)$
5. $(5y - 3x)(5x - 3y)$
7. $(a - b - c)(a + b + c)$
9. $(4x + 3y - 12z)(4x + 3y + 12z)$

Exercises 8, Page 18

1. $4x^2 - 12xy + 9y^2$
3. $4s^2 - 20st + 25t^2$
5. $a^2 - ab + b^2/4$
7. $16t + 4st + s^2/4$

Exercises 9, Page 19

1. $(x^2 + 4^2)^2$
3. $(4 - 3x)^2$
5. $(3 - 2x)^2$
7. $(x/3 - y/4)^2$
9. $(x + 2y)^2$

Exercises 10, Page 19

1. $(x^2 - 2x + 4)(x^2 + 2x + 4)$
3. $(x^2 - 2xy + 4y^2)(x^2 + 2xy + 4y^2)$
5. $(9 - 3x + x^2)(9 + 3x + x^2)$
7. $(x - y)(x + y)(x - 4y)(x + 4y)$
9. $(3t^2 - 3st + 5s^2)(3t^2 + 3st + 5s^2)$

Exercises 11, Page 20

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$
3. $y^3 + \frac{3}{2}xy^2 + \frac{3}{4}x^2y + \dfrac{x^3}{8}$
5. $\dfrac{y^3}{8} + \dfrac{y^2x}{4} + \dfrac{yx^2}{6} + \dfrac{x^3}{27}$
7. $64 - 48x + 12x^2 - x^3$
9. $a^3 + \frac{9}{2}a^2b + \frac{27}{4}ab^2 + \frac{27}{8}b^3$

Exercises 12, Page 20

1. $(2x + y)(4x^2 - 2xy + y^2)$
3. $(1 - 3xy)(1 + 3xy + 9x^2y^2)$
5. $(a^2 - 2)(a^4 + 2a^2 + 4)$
7. $(x^2y - 1)(x^4y^2 + x^2y + 1)$
9. $(x - a^2y^2)(x^2 + a^2xy + a^4y^4)$

Exercises 13, Page 20

1. $x^2 + 4y^2 + z^2 + 4xy + 2xz + 4yz$
3. $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$

5. $16 + x^2 + y^2 - 8x + 8y - 2xy$
7. $9x^2 + y^2 + 4z^2 - 6xy - 12xz + 4yz$

Exercises 14, Page 21

1. $(x-5)(x+2)$
3. $(x-3)(x-\sqrt{3})$
5. $(x-10)(x+3)$
7. $(x-a)(x+2a)$
9. $(x-a)(x-b)$
11. $(x-3)^2$

Exercises 15, Page 21

1. $(5x+1)(2x-3)$
3. $(15x-2)(x+5)$
5. $(5x+2)(3x-1)$
7. $(5-x)(2+x)$
9. $(3x-5y)(x-4y)$

Exercises 16, Page 21

1. $(a-b)(a+b)(x+y)$
3. $(x-1)(x+1)(x-3y)$
5. $(1-x)(1+x^2)$

Exercises 17, Page 22

1. $(x-4)(x-3)$
3. $(x-y)(x-2y)$
5. $(x+7)(x-3)$
7. $(x-6)(x+2)$
9. $4(y+9)(y+1)$
11. $(5y+4)(y+2)$
13. $(3x+2)(3x-4)$
15. $(3x+2y)^2$
17. $2(3y+2)(y+3)$
19. $(2x-1)(2x+1)(4x^2+3)$
21. $-(c-d)^2$
23. $a^2(a+1)(a-1)(a-1)$
25. $(x-4y)(x+4y)(x-y)(x+y)$
27. $(ax+by)(a^2x-3y)$
29. $(2x-y+3z)(2x+y-3z)$
31. $2(4+3x)(16-12x+9x^2)$
33. $(3x^2-5y^2+3xy)(3xy-3x^2+5y^2)$
35. $(x-2y)(x-2y-3z)$

Exercises 18, Page 23

1. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
3. $81a^4 + 540a^3b + 1350a^2b^2 + 1500ab^3 + 625b^4$
5. $\dfrac{729}{64} + \dfrac{243}{16}x + \dfrac{135}{16}x^2 + \dfrac{5}{2}x^3 + \dfrac{5}{12}x^4 + \dfrac{x^5}{27} + \dfrac{x^6}{729}$
7. $y^8 + 16y^7x + 112y^6x^2 + 448y^5x^3 + 1120y^4x^4 + 1792y^3x^5 + 1792y^2x^6 + 1024yx^7 + 256x^8$

Exercises 19, Page 24

3. HCF: none LCD: $a(a+b)^2(x-y)(a^2+ab+b^2)$
5. HCF: $x-1$ LCD: $(x-1)(x+1)(x^2+1)(x^4+1)$
7. HCF: $x-3$ LCD: $42(x-3)^2(x+1)(x+3)$

Exercises 20, Page 27

1. $\dfrac{a}{6x^4}$
3. $x+y$
5. $\dfrac{a(2x+3a)}{x(x-a)}$
7. $\dfrac{3x^2-8x-19}{(x+1)(x-1)(x+3)}$
9. $\dfrac{3x^2+6xy+3y^2-x-3y}{(x+y)^3}$
11. $\dfrac{6x^2-21x-5}{(x-2)(x-3)}$

13. $\dfrac{(a-b)^3}{a^2+b^2}$

15. $\dfrac{y-x}{x^3y}$

17. $\dfrac{x^2-y^2}{xy^2}$

19. $\dfrac{a-b}{a+b}$

21. $\dfrac{1}{1-a}$

23. $\dfrac{(ay-3y+a-8)(y-1)}{(2y+1)(y-4)}$

25. $\dfrac{-4}{a+x}$

27. $\dfrac{24(y^2-10xy+x^2)}{(y-5x)^3}$

29. $-\dfrac{x^2-8x+8}{2x^2(x-2)^2}$

31. $\dfrac{4a^2x^2}{(2x-a)^3}$

Exercises 21, Page 30

1. $f(0)=5;\ f(1)=9;\ f(-1)=3;\ f(a+b)=(a+b)^2+3(a+b)+5$
3. $F(1)=4;\ F(2)=13;\ F(a)=a^3+2a+1;\ F(w-1)=w^3-3w^2+5w-2$

Exercises 22, Page 32

3. Center is $(0, 0)$ and $r = 5$

Exercises 23, Page 35

1. $(2, 0)$
3. $(0, 0)$
5. $(0, 0)$
7. $(\frac{32}{3}, 0)$
9. $(\frac{3}{2}, 0)$
11. $(\frac{1}{3}, 0)$

Exercises 24, Page 38

1. $y = \frac{3}{4}x + \frac{3}{2}$
3. $y = -\frac{5}{2}x + \frac{7}{2}$
5. $y = \frac{4}{3}x$
7. $y = x$
9. $y = -2x + 7$
11. 28 square units

Exercises 25, Page 41

1. $x = 411.4$
3. $x = 26\frac{1}{4}$
5. $x = 5$
7. $x = \dfrac{a}{b+c}$
9. $x = 6$
11. $x = \dfrac{bcm - bc^2}{3c - b}$
13. $x = -\dfrac{21}{10}$
15. $x = \dfrac{a(a+1)}{a-1}$ $(a \neq 1)$
17. $x = \frac{5}{3}$
19. $x = 2$

Exercises 26, Page 43

1. 83
3. 5 hr
5. \$900 at $3\frac{1}{3}\%$
   \$1400 at $5\frac{1}{4}\%$
7. Mother: \$864
   Each daughter: \$288
   Each son: \$144
9. 10 ft from end
11. 42 men
13. $3\frac{1}{3}$ gal
15. 3 hr
17. \$1200 at 5%
    \$2000 at 6%
19. 2705 ft

Exercises 27, Page 48

1. $w = 17\frac{1}{2}$
3. 7.9 sec
5. 0.048
7. 8 lb per sq in.

9. 600 lb

11. 5.9 in.

13. $\frac{4}{3}$

15. 33%

Exercises 28, Page 54

1. $x = 5, y = 12$

3. $x = 7, y = 3$

5. $x = 4, y = 15$

7. $x = -\dfrac{m(k+m)}{k(k-m)}$

$y = \dfrac{k(k+m)}{m(k-m)}$

9. $x = 60, y = 40$

11. $x = 2, y = -2$

13. $x = 4, y = 2, z = -3$

15. $x = \frac{26}{17}, y = \frac{117}{28}, z = -\frac{156}{161}$

17. $7\frac{1}{2}$ hr up; $4\frac{1}{2}$ hr down

19. A: \$10; B: \$6

21. \$60,000 at 5%

Exercises 30, Page 57

1. $x = \frac{23}{20}$; $y = -\frac{5}{12}$;
consistent and independent

3. $x = 0$; $y = -\frac{9}{7}$;
consistent and independent

5. $x = 12$; $y = 31$;
consistent and independent

7. Consistent and dependent

9. $x = 5$; $y = 0$;
consistent and independent

11. $x = \frac{8}{3}$; $y = -\frac{8}{3}$;
consistent and independent

13. 13 in., $6\frac{1}{2}$ in., $19\frac{1}{2}$ in., or
$16\frac{1}{4}$ in., 13 in., $9\frac{3}{4}$ in.

15. A: 20 days; B: 30 days;
C: 60 days

17. A: 6.30 yd per second
B: 5.94 yd per second

19. 57.91 oz of 18.22% bar,
42.09 oz of 10.57% bar

21. 24 miles; 9 miles uphill

Exercises 32, Page 65

1. $x = 7, y = 5, z = 4$

3. $x = 5, y = 1, z = 3$

5. $x = \frac{29}{14}, y = \frac{11}{14}, z = \frac{87}{14}$

7. $x = \frac{2145}{271}, y = \frac{286}{3}, z = -\frac{429}{25}$

9. $r_1 = 18$ ohms
$r_2 = 9$ ohms
$r_3 = 12$ ohms

Exercises 33, Page 69

5. -600

7. $-\frac{529}{432}$

9. $x = \frac{1}{2}$, or $\frac{4}{9}$

11. $x = 5, y = -3, z = 4, w = -2$

Exercises 34, Page 74

1. $3\sqrt{2}$

3. $\frac{1}{2}$

5. 4

7. $\dfrac{\sqrt[4]{5}\sqrt[3]{x^2}\sqrt[8]{y^7}}{\sqrt[7]{z^2}}$

9. $\frac{1}{9}$

11. $\dfrac{7}{x^4}$

13. $\frac{50}{7}$

15. $\dfrac{5(x-2)}{x^3}$

17. $\dfrac{a^2 + 2a + 3}{a^3}$

19. $\dfrac{a^6}{16b^4c^8}$

21. $-\frac{1}{8}$

23. $\dfrac{b^5}{\sqrt[3]{a^2}}$

25. $\dfrac{b^{5/4}}{27a}$

27. $\dfrac{b^2}{(b-1)^2}$

29. $\dfrac{ab^2}{a^2-b^4}$

31. $\sqrt[4]{a^3}+\sqrt{a}-\sqrt{ab}-b$

33. $m^4+m^{8/3}+m^{4/3}n^{8/3}-m^{8/3}n^{4/3}-n^{8/3}-n^4$

Exercises 35, Page 78

1. $3\sqrt{2}$

3. $\dfrac{\sqrt[3]{18}}{3}$

5. $\dfrac{2\sqrt[3]{6}}{3}$

7. $\dfrac{\sqrt{6}}{6}$

9. $2\sqrt{2}$

11. $(x-3)\sqrt{2}$

13. $(a^2-b^2)\sqrt{a^2-b^2}$

15. $\sqrt{10}>\sqrt[3]{28}$, $\sqrt[3]{6}>\sqrt{3}$, $\sqrt{19}>\sqrt[3]{65}$

17. $-\frac{27}{7}\sqrt{7}$

19. $(6-c-\frac{1}{2})\sqrt{a-b}$

21. $2\sqrt{5}$

23. $\sqrt{10}-5$

25. $5-2\sqrt{6}$

27. $9\ -2\sqrt{3}+3\sqrt{15}-2\sqrt{5}$

29. $\sqrt[3]{4}-2\sqrt[3]{10}+\sqrt[3]{25}$

31. Yes

33. $4\sqrt{5}+8$

35. $\dfrac{27-7\sqrt{5}}{22}$

37. $\dfrac{a-\sqrt{a^2-x^2}}{x}$

39. $\dfrac{-ab}{x+\sqrt{a^2+x^2}}$

41. -4.832

Exercises 36, Page 80

1. $5+(2\sqrt{3}-4\sqrt{2})i$

3. $6-21\sqrt{6}+5\sqrt{3}i$

5. $14-5\sqrt{3}-(7\sqrt{5}+2\sqrt{15})i$

7. $\dfrac{10\sqrt{3}}{3}-1-2\sqrt{3}i$

9. $\dfrac{a^2-\sqrt{bc}}{a^2+b}-\dfrac{a(\sqrt{b}+\sqrt{c})i}{a^2+b}$

Exercises 37, Page 84

3. (a) $V(\frac{4}{3}, -\frac{25}{3})$; minimum $y=-\frac{25}{3}$
   (b) $V(\frac{4}{3}, -\frac{16}{3})$; minimum $y=-\frac{16}{3}$
   (c) $V(\frac{3}{4}, -\frac{31}{8})$; maximum $y=-\frac{31}{8}$
   (d) $V(-\frac{3}{4}, \frac{65}{8})$; maximum $y=\frac{65}{8}$
   (e) $V(0, -3)$; minimum $y=-3$
   (f) $V(\frac{5}{2}, -\frac{1}{2})$; minimum $y=-\frac{1}{2}$

5. $x=15$ ft

7. $-\frac{25}{8}$

9. $\frac{1}{2}$

Exercises 38, Page 88

1. $x=3, 4$

3. $x=\frac{1}{2}, \frac{1}{4}$

5. $x=\frac{2}{7}, -\frac{5}{3}$

7. $x=-\frac{1}{3}, -\frac{5}{2}$

9. $x = -\frac{5}{2}, \frac{3}{4}$

11. $x = \dfrac{a}{2}, 2a$

13. $x = \dfrac{a}{2p}, \dfrac{2a}{p}$

15. $x = \dfrac{2b}{a}, \dfrac{2b}{a}$

17. $1 \pm \sqrt{2}$

19. $\dfrac{7 \pm \sqrt{33}}{8}$

21. 1 and $\frac{1}{5}$

23. $\dfrac{5 \pm \sqrt{17}}{8}$

25. $\dfrac{1 \pm \sqrt{6}}{p}$

27. $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

29. 4.431 and 0.903

31. -4 and -2

33. 8.149 and 2.650

35. $\dfrac{17 \pm \sqrt{455}i}{12}$

37. 6 and $\frac{5}{11}$

39. $\frac{11}{2}$ and $-\frac{1}{2}$

41. $-\frac{1}{3}, -\frac{22}{57}$

Exercises 39, Page 92

1. $x = 5$

3. $x = 0$ and 4

5. $x = 3$

7. $x = \dfrac{25a}{16}$

9. $x = \dfrac{a^2}{a-1}$

11. $x = 9$

13. $x = 49$

15. $x = 37$

17. $x = \frac{5}{4}$ and $\frac{4}{5}$

19. $x = \pm \frac{3}{8}\sqrt{10}$

21. $x = 1$ and 4

23. $x = 3$ and $-\frac{5}{3}$

25. $x = \frac{1}{256}$ and 16

27. $x = 1$ and 4^n

29. 256, $\frac{400}{27}\sqrt[3]{1200}$

Exercises 40, Page 92

1. 37.26 acres
   40.26 acres

3. A: 6.26 mph; B: 4.26 mph

5. 14, 5 or $-\frac{19}{2}, -\frac{37}{2}$

7. 4 ft border; 20 ft side of square

9. Train: 40 mph; plane 120 mph

11. $4\frac{1}{3}$ hr

Exercises 41, Page 95

5. $q = \frac{9}{8}$

7. Real and equal: $q = \pm 6$;
   not real: $-6 < q < 6$

9. $x_1 + x_2 = \frac{1}{5}$; $x_1x_2 = -\frac{1}{5}$

Exercises 42, Page 98

1. $(2, 5), (-4, -1)$

3. $(1, 2), \frac{37}{19}, \frac{26}{19}$

5. $(8, 16), (-3, \frac{4}{3})$

7. $(3, 1), (-\frac{1}{15}, -\frac{41}{5})$

9. $(4, -3), (\frac{144}{5}, \frac{78}{5})$

Exercises 43, Page 100

1. $(2, 4), (-2, -4), (\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}), (-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10})$

3. $(4, 5), (-4, -5), (3\sqrt{3}, \sqrt{3}), (-3\sqrt{3}, -\sqrt{3})$

5. $(6, 2), (-6, -2), (4i, -3i), (-4i, 3i)$

7. $(2, -3), (-2, 3), (\frac{50}{23}\sqrt{23}, \frac{11}{23}\sqrt{23}), (-\frac{50}{23}\sqrt{23}, -\frac{11}{23}\sqrt{23})$

9. $(5, \frac{1}{2}), (-5, -\frac{1}{2}), (\frac{3}{2}\sqrt{2}i, \frac{13}{8}\sqrt{2}i), (-\frac{3}{2}\sqrt{2}i, -\frac{13}{8}\sqrt{2}i)$

Exercises 44, Page 102

1. $(9, 3), (3, 9), \left(-\frac{13 + \sqrt{39}i}{2}, -\frac{13 - \sqrt{39}i}{2}\right), \left(\frac{-13 - \sqrt{39}i}{2}, \frac{-13 + \sqrt{39}i}{2}\right)$

3. $(\frac{3}{5}, \frac{3}{4}), (\frac{3}{4}, \frac{3}{5})$

5. $(2, -6), (-6, 2), \left(\frac{7 + \sqrt{35}i}{2}, \frac{7 - \sqrt{35}i}{2}\right), \left(\frac{7 - \sqrt{35}i}{2}, \frac{7 + \sqrt{35}i}{2}\right)$

7. $(9, -5), (-5, 9), \left(-\frac{49 + 7\sqrt{41}i}{4}, -\frac{49 - 7\sqrt{41}i}{4}\right),$
$\left(\frac{-49 - 7\sqrt{41}i}{4}, \frac{-49 + 7\sqrt{41}i}{4}\right)$

9. $(7, -3), (-3, 7), \left(\frac{9 + \sqrt{65}}{2}, \frac{9 - \sqrt{65}}{2}\right), \left(\frac{9 - \sqrt{65}}{2}, \frac{9 + \sqrt{65}}{2}\right)$

Exercises 45, Page 104

1. $(1, 2), (2, 1)$

3. $(9, 3), (-9, -3)$

5. $(2, -3), (6, -1)$

7. $\left(\frac{6\sqrt{33}}{11}, \frac{2\sqrt{33}}{11}\right), \left(\frac{-6\sqrt{33}}{11}, \frac{-2\sqrt{33}}{11}\right), (-2, 2), (2, -2)$

9. $(\frac{1}{2}\sqrt[3]{484}, \frac{1}{22}\sqrt[3]{484})$

11. $(4, 1), (2, 2)$

13. $\left(\frac{5i}{2}, \frac{-17i}{6}\right), \left(\frac{-5i}{2}, \frac{17i}{6}\right)$

Exercises 46, Page 104

1. $(4, 5), (-4, -5), (-19, 5), (19, -5)$

3. $(0, 0), (4, 2), (-4, -2)$

5. $\left(\sqrt{158}, \frac{67}{1 + \sqrt{158}}\right), \left(-\sqrt{158}, \frac{67}{1 - \sqrt{158}}\right), \left(\sqrt{158}, \frac{-\sqrt{158} - \sqrt{426}}{2}\right),$
$\left(-\sqrt{158}, \frac{\sqrt{158} + \sqrt{426}}{2}\right)$

7. $(2, 1), (-2, -1), \left(-\frac{2i}{\sqrt{19}}, \frac{9i}{\sqrt{19}}\right), \left(\frac{2i}{\sqrt{19}}, \frac{9i}{\sqrt{19}}\right)$

9. $(-4.999, -0.1262), (1.279, -4.834)$ approximately

11. $(1.300, 2.280), (1.300, -2.280)$ approximately

13. $(2a, a), (-2a, a), (2\sqrt{2a}i, -2a), (-2\sqrt{2a}i, -2a)$

15. $(3, -4), (4, 3)$

17. $(2, 3)$

19. 1.65 in. and 30.35 in.

21. 92.5 mph and 112.5 mph

23. 25 lots at $192 each

Exercises 48, Page 114

1. Ellipse
3. Ellipse
5. Hyperbola
7. Hyperbola
9. Hyperbola
11. Ellipse
13. Hyperbola
15. Hyperbola

Exercises 50, Page 118

1. $f(1) = 3$; $f(2) = 5$; $f(3) = 11$; $f(-1) = 11$; $f(0) = -1$
3. $f(-1) = -15$; $f(2) = 114$; $f(5) = 13{,}455$; $f(10) = 462{,}810$
5. $f(2) = 53$; $f(-1) = -13$; $f(3) = 231$; $f(-3) = -567$

Exercises 51, Page 120

3. -1
7. -28
11. $(x+1)(x+2)(x-1)$
13. $(x-4)(x-2)(x-3)(x-1)$

Exercises 52, Page 126

1. $2, 3, -4$
3. $1, -1, 7, 3$
5. $-3, 1, 2$
7. $1, 1, 1, -4$
9. $-\frac{1}{2}, -1, -\frac{2}{3}$
11. $2, -\frac{1}{2}, \frac{1}{2}$
13. $1, \frac{3}{2}, -\frac{3}{2}$
15. $-\frac{3}{2}, -\frac{3}{2}, \dfrac{1 \pm \sqrt{5}}{2}$
17. $-1, 5, \frac{4}{5}$
19. $\frac{5}{3}, \pm\sqrt{2}$
21. 28 ft, 21 ft

Exercises 53, Page 129

1. 4.23
3. -2.43
5. $0.409, -1.11, 2.20$
7. $1.75, -0.331$
9. 2.38

Exercises 54, Page 133

1. $0.27, 3.36, -1.63$
3. $0.28, 0.69, -0.97$
5. 1.77
7. 19.34 in. or 3.36 in.
9. $r = 3.59$ in.; $h = 5.51$ in.; or $r = 6.75$ in., $h = 1.56$ in.
11. 0.32 ft

Exercises 55, Page 135

1. (*a*) $\log_2 8 = 3$; (*b*) $\log_5 25 = 2$; (*c*) $\log_2 \frac{1}{2} = -1$; (*d*) $\log_3 1 = 0$; (*e*) $\log_8 4 = \frac{2}{3}$
3. (*a*) 6; (*b*) 2; (*c*) -4; (*d*) -1
7. (*a*) 3; (*b*) 3; (*c*) 625; (*d*) $\frac{1}{125}$

Exercises 59, Page 139

1. 0.66680
3. 2.31027
5. 1.94929
7. $0.48897 - 2$
9. 0.02284
11. 1.73030
13. $0.99564 - 3$
15. $0.77830 - 5$
17. $0.84510 - 1$
19. 3.79000
21. 322.1
23. 9570.
25. 5.9204
27. 0.00303
29. 9.991

Exercises 60, Page 143

1. 1.10497
3. 2.97392
5. 2.30203
7. 7.12209 − 10
9. 70148.
11. 45.503
13. 60.473
15. 0.029182

Exercises 61, Page 145

1. 0.11203
3. 37.816
5. 3.006
7. 15.78
9. 1.554
11. −0.9788
13. 1.0092
15. 5.39078
17. 0.48474
19. 1.7094
21. 12.29
23. 34.885
25. 11.23
27. $2,513
29. 1.286×10^{10}
31. 0.608
33. 4.578

Exercises 62, Page 147

1. −1.9554
3. −0.00002493
5. −0.015218
7. 3.2314

Exercises 63, Page 149

1. 2.7604
3. −3.170
5. 0.1 or 3.162
7. 1.442
9. 100 or 0.001
11. 0.1704
13. 11.923
15. 4.5913
17. 2.4012
19. −0.18551
21. 0.049
23. 0.042

Exercises 64, Page 150

3. 50
5. 4.3731 − 10; 9.0837 − 10; 2.3322; 8.6907 − 10; 1; 1.9149; 5.951; 2.26593
7. (*a*) 3.2188
   (*b*) 5.52147
   (*c*) 7.82406
   (*d*) 8.61370 − 10
   (*e*) 6.31111 − 10
   (*f*) 1.83258

Exercises 66, Page 154

3. $a_{10} = 35$; $s_{20} = 740$
5. $112\frac{1}{2}$
7. $a_n = 22 + \sqrt{2}$; $s_n = 12\sqrt{2} + 132$
9. $n = 9$; $s_n = 477$
11. $14\frac{1}{2}$
13. $139
15. 2,300 ft
17. $s = \frac{1}{2}gt^2$
21. $\frac{3}{16}$; $\frac{1}{12}$

Exercises 68, Page 157

1. $a_1 = \dfrac{a_n}{r^{n-1}}$; $n = 1 + \dfrac{\log a_n - \log a}{\log r}$; $r = \left(\dfrac{a_n}{a_1}\right)^{1/(n-1)}$

3. $a_q = -\frac{1}{256}$; $s_q = -\frac{171}{256}$
5. $a_n = \frac{7}{729}$; $s_n = 31\frac{361}{729}$
7. \$2412.10
9. \$3306.80
11. $0.34866w$; 37 strokes
13. 1.8447×10^{19}

Exercises 69, Page 160

1. 2
3. $\frac{5}{3}$
5. 9
7. $\frac{4}{33}$
9. $\frac{267}{550}$
11. 120°
13. $1/x$

Exercises 71, Page 168

1. 15
3. 150
5. 486,486,000
7. 144
9. 9

Exercises 72, Page 170

1. 1680
3. 720
5. 840
7. 96
9. 126

Exercises 73, Page 171

1. 120
3. $3({}_{27}C_8) = 6{,}660{,}225$
5. 637
7. 28
9. 75,287,520
11. ${}_{52}C_{13}$

Exercises 74, Page 171

1. 600
3. 455
5. 455
7. 1225
9. 120
11. 190
13. 151,200
15. $(\underline{|12})^2$
17. $6\underline{|10}$

Exercises 75, Page 174

1. $\frac{1}{5}$
3. (a) $\frac{723}{92{,}637}$; (b) $\frac{91{,}914}{92{,}637}$
5. $\frac{7}{20}$
7. $\frac{1}{7}$
9. $\frac{1}{6}$

Exercises 76, Page 175

1. (a) $\frac{1}{28{,}561}$; (b) $\frac{1}{27{,}025}$
3. (a) $\frac{256}{625}$; (b) $\frac{16}{625}$
5. 0.568; 0.185
7. $\frac{7}{228}$
9. $\frac{1}{20}$
11. (a) $\frac{1}{13}$; (b) $\frac{4}{13}$
13. $\frac{7}{30}$; $\frac{1}{60}$

Exercises 77, Page 177

1. \$0.06$\frac{2}{3}$

3. \$0.75; \$1.50; \$3.75

5. \$4805

Exercises 78, Page 177

1. (a) $\frac{2}{91}$; (b) $\frac{4}{455}$; (c) $\frac{2}{65}$; (d) $\frac{24}{91}$; (e) $\frac{53}{65}$

3. $\frac{47}{95}$

5. (a) 0.94; (b) 0.44; (c) 0.06; (d) 0.14

7. $\frac{4}{17}$; $\frac{1}{2652}$; $\frac{1}{221}$

9. \$0.059

Exercises 79, Page 182

1. $\dfrac{2}{x-1} - \dfrac{9}{x-2} + \dfrac{7}{x-3}$

3. $\dfrac{5}{3(x-1)^2} - \dfrac{5}{9(x-1)} + \dfrac{5}{9(x+2)}$

5. $\dfrac{x-2}{5(x^2+1)} + \dfrac{29}{5(x+2)}$

7. $8x + 29 + \dfrac{91}{x-2} - \dfrac{20}{x-1}$

9. $\dfrac{3}{5(x+1)} + \dfrac{1}{5(3x-2)}$

11. $3x + 15 + \dfrac{89}{x-3} - \dfrac{32}{x-2}$

13. $\dfrac{3}{4x^2} + \dfrac{4x-3}{4(x^2+4)}$

15. $\dfrac{35}{16(x-1)} + \dfrac{7}{4(x-1)^2} - \dfrac{35x-14}{16(x^2+x+2)}$

17. $\dfrac{1}{w-1} + \dfrac{2}{(w-1)^2} + \dfrac{1}{(w-1)^3}$

19. $\dfrac{2x}{(x^2+1)^2} + \dfrac{x}{x^2+1}$

Exercises 81, Page 187

1. $x > 1$

3. $V < \frac{3}{2}$

5. $x < 1$; $x > 3$

7. $-\frac{3}{2} < x < 1$

9. $-3 < x < 2$, $x > 7$

11. $x \leq -3$, $x \geq 0$

13. $x < \frac{3}{2}$

15. $x < \dfrac{3-\sqrt{13}}{2}$, $x > \dfrac{3+\sqrt{13}}{2}$

17. $x > 1$; $x < 0$

19. $x < 0$

Exercises 82, Page 188

1. 0.989

3. $n = \dfrac{Cr}{E - Cr}$; $r = \dfrac{nE}{C(n+1)}$; $E = \dfrac{Cr(n+1)}{n}$

5. $x^{13/2} - \frac{13}{3}x^6y + \frac{26}{3}x^{11/2}y^2 - \frac{286}{27}x^5y^3 + \cdots$; ninth term is $\frac{143}{729}x^{5/2}y^8$

7. $\dfrac{2a^{11}x^{1/2}}{5b^{3/2}c^{9/4}}$

9. $\frac{18}{31}$

11. $x = -\frac{3}{2}$

13. $x = -6;\; y = 14;\; z = 5$

15. $142\frac{6}{7}$ miles

17. $x = \dfrac{10 \pm \sqrt{91}}{3}$

19. (*a*) $K = \pm\sqrt{3}$; (*b*) anything but $\pm\sqrt{3}$

21. $x = 0$

23. $6x^{1/4}y^{-1} - 2x^{-1/4} - 3x^{-3/4}y$

25. $x > 7$ and $x < -\frac{3}{2}$

27. $\frac{27}{8}\sqrt{3A}$

29. $x = 4$

31. $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{5\sqrt{2}}{2}\right), \left(-\dfrac{\sqrt{2}}{2}, \dfrac{5\sqrt{2}}{2}\right), (3, 2), (-3, -2)$

33. (*a*) yes; (*b*) yes

35. $-3, -4, 1 \pm \sqrt{2}$

37. $2x^4 - 11x^3 - 23x^2 + 14x = 0$

39. 2.756; 5.190; -1.912

41. 3

43. 19.61

47. 32.015

49. \$1591.8

51. 273,700

BOOK II

Exercises 1, Page 197

1. 90°; 60°; 15°; 38°11′50″; 85°56′37″
3. 0.6698; 1.2613; 2.208; 1.5132; 2.483; 0.8269
5. 12.57 ft
7. 5.094 in.
9. 9.48 in.

Exercises 2, Page 198

1. $y = 150$ ft; $r = 212.13$ ft
3. $\frac{a}{2}$; $\frac{a\sqrt{3}}{2}$
5. 12.69 ft

Exercises 3, Page 204

1.

	sin A	cos A	tan A	csc A	sec A	cot A
(a)	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
(b)	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
(c)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Exercises 4, Page 206

3. 2.2802

5. $x = 2.75$

Exercises 6, Page 210

	sin A	cos A	tan A	csc A	sec A	cot A
1	$\frac{5}{13}$	$\pm\frac{12}{13}$	$\pm\frac{5}{12}$	$\frac{13}{5}$	$\pm\frac{13}{12}$	$\pm\frac{12}{5}$
3	$-\frac{3}{5}$	$\frac{4}{5}$	$-\frac{3}{4}$	$-\frac{5}{3}$	$\frac{5}{4}$	$-\frac{4}{3}$
5	$\pm\frac{5}{74}$	$\pm\frac{7}{74}$	$\frac{5}{7}$	$\pm\frac{5\sqrt{74}}{74}$	$\pm\frac{7\sqrt{74}}{74}$	$\frac{7}{5}$

7. (a) 5.7683; (b) 1.799
9. (a) 4.7669; (b) $\frac{169}{144}$

Exercises 10, Page 226

27. $\dfrac{\sin^7\theta - 2\sin^5\theta + 2\sin^3\theta + \sin^2\theta - 1}{\sin^2\theta(1 - \sin^2\theta)}$

29. $\dfrac{1 + \cos A - \cos^2 A - \cos^3 A}{\cos^2 A}$

Exercises 11, Page 233

1. 0.45865
3. 0.73010
5. 0.56666
7. 0.13710
9. 1.0844
11. 24°15′
13. 51°50′22″
15. 39°35′25″
17. 39°39′46″
19. 33°53′45″

Exercises 12, Page 233

The general solutions are obtained by adding $n \cdot 360°$, where $n = 0, 1, 2, 3, \cdots$, to the values given.

1. (*a*) 30°; 150°
   (*b*) No solution
   (*c*) 80°32′16″; 260°32′16″
3. 45°; 225°
5. 80°32′16″; 260°32′16″
7. 36°52′12″
9. 60°; 120°
11. 0°; 180°; 45°; 225°
13. 32°37′; 147°23′; 327°23′; 212°37′
15. 60°; 180°; 300°
17. 90°; 60°; 300°
19. 72°24′49″; 220°12′28″
21. $(\sqrt{13}, \tan^{-1} \frac{2}{3})$, θ in first quadrant; $(-\sqrt{13}, \tan^{-1} \frac{2}{3})$, θ in third quadrant
23. $(9.30075, 98°2'22'' \pm n \cdot 360°)$; $(9.30075, 261°57'38'' \pm n \cdot 360°)$
25. $r = \pm \sqrt{10}$; $\theta = \tan^{-1}(-2)$; $\theta = \frac{\pi}{4} 2n\pi$

Exercises 13, Page 236

1. $\sin(A + B) = -\dfrac{4\sqrt{5} + 5}{15}$ $\qquad \sin(A - B) = \dfrac{4\sqrt{5} - 5}{15}$

   $\cos(A + B) = \dfrac{2\sqrt{5} - 10}{15}$ $\qquad \cos(A - B) = \dfrac{-2\sqrt{5} - 10}{15}$

   $\tan(A + B) = \dfrac{9 + 5\sqrt{5}}{8}$ $\qquad \tan(A - B) = \dfrac{9 - 5\sqrt{5}}{8}$

5. $\cos B$

Exercises 14, Page 238

1. $\sin 2A = \dfrac{\sqrt{3}}{2}$; $\cos 2A = -\frac{1}{2}$, $\tan 2A = -\sqrt{3}$
3. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$
5. $\cos 3A = 4 \cos^3 A - 3 \cos A$
7. $\sin \dfrac{A}{2} = \sqrt{\dfrac{5}{6}}$; $\cos \dfrac{A}{2} = \sqrt{\dfrac{1}{6}}$; $\tan \dfrac{A}{2} = \sqrt{5}$

Exercises 16, Page 241

In each of the following answers $n = 0, 1, 2, 3, \cdots$.

1. $30° \pm n \cdot 180°$
   $150° \pm n \cdot 180°$
3. $n \cdot 360° \pm 45°$
5. $90° + n \cdot 360°$; $18° + n \cdot 360°$; $162° + n \cdot 360°$; $234° + n \cdot 360°$; $306° + n \cdot 360°$
7. $\frac{\pi}{10} \pm \frac{2n\pi}{5}$; $-\frac{\pi}{6} \pm 2n\pi$
9. $270° \pm n \cdot 360°$
11. $\pm n \cdot 180°$; $45° \pm n \cdot 180°$; $90° \pm n \cdot 180°$
13. $180° \pm n \cdot 360°$; $60° \pm n \cdot 360°$; $300° \pm n \cdot 360°$
15. $\pm n \cdot 180°$; $n \cdot 180° \pm 60°$
17. $161°33'54'' + n \cdot 180°$
19. $r = 4.3149$; $\theta = 22°1'27''$; $r = -4.3149$; $\theta = 157°58'33''$
21. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
23. $x^3 + xy^2 - 2ay^2 = 0$
25. $\frac{x^2}{a^2} + \frac{y^{2/3}}{b^{2/3}} = 1$
27. $y^2 + 2ax = 2a^2$
29. $\theta = 121°17'10''$
31. $\theta = 44°37'24''$

Exercises 17, Page 244

1. $x = \frac{1}{2}\sqrt{3}$
3. $x = \frac{\sqrt{21}}{14}$
5. $x = 0$; $x = \pm 1$
7. $x = 0, x = \pm 1$
9. $x = \frac{1}{6}$

Exercises 19, Page 247

1. 9.52421 − 10
3. 0.50040
5. 0.38694
7. 9.88095 − 10
9. 0.32389
11. $22°29'$
13. $54°12'$
15. $35°38'32''$
17. $40°37'31''$
19. $22°45'30''$

Exercises 20, Page 249

1. $A = 62°4'58''$; $B = 27°55'2''$; $c = 303.29$
3. $A = 60°14'41''$; $B = 29°45'19''$; $a = 313.86$
5. $A = 70°43'22''$; $b = 161.37$; $c = 488.78$
7. 7.6537 in.; area = 282.84 sq in.
9. 15.867 in.
11. 43.333 sq in.
13.

6 in.: 10.10 cu ft	36 in.: 123.0 cu ft
12 in.: 27.94 cu ft	42 in.: 146.8 cu ft
18 in.: 49.53 cu ft	48 in.: 168.4 cu ft
24 in.: 73.32 cu ft	54 in.: 186.2 cu ft
30 in.: 98.15 cu ft	60 in.: 196.3 cu ft

15. 153.96 ft

17. $B = 58°0'8''$ $\quad B' = 121°59'52''$
$C = 80°22'27''$ $\quad C' = 16°22'43''$
$c = 553.58$ $\quad c' = 158.33$

19. $B = 53°51'34''$; $C = 57°40'51''$; $c = 876.84$

21. $A = 16°46'35''$; $C = 24°45'33''$; $b = 636.79$

Exercises 21, Page 257

1. $A = 56°19'$; $b = 838.0$; $c = 786.7$
3. $A = 47°32'$; $C = 80°9'$; $c = 514.2$
5. No solution

Exercises 22, Page 260

1. $A = 51°29'36''$; $B = 89°13'24''$; $c = 291.28$
3. $A = 32°28'17''$; $C = 57°31'43''$; $b = 65.19$
5. 777.68 miles; 425.62 miles from other road
7. 135.8 miles apart
9. $54°44'6''$

Exercises 23, Page 268

1. $C = 66°39'24''$; $b = 592.74$; $c = 580.11$
3. $A = 34°32'6''$; $a = 0.1426$; $c = 0.2510$
5. $A = 72°41'24''$; $B = 52°9'56''$; $C = 55°8'42''$
7. $A = 40°29'$; $B = 59°6'$; $c = 1113$
9. $A = 46°52'10''$; $C = 111°53'25''$; $c = 883.65$
$A' = 133°7'50''$; $C' = 25°37'45''$; $c' = 411.92$
11. $B = 23°18'21''$; $C = 141°8'57''$; $c = 241.57$
$B' = 156°41'39''$; $C' = 7°45'39''$; $c' = 52.006$
13. $A = 66°22'42''$; $C = 72°20'0''$; $b = 0.69757$
15. 263.5 miles
17. 385.9 rods
19. 10.7 in.

Exercises 24, Page 271

1. $B = 46°12'45''$; $C = 51°29'51''$; $a = 527.44$; area $= 79.300$
3. $A = 86°45'15''$; $b = 700.67$; $c = 792.42$; area $= 277160$
5. $A = 29°46'49''$; $B = 87°57'36''$; $b = 921.96$; area $= 186{,}940$
7. 0.82042
9. 76.41; 179.9
11. $113°1'37''$; $66°58'23''$; area $= 137.58$ sq ft
13. One diagonal $= 165.5$; other $= 409.2$; area $= 29{,}677$
15. 5800.7 ft
17. 1584.4 ft
19. $538 \sin 31°27' > .237$. $\therefore$ BC is too short
21. $AB = 536.76$ ft
23. $AB = 405.6$ ft
25. $\angle ABD = 139°47'23''$; $BD = 897.16$ ft

27. 279.5 ft = horizontal distance; 6859.7 ft = height of cliff

29. 4000.5 miles; 178.94 miles

31. 674.5 sq ft

33. 5.82 in.

35. 2.32 miles

37. 122.7; 148.2; $52°39'54''$; $53°1'30''$

39. 14.72 ft

Exercises 25, Page 278

1. (*a*) $2(\cos 60° + i \sin 60°)$ (*b*) $2(\cos 330° + i \sin 330°)$
(*c*) $2\sqrt{2}(\cos 225° + i \sin 225°)$ (*d*) $2(\cos 90° + i \sin 90°)$
(*e*) $3(\cos 0° + i \sin 0°)$ (*f*) $(\cos 270° + i \sin 270°)$
(*g*) $\sqrt{34}(\cos\theta + i \sin\theta)$, where $\theta = \tan^{-1}(-\frac{5}{3})$ in the 4th quadrant (*h*) $\sqrt{2}(\cos 315° + i \sin 315°)$

3. (*a*) $3 + i$ (*b*) $1 - 7i$
(*c*) $14 + 5i$ (*d*) $-\frac{10}{17} - \frac{11}{17}i$

Exercises 26, Page 282

1. $6(\cos 60° + i \sin 60°)$

5. $\frac{3}{8}i$

7. $4 + 4\sqrt{3}i$

9. $2\sqrt{2}(\cos 345° + i \sin 345°)$

11. (*a*) ± 1; (*b*) $1, -\dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$; (*c*) $\pm 1, \pm i$

Exercises 27, Page 284

1. $4 - i$ **3.** $-3 - 11i$

5. $-\frac{3}{10} + \frac{11}{10}i$ **7.** $z = \dfrac{1 \pm \sqrt{15}i}{2}$

11. $5\sqrt{13}(\cos 70°33'39'' + i \sin 70°33'39'')$

15. $\dfrac{1}{5}(\cos 330° + i \sin 330°)$; $\dfrac{3}{25} - \dfrac{4}{25}i$; $\dfrac{1}{10}(\cos 45° + i \sin 45°)$

17. $\sqrt{5}(\cos 15° + i \sin 15°)$ and $\sqrt{5}(\cos 195° + i \sin 195°)$; $2 + i$ and $-2 - i$; $\sqrt{10}(\cos 22°30' - i \sin 22°30')$ and $\sqrt{10}(\cos 157°30' + i \sin 157°30')$

19. $\pm 2i$

21. $1.697(\cos 148°49'32'')$; $1.697(\cos 328°49'32'')$

23. $2(\cos 60° + i \sin 60°)$; $2(\cos 180° + i \sin 180°)$; $2(\cos 300° + i \sin 300°)$

25. $\sqrt[3]{5}(\cos 47°42'36'' + i \sin 47°42'36'')$; $\sqrt[3]{5}(\cos 167°42'36'' + i \sin 167°42'36'')$; $\sqrt[3]{5}(\cos 287°42'36'' + i \sin 287°42'36'')$

27. $2(\cos 0° + i \sin 0°)$; $2(\cos 72° + i \sin 72°)$; $2(\cos 144° + i \sin 144°)$; $2(\cos 216° + i \sin 216°)$; $2(\cos 288° + i \sin 288°)$.

31. $z_1 = \sqrt{34}(\cos 59°2'9'' + i \sin 59°2'9'')$; $z_2 = \sqrt{13}(\cos 303°41'24'' + i \sin 303°41'24'')$; $z = \sqrt{29}(\cos 21°48'5'' + i \sin 21°48'5'')$

33. $\frac{41}{17} - \frac{12}{17}i$

35. in series $4.571(\cos 334°3' + i \sin 334°3')$; in parallel $1.718(\cos 334°49' + i \sin 334°49')$

Review Exercises 28, Page 286

5. 4.59 ft
7. 616.6 ft
13. (*a*) $\theta = 30°$; $150°$; $135°$; $315°$
    (*b*) $\theta = 48°35'25''$; $131°24'35''$; $194°28'39''$; $345°31'21''$.
15. $\sin(A+B) = \dfrac{220}{221}$; $\cos(A-B) = \dfrac{171}{221}$; $\cos 2A = \dfrac{119}{169}$; $\sin\dfrac{A}{2} = \dfrac{\sqrt{26}}{26}$
17. (*e*) $\dfrac{\cos 4\theta + 4\cos 2\theta + 3}{8}$
19. $\dfrac{\pm 25\sqrt{3} + 48}{39}$
23. (*a*) 120°, 150°, 300°, 330°
    (*b*) 0, 90°, 180°, 270°
25. $y^2 = 4x^2(1 - x^2)$
29. $\sqrt[3]{17}(\cos 99°21'27'' + i\sin 99°21'27'')$; $\sqrt[3]{17}(\cos 219°21'27'' + i\sin 219°21'27'')$; $\sqrt[3]{17}(\cos 339°21'27'' + i\sin 219°21'27'')$

BOOK III

Exercises 1, Page 292

3. $AB = 9$; $BC = \sqrt{130}$; $AC = \sqrt{85}$; altitude $= 9$; area $= \frac{81}{2}$
9. $6x - 4y + 13 = 0$

Exercises 2, Page 294

1. $(\frac{1}{2}, -\frac{3}{2})$
3. $(-\frac{2}{3}, -5)$; $(-\frac{7}{3}, -4)$
5. (a) $AB = 5\sqrt{10}$; $BC = \sqrt{173}$; $AC = \sqrt{13}$
   (b) to $BC = \frac{1}{2}\sqrt{353}$; to $AC = \frac{7}{2}\sqrt{17}$; to $AB = \frac{1}{2}\sqrt{122}$
   (c) $\frac{1}{2}\sqrt{13}$
   (d) $(\frac{2}{3}, -\frac{13}{3})$

Exercises 3, Page 295

1. 12
5. $\frac{29}{5}$ units
7. (b) 60; (c) 15, $\sqrt{89}$
   (d) $2\sqrt{29}$; (e) $\dfrac{30\sqrt{29}}{29}$

Exercises 4, Page 297

3. $(2, 60°)$; $(\sqrt{2}, 45°)$; $(\sqrt{74}, 234°27'45'')$
5. $r = 5$
7. $x^2 + y^2 - y = 0$
9. $y^2 = 4(x + 1)$
11. 5.33

Exercises 8, Page 311

1. $(4, 3)$
3. $(2, 4)$; $(50, -20)$
5. $(\frac{9}{2}, -6)$
7. $(2, 2\sqrt{3})$; $(2, -2\sqrt{3})$
9. $(2, 10)$; $(10, 2)$
11. $(0, 2)$; $(0, -2)$
13. $(0, 0)$; $(2, 4)$
15. $\left(\dfrac{\pi}{4} \pm n\pi, \dfrac{1}{2}\right)$; $\left(\dfrac{3\pi}{4} \pm n\pi, -\dfrac{1}{2}\right)$
17. $\left(10\sqrt{2}, \dfrac{\pi}{4}\right)$
19. $(1, 0)$; $(1, \pi)$
21. $(1, 0)$; $(0, 0)$
23. $\left(\dfrac{a}{2}, \dfrac{\pi}{6}\right)$; $\left(\dfrac{a}{2}, \dfrac{\pi}{3}\right)$; $\left(\dfrac{a}{2}, \dfrac{2\pi}{3}\right)$; $\left(\dfrac{a}{2}, \dfrac{5\pi}{6}\right)$; $\left(\dfrac{a}{2}, \dfrac{7\pi}{6}\right)$; $\left(\dfrac{a}{2}, \dfrac{4\pi}{3}\right)$; $\left(\dfrac{a}{2}, \dfrac{5\pi}{3}\right)$; $\left(\dfrac{a}{2}, \dfrac{11\pi}{6}\right)$

Exercises 9, Page 314

1. $2x - 12y - 29 = 0$
3. $x^2 + y^2 - 2x - 5y = 25$
5. $y^2 = 8x - 16$
7. $x^2 - 4x - 6y + 13 = 0$
9. $19x^2 - 18xy + 99y^2 - 50x - 450y + 175 = 0$

11. $44x^2 - 100y^2 = 275$

13. $3x^2 + 4y^2 - 40x + 100 = 0$

15. $r = 10 \cos \theta$

17. $r^2 - 8r \cos \left(\theta - \frac{\pi}{6}\right) = 84$

Exercises 10, Page 319

1. $y = -3$
3. $2x + 3y - 19 = 0$
5. $2x + 5y - 10 = 0$
7. $\sqrt{3}x - y - 3 - 2\sqrt{3} = 0$
9. $x + y - 2 = 0$
11. $x + \sqrt{3}y - 5 = 0$
15. (a) $2x + y - 5 = 0$; $x - 5y - 19 = 0$; $5x - 3y - 7 = 0$
    (b) $9x - y - 17 = 0$; $3x + 7y + 9 = 0$; $3x - 4y - 13 = 0$
    (c) 26
    (d) 11

Exercises 11, Page 323

1. $y = \frac{3}{4}x - \frac{5}{2}$; y intercept $= -\frac{5}{2}$
3. $\frac{3}{5}x + (-\frac{4}{5})y = 2$; distance $= 2$
5. Altitudes: to $AB = 7$; to $BC = \frac{35\sqrt{58}}{58}$; to $AC = \frac{35\sqrt{53}}{53}$; area $= 17\frac{1}{2}$
7. $63°26'4''$
9. $123°41'24''$
11. $3x + 2y - 23 = 0$
13. $\frac{7\sqrt{194}}{194}$

Exercises 12, Page 326

3. $(-\frac{7}{3}, -4)(-\frac{2}{3}, -5)$
5. $90°$
7. (a) $AB = 5$; $BC = 5$; $AC = \sqrt{10}$
    (b) $y = 0$; $3x + 4y - 15 = 0$; $3x - y = 0$
    (c) 3; 3; $\frac{3}{2}\sqrt{10}$
    (d) $71°33'54''$; $71°33'54''$; $36°52'12''$
    (e) $x = 1$; $4x - 3y = 0$; $x + 3y - 6 = 0$
    (f) $(1, \frac{4}{3})$
    (g) $2x + y - 5 = 0$; $x - 2y = 0$; $x + 3y - 5 = 0$
    (h) $7\frac{1}{2}$
9. $x + 2y + 6 = 0$
17. $\frac{22\sqrt{17}}{17}$; $\frac{22\sqrt{29}}{29}$; $\frac{11\sqrt{10}}{10}$
19. $2x + y \pm 5\sqrt{5} = 0$

Exercises 13, Page 330

1. $x^2 + y^2 - 4x + 6y - 23 = 0$
3. $x^2 + y^2 + 6x - 8y = 0$
5. $x^2 + y^2 - 10y = 0$
7. $11x^2 + 11y^2 - 21x - 47y + 10 = 0$; $r = \frac{\sqrt{2210}}{22}$; $C\left(\frac{21}{22}, \frac{47}{22}\right)$

9. $x^2 + y^2 - 18x + 6y + 9 = 0$
11. $89x^2 + 89y^2 + 1246x - 356y - 324 = 0$
13. $x^2 + y^2 = 16$
15. $2x^2 + 2y^2 - 2(a + c)x - 2(b + d)y + a^2 + b^2 + c^2 + d^2 - k = 0$
17. $296x^2 + 296y^2 - 1440x - 31y - 8676 = 0.$
19. $2\sqrt{73}$

Exercises 14, Page 332

1. $r = 10 \cos \theta$
3. $r^2 + 10r \sin \theta = 75$
5. (a) $C(0, 0)$; $r = 7$
   (b) $C(3, 0)$; $r = 3$
   (c) $C(5, \pi/2)$; $r = 5$
   (d) $C(2, \pi/4)$; $r = 2$
   (e) $C(\frac{1}{2}, \pi/6)$; $r = \frac{1}{2}$
   (f) $C(6, 0)$; $r = 6$
   (g) $C(\frac{5}{2}, \cos^{-1} \frac{3}{5})$; $r = \frac{5}{2}$

Exercises 15, Page 339

1. $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$
3. $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$
5. $\dfrac{x^2}{64} + \dfrac{y^2}{39} = 1$
7. $\dfrac{x^2}{5} + \dfrac{y^2}{8} = 1$
9. (a) $e = \dfrac{\sqrt{5}}{3}$; $F(\pm 2\sqrt{5}, 0)$; $x = \pm \dfrac{18\sqrt{5}}{5}$
   (b) $e = \frac{4}{5}$; $F(0, \pm 4)$; $y = \pm \frac{25}{4}$
   (c) $e = \dfrac{2\sqrt{5}}{5}$; $F(0, \pm 2\sqrt{5})$; $y = \pm \dfrac{5\sqrt{5}}{2}$
11. Length of side $= \dfrac{15\sqrt{34}}{17}$
17. $A = \dfrac{50\pi\sqrt{2}}{3}$

Exercises 16, Page 342

1. $\dfrac{(x - 5)^2}{36} + \dfrac{(y + 3)^2}{27} = 1$
3. $\dfrac{(x - 2)^2}{16} + \dfrac{(y - 9)^2}{25} = 1$
5. $\dfrac{4(x - 3)^2}{75} + \dfrac{(y - 4)^2}{25} = 1$
7. $\dfrac{(x - 5)^2}{25} + \dfrac{16y^2}{25} = 1$; $e = \dfrac{\sqrt{15}}{4}$; $F\left(5 \pm \dfrac{5\sqrt{15}}{4}, 0\right)$; L.R. $= \frac{5}{8}$
9. $C(0, 1)$; $a = 5$; $b = \sqrt{5}$; $F(\pm 2\sqrt{5}, 1)$; $x = \pm\frac{5}{2}\sqrt{5}$
11. $C(3, -1)$; $a = 5\sqrt{2}$; $b = 5$; $F(3 \pm 5, -1)$; $x = 3 \pm 10$
13. $C(0, 3)$; $a = 4\sqrt{3}$; $b = 12$; $F(0, 3 \pm 4\sqrt{6})$; $y = 3 \pm 6\sqrt{6}$

15. (a) $r = \dfrac{6}{2 - \cos\theta}$; (b) $r = \dfrac{6}{2 - \cos\theta}$; (c) $r = \dfrac{6}{2 + \sin\theta}$

17. $r = \dfrac{ek}{1 - e\cos\theta}$

Exercises 17, Page 347

1. $e = \dfrac{\sqrt{34}}{5}$; $F(\pm\sqrt{34}, 0)$; $\dfrac{x}{5} \pm \dfrac{y}{3} = 0$

3. $e = \dfrac{\sqrt{6}}{2}$; $F(\pm 2\sqrt{3}, 0)$; $\dfrac{x}{2\sqrt{2}} \pm \dfrac{y}{2} = 0$

5. $e = \frac{1}{2}\sqrt{6}$; $F(\pm 2\sqrt{3}, 0)$; $\dfrac{x}{\sqrt{2}} \pm y = 0$

7. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

9. $\dfrac{x^2}{5} - \dfrac{y^2}{20} = 1$

11. $\sqrt{2}$

13. $\dfrac{x^2}{9} - \dfrac{y^2}{18} = 1$

Exercises 18, Page 351

1. (a) $\dfrac{(x-2)^2}{9} - \dfrac{(y+3)^2}{25} = 1$; $C(2, -3)$; $F(2 \pm \sqrt{34}, -3)$; $x = 2 \pm \dfrac{9\sqrt{34}}{34}$;
   $5x - 3y = 19$; $5x + 3y = 1$

   (b) $\dfrac{(y+6)^2}{25} - \dfrac{(x-1)^2}{9} = 1$; $C(1, -6)$; $F(1, -6 \pm \sqrt{34})$; $y = -6 \pm \dfrac{25\sqrt{34}}{34}$;
   $5x - 3y - 23 = 0$; $5x + 3y + 13 = 0$

   (c) Two straight lines: $5x - 3y + 13 = 0$; $5x + 3y - 23 = 0$

   (d) $\dfrac{(x-3)^2}{\frac{51}{2}} - \dfrac{(y-3)^2}{102} = 1$; $C(3, 3)$; $F\left(3 \pm \dfrac{\sqrt{510}}{2}, 3\right)$; $x = 3 \pm \dfrac{51}{\sqrt{510}}$;
   $2x - y - 3 = 0$; $2x + y - 9 = 0$

   (e) $\dfrac{(x+1)^2}{\frac{13}{2}} - \dfrac{(y-3)^2}{\frac{13}{2}} = 1$; $C(-1, 3)$; $F(-1 \pm \sqrt{13}, 3)$; $x = -1 \pm \dfrac{\sqrt{13}}{2}$;
   $x - y + 4 = 0$; $x + y - 2 = 0$

3. $9x^2 - 3y^2 - 36x + 24y - 16 = 0$

5. $r = \dfrac{ek}{1 + e\cos\theta}$

7. $56x^2 - 25y^2 - 168x - 224 = 0$

9. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{20} = 1$

11. $b^2(x-k)^2 - a^2(y-k)^2 = 0$

Exercises 19, Page 355

3. $4p$

5. $y^2 = 24x$

7. (a) $V(3, 0)$; $F(5, 0)$; $x = 1$
   (b) $V(5, -2)$; $F(8, -2)$; $x = 2$
   (c) $V(1, 0)$; $F(1, \frac{3}{2})$; $2y + 3 = 0$

9. $x^2 = 20y$

11. $y^2 - 6y - 9x + 27 = 0$

13. $y^2 - 10x + 45 = 0$

17. $6\frac{9}{16}$ ft

Exercises 20, Page 361

1. $3x' + 7y' - 6\sqrt{2} = 0$

3. $(3 - 5\sqrt{3})x' - (5 - 3\sqrt{3})y' - 14 = 0$

5. $x'^2 + y'^2 = 36$

7. $x'^2 - y'^2 = 12$; $e = \sqrt{2}$

9. $e = \dfrac{2\sqrt{5}}{5}$

11. $2y'^2 + 3\sqrt{2}x' - 3\sqrt{2}y' = 0$

Exercises 21, Page 363

1. $x'^2 - y'^2 = 14$; $\theta = 45°$

3. $\dfrac{x'^2}{3} + \dfrac{y'^2}{2} = 1$; $\theta = 45°$

5. $x'^2 = -4y'$; $\theta = \tan^{-1} \frac{4}{3}$

7. $\left(y' + \dfrac{3\sqrt{13}}{13}\right)^2 = -\frac{25}{13}$; $\theta = \tan^{-1} \frac{3}{2}$

9. $\dfrac{y''^2}{2} - \dfrac{x''^2}{48} = 1$; $\theta = \tan^{-1} \frac{4}{3}$

11. $\dfrac{x''^2}{35} + \dfrac{y''^2}{10} = 1$

Exercises 22, Page 366

1. $(x - y + 2)(2x - y + 3) = 0$

3. Not factorable

5. $(x - y + 1)(x + 7) = 0$

7. $(x - y + 2)^2 = 0$

9. $(x - 7)(2x + y) = 0$

Exercises 23, Page 372

1. (a) $\frac{1}{2}$; (b) $-\frac{3}{4}$; (c) 1; (d) -2

3. (a) $x - y - 1 = 0$; (b) $x + y - 3 = 0$

7. (a) $3x + 5y - 25 = 0$; (b) $25x - 15y - 27 = 0$

11. $10x \pm 9y - 48 = 0$

13. $x - y - 2 = 0$; $x + y + 2 = 0$

Exercises 24, Page 374

1. $2x - y + 1 = 0$

3. $y = 2x - 6$

5. $(6, 6)$

7. $y = \dfrac{-20 \pm 4\sqrt{34}}{9}x + \dfrac{125 \mp 16\sqrt{34}}{9}$

9. $y = mx \pm 2\sqrt{-mk}$

11. $y = -\frac{2}{3}x \pm \sqrt{7}$

Exercises 26, Page 380

1. $10x + 3y + 1 = 0$
3. $y = 20(2^x)$
5. $y = -\frac{5}{6}x + \frac{5}{6}x^2$
7. (a) $y = \frac{9}{2} - \frac{3}{2}x$ (b) $y = -\frac{3}{2} + \dfrac{\frac{9}{2}}{x}$
   (c) Cannot be done (d) Cannot be done
   (e) Not a unique answer
9. $y = \sqrt[5]{10}\,(\sqrt[5]{0.01})^x$

Exercises 27, Page 385

1. $S = 72.9 + 0.727t$
3. $P = 2.36 + 0.12W$
5. $P = 0.75 + 0.19R$
7. $T = -17.6 + 4.32I$

Exercises 28, Page 388

1. $y = 3 - \frac{15}{4}x + \frac{7}{8}x^2$
3. $y = \frac{9}{8} + \frac{7}{8}x + \frac{7}{8}x^2$
5. $y = 1.214 - 0.672x + 0.213x^2$

Exercises 29, Page 390

1. $y = \dfrac{34}{3} - \dfrac{50}{3x}$
3. $y = -3 + \dfrac{4}{x}$

Exercises 34, Page 409

5. $y = 2x - 1$
9. About 66°
11. (a) $xy = 12$ (b) $x^2 - y^2 + 4 = 0$
    (c) $y^2 = 16x - 8x^2 + x^3$ (d) $y = 1 - 2x^2$
    (e) $y^2 = x + 1$ (f) $y = 3x + 1$
    (g) $5x = 10 + 2y$ (h) $y = 2x^2$

Exercises 36, Page 418

1. $\sqrt{34}, \sqrt{77}, \sqrt{53}$
3. $ABC = 55°28'37''$; $BCA = 41°17'21''$; $BAC = 83°14'6''$
5. $\dfrac{\sqrt{23}}{6}$
7. $x^2 + y^2 + z^2 - 4x - 10y + 2z + 5 = 0$

Exercises 37, Page 422

1. $\frac{x}{2} + \frac{2y}{3} + \frac{\sqrt{11}}{6}z = 7$

5. $\frac{x}{\frac{20}{3}} + \frac{y}{-4} + \frac{z}{2} = 1$

7. $54°44'10''$

9. $2x + 6y + 3z = 98$

11. $-3x + 2y + 6z = 14$

13. $2x - y + 9z = 10$

Exercises 38, Page 424

3. $(5, -1, 2)$

5. $4x + 3y = 13$, $\sqrt{11}x - 3z = 15 + \sqrt{11}$

7. (a) $\cos\alpha = \frac{1}{3\sqrt{10}}, \cos\beta = \frac{8}{3\sqrt{10}}, \cos\gamma = \frac{5}{3\sqrt{10}}$

   (b) $\cos\alpha = \frac{2}{7\sqrt{6}}, \cos\beta = -\frac{1}{7\sqrt{6}}, \cos\gamma = -\frac{17}{7\sqrt{6}}$

9. $\theta = 21°39'$

Exercises 39, Page 427

1. $y^2 + z^2 = 6x$; $y^4 = 36(x^2 + z^2)$

3. $\frac{x^2}{9} - \frac{y^2 + z^2}{4} = 1$; $\frac{x^2 + z^2}{9} - \frac{y^2}{4} = 1$

Exercises 41, Page 433

1. (a) $r = 2$; (b) $r^2 + z^2 = 4$

3. (a) $r \sin^2\beta = 2a\cos\beta$; (b) $r^2 = 2az$

Index

(The numbers refer to pages.)